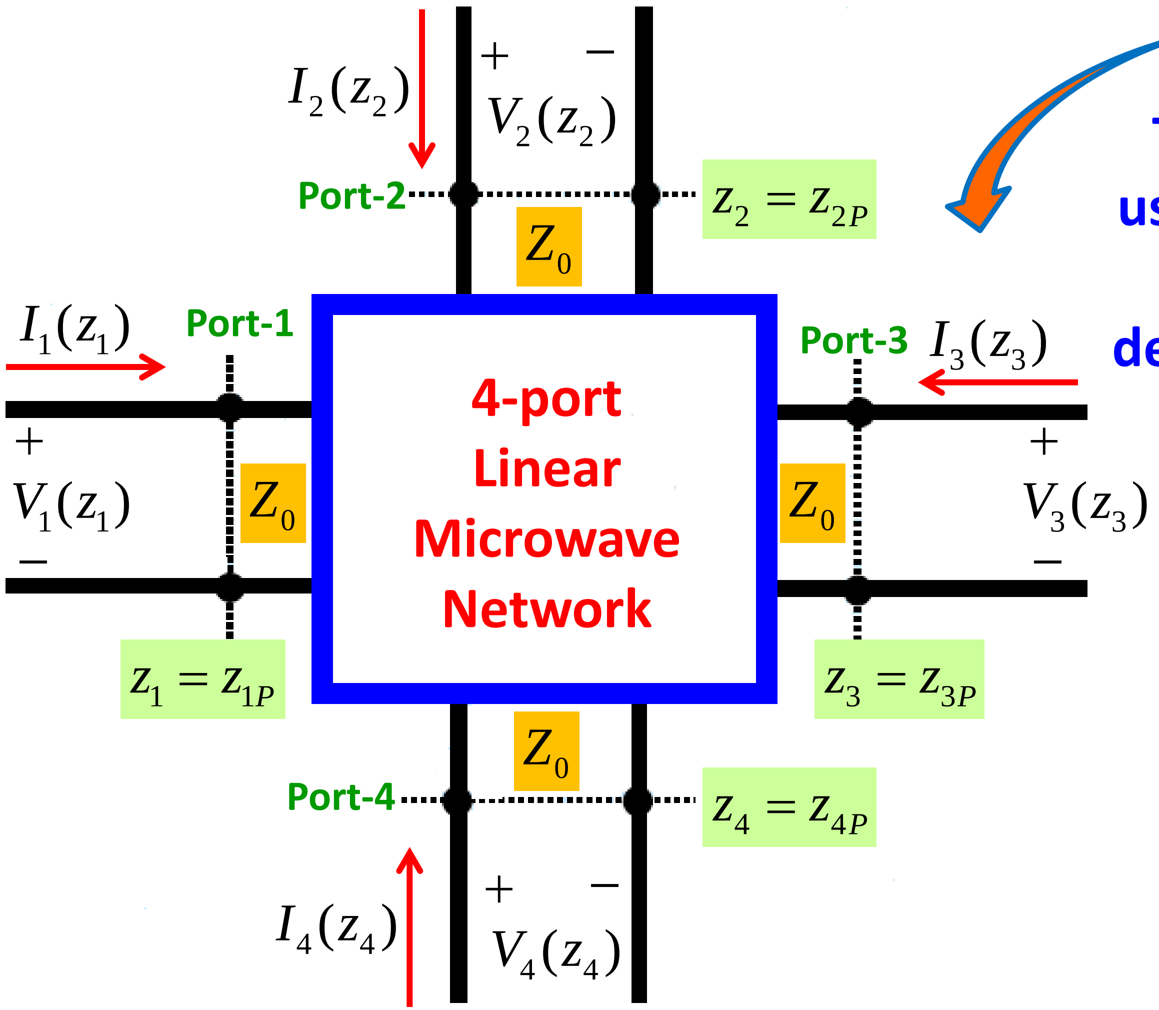
Lecture – 10

Date: 04.09.2014

- The Admittance Matrix
- Lossless and Reciprocal Networks
- Examples
- The Scattering Matrix
- Matched, Lossless, Reciprocal Devices
- Examples

The Admittance Matrix

- Let us consider the 4-port network again:



This can be characterized using admittance matrix – if currents are taken as dependent variables instead of voltages

The elements of admittance matrix are called trans-admittance parameters Y_{mn}

The Admittance Matrix (contd.)

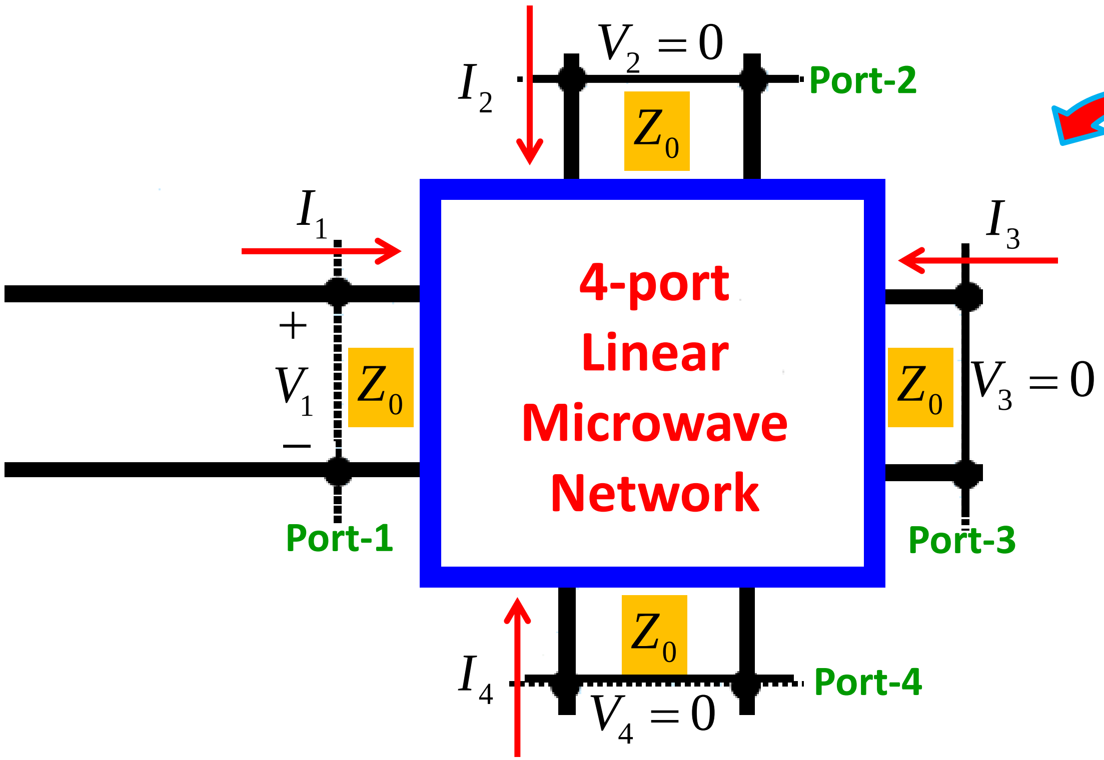
- The trans-admittances Y_{mn} are defined as:

$$Y_{mn} = \frac{I_m}{V_n}$$ **(given that $V_k = 0$ for all $k \neq n$)**

Important

$$Y_{mn} \neq \frac{1}{Z_{mn}}$$

- It is apparent that the voltage at all but one port must be equal to zero. This can be ensured by short-circuiting the voltage ports.



The ports should be short-circuited! not the TL connected to the ports

The Admittance Matrix (contd.)

- Once we have defined the terms by shorting various ports, it is time to formulate the admittance matrix.
- Since the network is **linear**, the current at any **one** port due to all the port voltages is simply the coherent **sum** of the currents at that port due to **each** of the port voltages.
- For example, the current at **port-3** is:

$$I_3 = Y_{34}V_4 + Y_{33}V_3 + Y_{32}V_2 + Y_{31}V_1$$

- Therefore we can generalize the current for **N-port** network as:

$$I_m = \sum_{n=1}^{N} Y_{mn}V_n \quad \Rightarrow \mathbf{I} = \mathbf{YV}$$

- Where **I** and **V** are vectors given as:

$$\mathbf{V} = \left[\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, ..., \mathrm{V}_\mathrm{N}\right]^T \qquad \mathbf{I} = \left[\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, ..., \mathrm{I}_\mathrm{N}\right]^T$$

The Admittance Matrix (contd.)

- The term **Y** is matrix given by:

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & & & \vdots \\ \vdots & & & \\ Y_{m1} & Y_{m2} & \cdots & Y_{mn} \end{bmatrix}$$

Admittance Matrix

- The values of elements in the admittance matrix are frequency dependents and often it is advisable to describe admittance matrix as:

$$\boldsymbol{Y(\omega)} = \begin{bmatrix} Y_{11}(\omega) & Y_{12}(\omega) & \cdots & Y_{1n}(\omega) \\ Y_{21}(\omega) & & & \vdots \\ \vdots & & & \\ Y_{m1}(\omega) & Y_{m2}(\omega) & \cdots & Y_{mn}(\omega) \end{bmatrix}$$

The Admittance Matrix (contd.)

You said that

$$Y_{mn} \neq \frac{1}{Z_{mn}}$$

Is there any relationship between admittance and impedance matrix of a given device?

Answer: Let us see if we can figure it out!

The Admittance Matrix (contd.)

- Recall that we can determine the inverse of a matrix. Denoting the matrix inverse of the admittance matrix as $\mathbf{Y}^{-1}$, we find:

$$\mathbf{I} = \mathbf{Y}\mathbf{V}$$

$$\Rightarrow \mathbf{Y}^{-1}\mathbf{I} = \mathbf{Y}^{-1}(\mathbf{Y}\mathbf{V}) \quad \rightarrow \quad \mathbf{Y}^{-1}\mathbf{I} = (\mathbf{Y}^{-1}\mathbf{Y})\mathbf{V}$$

$$\mathbf{Y}^{-1}\mathbf{I} = \mathbf{V}$$

- We also know: $\mathbf{V} = \mathbf{Z}\mathbf{I}$

$$\mathbf{Z} = \mathbf{Y}^{-1}$$

OR

$$\mathbf{Y} = Z^{-1}$$

Reciprocal and Lossless Networks

- We can **classify** multi-port devices or networks as either **lossless** or **lossy**; **reciprocal** or **non-reciprocal**. Let's look at each classification individually.

Lossless Network

- A **lossless** network or device is simply one that **cannot** absorb power. This does **not** mean that the delivered power at **every port** is zero; rather, it means the total power flowing **into** the **device** must equal the total power **exiting** the **device**.

- A lossless device exhibits an impedance matrix with an interesting **property**. Perhaps not surprisingly, we find for a lossless device that the **elements** of its impedance matrix will be **purely reactive**:

$$\mathrm{Re}(Z_{mn}) = 0$$

For a lossless device

Reciprocal and Lossless Networks (contd.)

- If the device is lossy, then the elements of the impedance matrix must have **at least** one element with a real (i.e., resistive) component.
- Furthermore, we can similarly say that if the elements of an **admittance** matrix are **all** purely imaginary (i.e., Re{Y_{mn}} =0), then the device is lossless.

Reciprocal Network

- Ideally, most **passive, linear** microwave components will turn out to be **reciprocal**—regardless of whether the designer **intended** it to be or not!
- Reciprocity is a tremendously important characteristic, as it greatly **simplifies** an impedance or admittance matrix!
- Specifically, we find that a reciprocal device will result in a **symmetric** impedance and admittance **matrix**, meaning that:

$$Z_{mn} = Z_{nm} \qquad Y_{mn} = Y_{nm}$$

For a reciprocal device

Reciprocal and Lossless Networks (contd.)

- **For example**, we find for a reciprocal device that $Z_{23} = Z_{32}$, and $Y_{12} = Y_{21}$.

$$\mathbf{Z} = \begin{bmatrix} j2 & 0.1 & j3 \\ -j & -1 & 1 \\ 4 & -2 & 0.5 \end{bmatrix}$$

neither lossless nor reciprocal

lossless, but not reciprocal

$$\mathbf{Z} = \begin{bmatrix} j2 & j0.1 & j3 \\ -j & -j1 & j1 \\ j4 & -j2 & j0.5 \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} j2 & -j & 4 \\ -j & -1 & -j2 \\ 4 & -j2 & j0.5 \end{bmatrix}$$

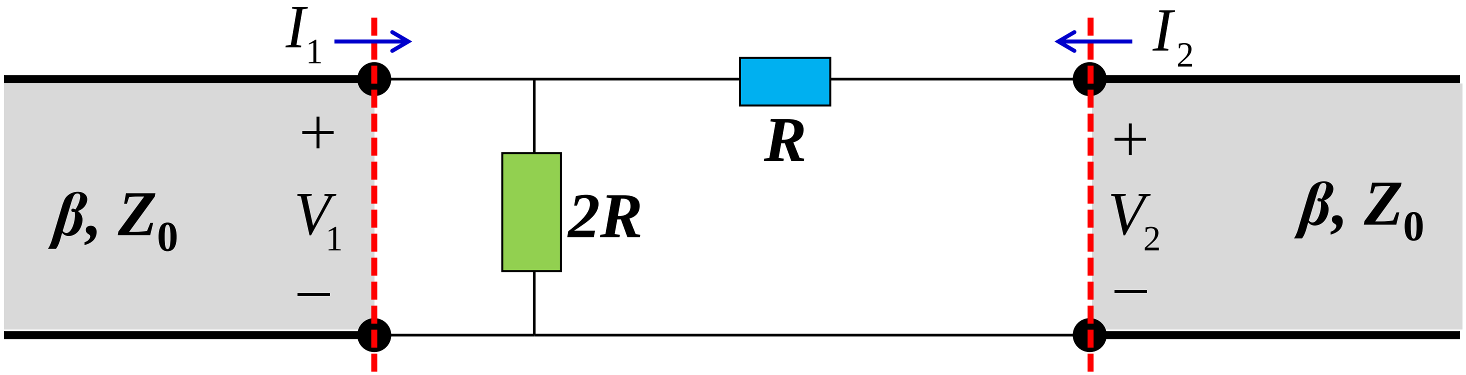
reciprocal, but not lossless

lossless and reciprocal

$$\mathbf{Z} = \begin{bmatrix} j2 & -j & j4 \\ -j & -j & -j2 \\ j4 & -j2 & j0.5 \end{bmatrix}$$

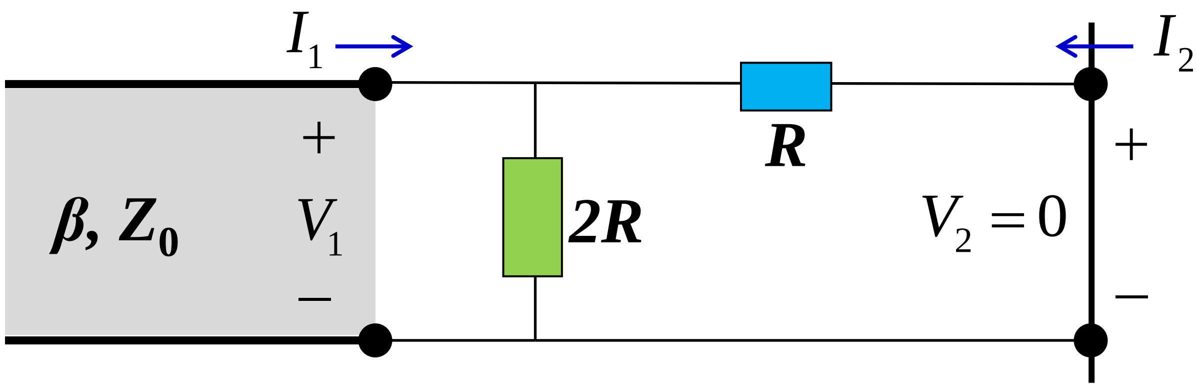
Example – 1

- determine the **admittance matrix** of the following two-port device.



Example – 1 (contd.)

Step-1: Place a **short** at port 2



Step-2: Determine currents I_1 and I_2

- Note that **after** the short was placed at port 2, both resistors are in **parallel**, with a potential V_1 across each

Therefore current I_1 is

$$I_1 = \frac{V_1}{2R} + \frac{V_1}{R} = \frac{3V_1}{2R}$$

- The current I_2 equals the portion of current I_1 through R but with opposite sign

$$I_2 = -\frac{V_1}{R}$$

Example – 1 (contd.)

Step-3: Determine the trans-admittances Y_{11} and Y_{21}

$$Y_{11} = \frac{I_1}{V_1} = \frac{3}{2R}$$

$$Y_{21} = \frac{I_2}{V_1} = -\frac{1}{R}$$

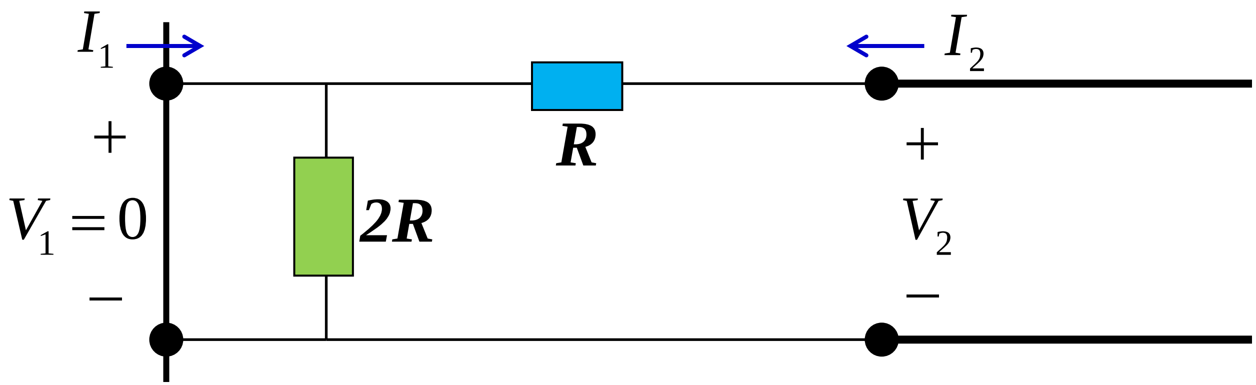
Note that Y_{21} is real and negative

This is **still** a valid physical result, **although** you will find that the **diagonal** terms of an impedance or admittance matrix (e.g., Y_{22} , Z_{11}, Y_{44}) will **always** have a real component that is **positive**

To find the **other two** trans-admittance parameters, we must **move** the short and then **repeat** each of our previous steps!

Example – 1 (contd.)

Step-1: Place a **short** at port 1



Step-2: Determine currents I_1 and I_2

- Note that **after** a short was placed at port **1**, resistor 2R has **zero** voltage across it—and thus **zero current** through it!

Therefore:

$$I_2 = \frac{V_2}{R}$$

$$I_1 = -I_2 = -\frac{V_2}{R}$$

Example – 1 (contd.)

Step-3: Determine the trans-admittances Y_{12} and Y_{22}

$$Y_{12} = \frac{I_1}{V_2} = -\frac{1}{R}$$

$$Y_{22} = \frac{I_2}{V_2} = \frac{1}{R}$$

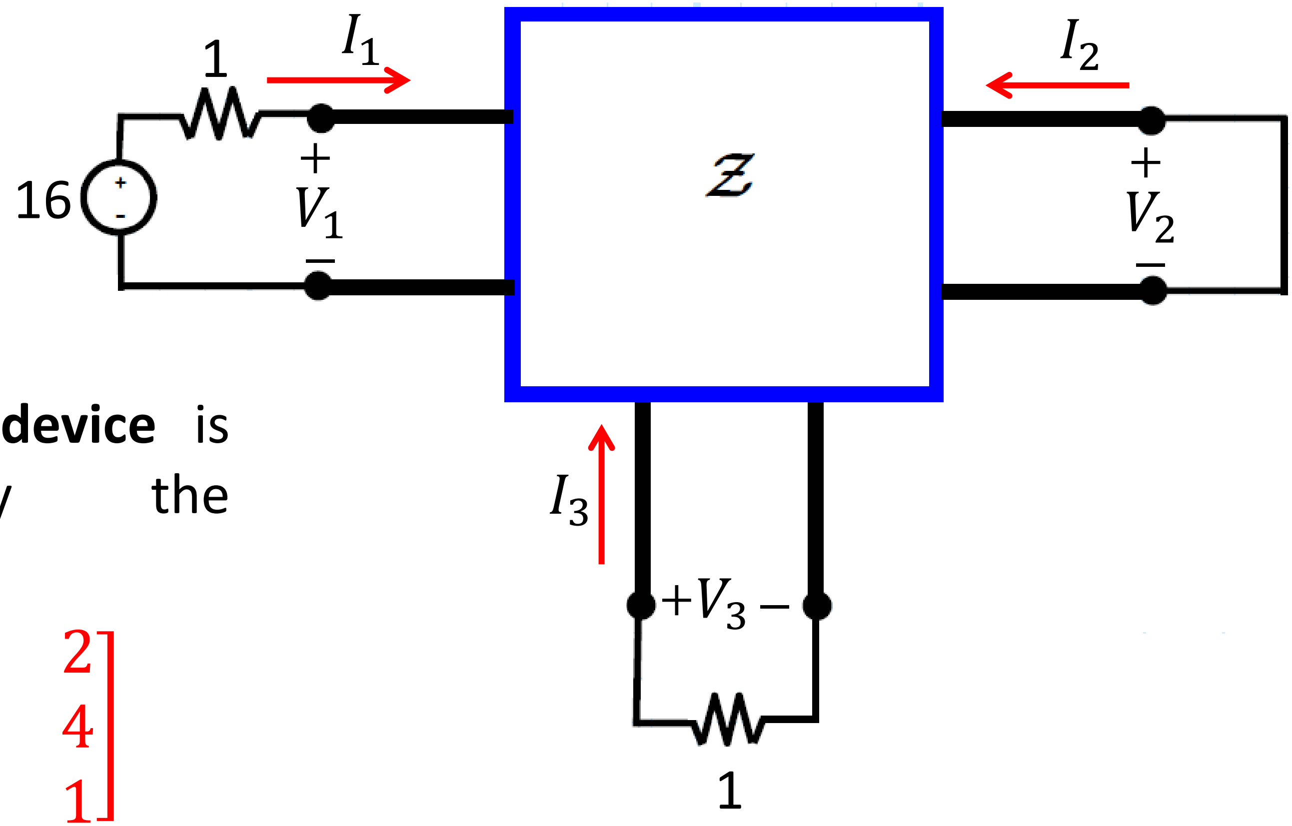
Therefore the admittance matrix is:

$$\mathbf{Y} = \begin{bmatrix} 3/2R & -1/R \\ -1/R & 1/R \end{bmatrix}$$

Is it lossless or reciprocal?

Example – 2

- Consider this circuit:



- Where the 3-port **device** is characterized by the **impedance matrix**:

$$\mathbf{Z}=\begin{bmatrix}2 & 1 & 2\\ 1 & 1 & 4\\ 2 & 4 & 1\end{bmatrix}$$

- determine all port **voltages** V_1, V_2, V_3 and all **currents** I_1, I_2, I_3.

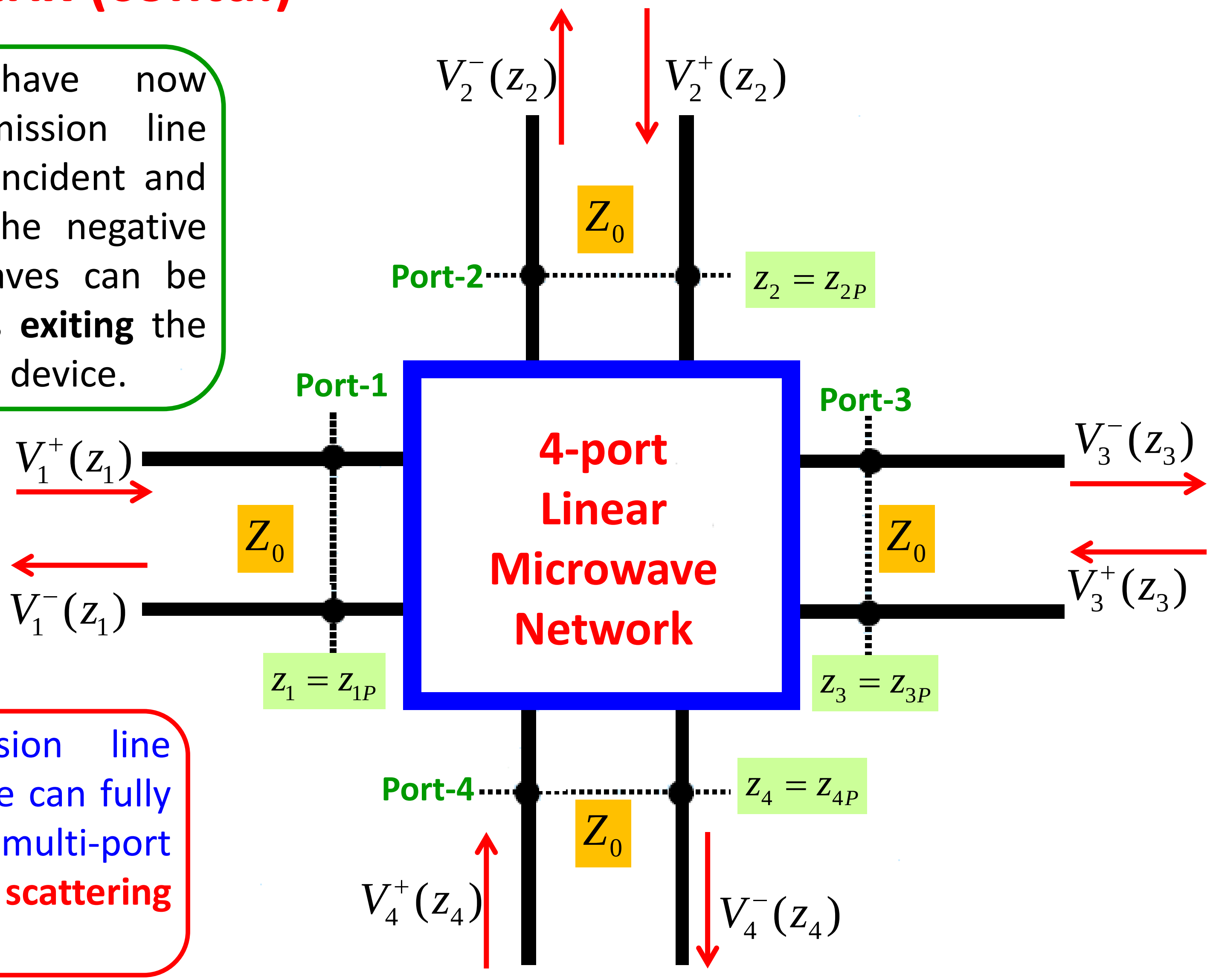
HA # 2

Scattering Matrix

- At **"low"** frequencies, a **linear** device or network can be fully characterized using an **impedance or admittance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.
- But, at high frequencies, it is **not feasible** to measure total currents and voltages!
- Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^+(z)$ and $V^-(z)$ → enables determination of relationship between the incident and reflected waves at **each** device terminal to the incident and reflected waves at **all** other terminals
- These relationships are completely represented by the **scattering matrix** that **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω, and a given line impedance Z_0

Scattering Matrix (contd.)

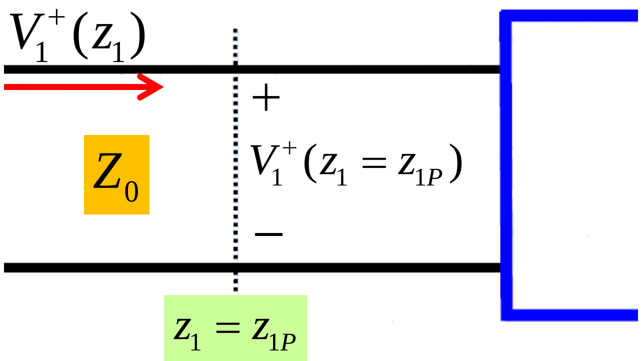
Note that we have now characterized transmission line activity in terms of incident and "reflected" waves. The negative going "reflected" waves can be viewed as the waves **exiting** the multi-port network or device.



Viewing transmission line activity this way, we can fully characterize a multi-port device by its **scattering parameters**!

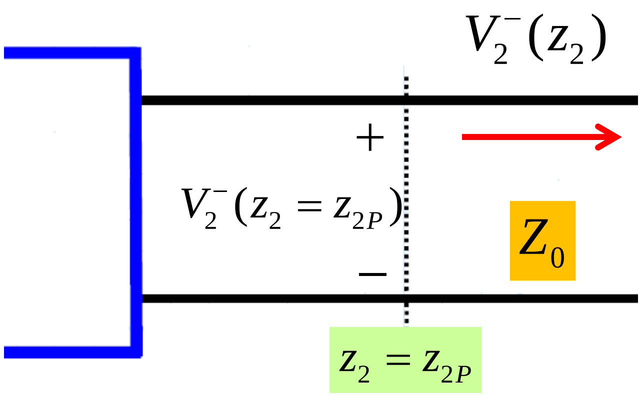
Scattering Matrix (contd.)

- Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).



Say we measure/determine the voltage of the wave flowing **into port 1,** at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).

Say we then measure/determine the voltage of the wave flowing **out** of **port 2,** at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).



The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is known as the **scattering parameter** S_{21}

Scattering Matrix (contd.)

Therefore:

$$S_{21} = \frac{V_2^-(z_2 = z_{2P})}{V_1^+(z_1 = z_{1P})} = \frac{V_2^- e^{+j\beta z_{2P}}}{V_1^+ e^{-j\beta z_{1P}}} = \frac{V_2^-}{V_1^+} e^{+j\beta(z_{2P} + z_{1P})}$$

Similarly:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})}$$

$$S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

- We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3) and $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4), given that the input to all other ports (1,2, and 3) are zero
- Thus, more **generally**, the ratio of the wave incident on port **n** to the wave emerging from port **m** is:

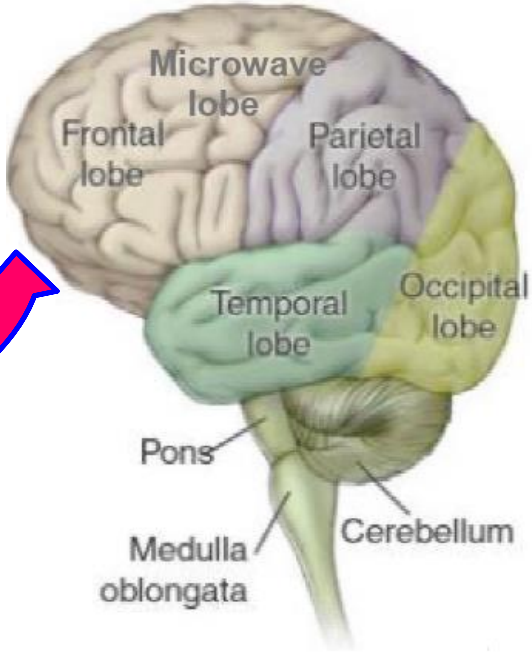
$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad V_k^+(z_k) = 0 \quad \textbf{for all } k \neq n$$

Scattering Matrix (contd.)

- Note that, frequently the port positions are assigned a **zero** value (e.g., z_{1P}=0, z_{2P}=0). This of course **simplifies** the scattering parameter calculation:

$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_m^+ e^{+j\beta 0}}{V_n^- e^{-j\beta 0}} = \frac{V_m^+}{V_n^-}$$

- We will **generally assume** that the port locations are defined as z_{nP}=0, and thus use the **above** notation. But **remember** where this expression came from!

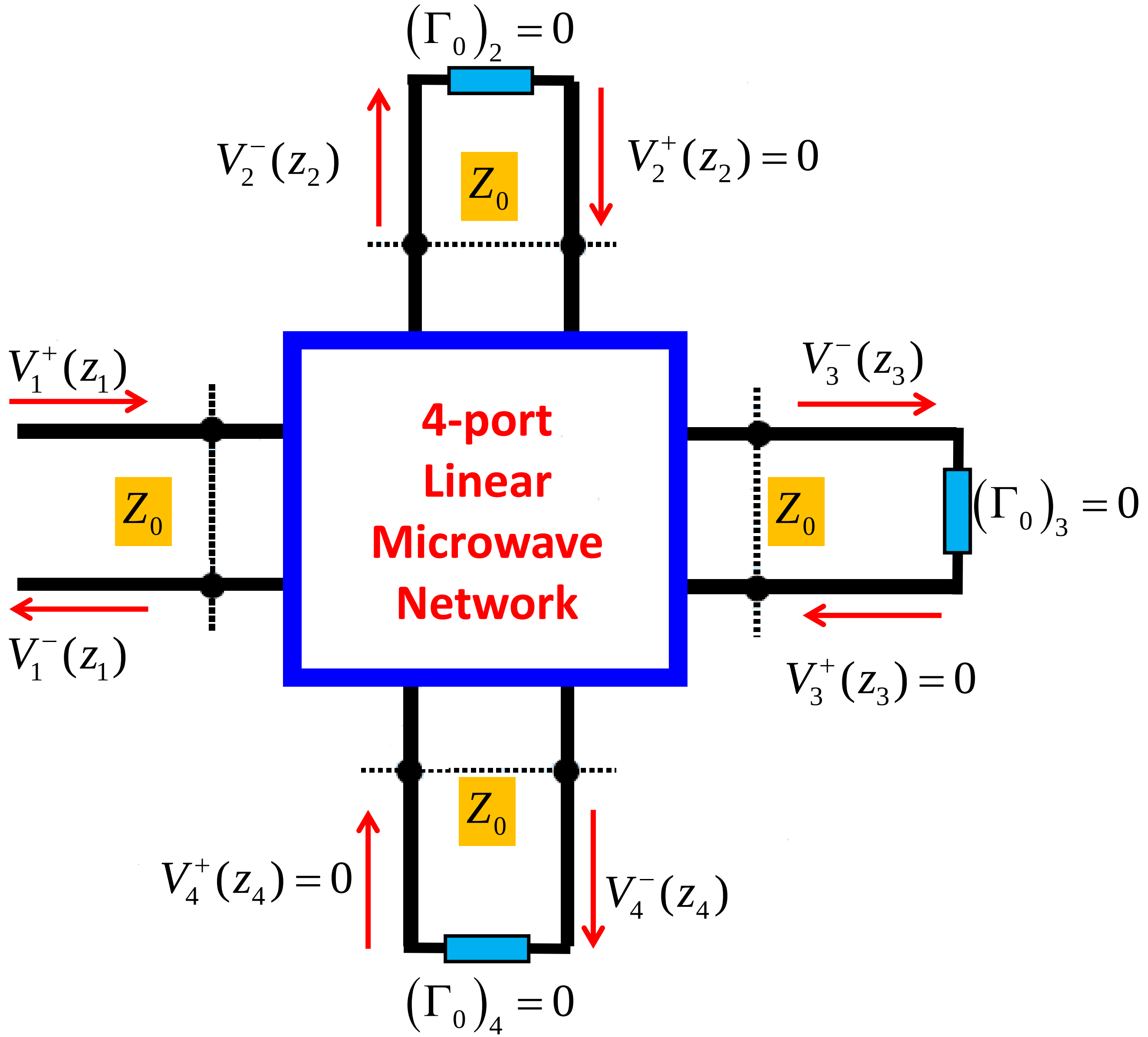


Scattering Matrix (contd.)

Q: How do we ensure that **only one** **i**ncident wave is non-zero ?

A: **Terminate** all other ports with a **matched load**!

Scattering Matrix (contd.)



Scattering Matrix (contd.)

- Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $(\Gamma_0)_n = 0$ and therefore:

$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Scattering Matrix (contd.)

Just between you and me, I think you've messed this up! **In all** previous slides you said that if $\Gamma_0 = 0$, the wave in the **minus** direction would be zero:

$$V^{-}(z) = 0 \qquad \text{if} \qquad \Gamma_0 = 0$$

but just **now** you said that the wave in the **positive** direction would be zero:

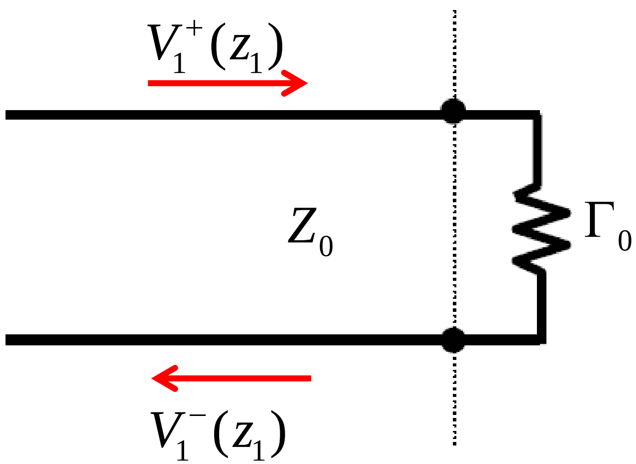
$$V^{+}(z) = 0 \qquad \text{if} \qquad \Gamma_0 = 0$$

Obviously, there is **no way** that **both** statements can be correct!

Scattering Matrix (contd.)

Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:

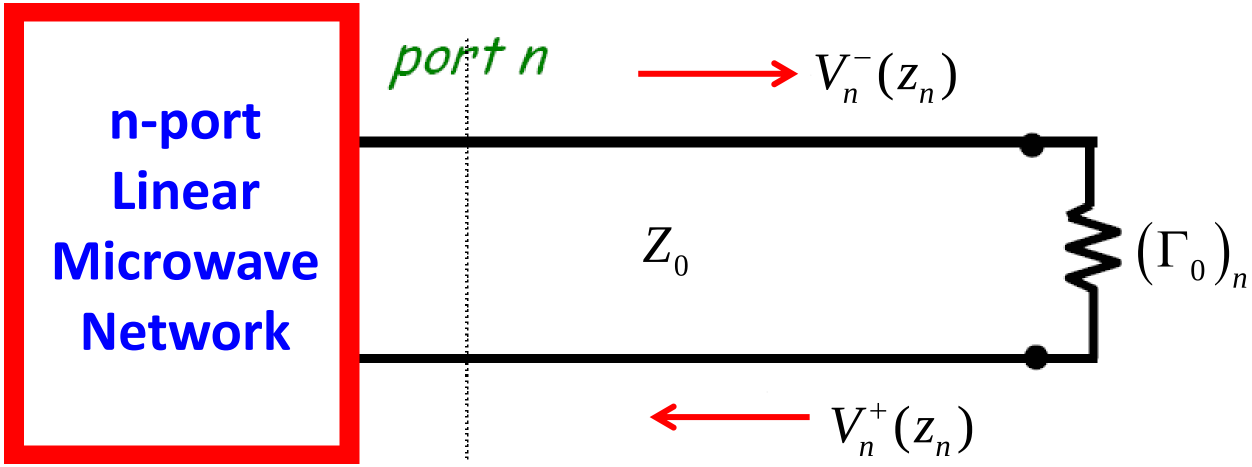


$V^-(z\ ) = 0$ if $\Gamma_0 = 0$

In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Scattering Matrix (contd.)

Contrast this with the case we are **now** considering:



- For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port n), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port n).

Scattering Matrix (contd.)

- **back** to our discussion of **S-parameters**. We found that **if** z_{nP} = 0 for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_n^+ and V_m^-

$$S_{mn} = \frac{V_m^-}{V_n^+} \qquad V_k^+(z_k) = 0 \quad \textbf{for all } k \neq n$$

- Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_m^-}{V_n^+}$$ **(for all ports, except port *n*, are terminated in matched loads)**

- One more **important** note—notice that for the ports terminated in matched loads (i.e., those ports with **no** incident wave), the voltage of the exiting **wave** is also the **total** voltage!

Scattering Matrix (contd.)

$$V_m(z_m) = V_m^+ e^{-j\beta z_m} + V_m^- e^{+j\beta z_m} = 0 + V_m^- e^{+j\beta z_m} = V_m^- e^{+j\beta z_m}$$

For all terminated ports!

- Therefore, the value of the exiting wave **at** each terminated **port** is the value of the total voltage **at** those ports:

$$V_m(0) = V_m^+ + V_m^- = 0 + V_m^- = V_m^-$$

For all terminated ports!

- And so, we can express **some** of the scattering parameters equivalently as:

$$S_{mn} = \frac{V_m(0)}{V_n^+}$$

(for terminated port *m*, i.e., for m ≠ n)

This result could be **helpful** for determining scattering parameters where m≠n (e.g., S_{21}, S_{43}, S_{13}), as we can often use traditional **circuit theory** to easily determine the **total** port voltage $V_m(0)$

However, we **cannot** use the expression above to determine the scattering parameters when m=n (e.g., S_{11}, S_{22}, S_{33}).

Scattering Matrix (contd.)

- We can use the scattering matrix to determine the solution for a more **general** circuit—one where the ports are **not** terminated in matched loads!
- Since the device is **linear**, we can apply **superposition**. The output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** wave!
- For example, the **output** wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_3^- = S_{34}V_4^+ + S_{33}V_3^+ + S_{32}V_2^+ + S_{31}V_1^+$$

- More **generally**, the output at port m of an N-port device is:

$$V_m^- = \sum_{n=1}^{N} S_{mn}V_n^+ \qquad \mathbf{z_{nP} = 0}$$

Scattering Matrix (contd.)

- This expression of Scattering parameter can be written in **matrix** form as:

$$V^{-} = SV^{+}$$

Row Vector

Row Vector

Scattering Matrix

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & & & \vdots \\ \vdots & & & \\ S_{m1} & S_{m2} & \cdots & S_{mn} \end{bmatrix}$$

The scattering matrix is N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_0 describes a single-port device (e.g., a load)!

Scattering Matrix (contd.)

- The values of the scattering matrix for a particular device or network, just like Γ_0, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\boldsymbol{S}(\omega) = \begin{bmatrix} S_{11}(\omega) & S_{12}(\omega) & \dots & S_{1n}(\omega) \\ S_{21}(\omega) & & & \vdots \\ \vdots & & & \\ S_{m1}(\omega) & S_{m2}(\omega) & \dots & S_{mn}(\omega) \end{bmatrix}$$

- Also realize that—also just like Γ_0—the scattering matrix is dependent on **both** the **device/network** and the Z_0 value of the **transmission lines connected** to it.

- Thus, a device connected to transmission lines with $Z_0 = 50\Omega$ will have a **completely different scattering matrix** than that same device connected to transmission lines with $Z_0 = 100\Omega$

Matched, Lossless, Reciprocal Devices

- A device can be **lossless** or **reciprocal**. In addition, we can also classify it as being **matched.**
- Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix.**

Matched Device

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will come out from a port if a signal is incident on that port (but **only** that port!).

In other words,

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{For all m}$$

When all the ports 'm' are matched

Matched, Lossless, Reciprocal Devices (contd.)

- It is apparent that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

$$\mathbf{S}=\begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

Scattering matrix for a matched, three-port device

Lossless Device

- For a lossless device, all of the power that is delivered to each device port must eventually find its way **out**!
- In other words, power is not **absorbed** by the network—no power to be **converted to heat**!
- The **power incident** on some port m is related to the amplitude of the **incident wave** (V_m^+) as:

$$P_m^+ = \frac{|V_m^+|^2}{2Z_0}$$

Matched, Lossless, Reciprocal Devices (contd.)

- The power of the **wave exiting** the port is:

$$P_m^- = \frac{|V_m^-|^2}{2Z_0}$$

- the power absorbed by that port is the **difference** of the incident power and reflected power:

$$\Delta P_m = P_m^+ - P_m^- = \frac{|V_m^+|^2}{2Z_0} - \frac{|V_m^-|^2}{2Z_0}$$

- For an N-port device, the total incident power is:

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{1}{2Z_0}\sum_{m=1}^{N}|V_m^+|^2$$

$$|V_m^+|^2 = (\mathbf{V}^+)^H \mathbf{V}^+$$

$(V^+)^H$ is the conjugate transpose of the row vector V^+

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{(\mathbf{V}^+)^H \mathbf{V}^+}{2Z_0}$$

Similarly, the total reflected power

$$P^- = \sum_{m=1}^{N} P_m^- = \frac{(\mathbf{V}^-)^H \mathbf{V}^-}{2Z_0}$$

Matched, Lossless, Reciprocal Devices (contd.)

- Recall that the incident and reflected wave amplitudes are **related** by the **scattering matrix** of the device as:

$$\mathbf{V}^- = \mathbf{S}\mathbf{V}^+$$

- Therefore,

$$P^- = \frac{(\mathbf{V}^-)^H \mathbf{V}^-}{2Z_0} = \frac{(\mathbf{V}^+)^H \mathbf{S}^H\mathbf{S}\mathbf{V}^+}{2Z_0}$$

- Therefore the **total power delivered** to the N-port device is:

$$\Delta P = P^+ - P^- = \frac{(\mathbf{V}^+)^H \mathbf{V}^+}{2Z_0} - \frac{(\mathbf{V}^+)^H \mathbf{S}^H\mathbf{S}\mathbf{V}^+}{2Z_0}$$

$$\Rightarrow \Delta P = \frac{(\mathbf{V}^+)^H}{2Z_0}\left(\boldsymbol{I} - \mathbf{S}^H\mathbf{S}\right)\mathbf{V}^+$$

- For a lossless device: **ΔP=0** $\Rightarrow \frac{(\mathbf{V}^+)^H}{2Z_0}\left(\boldsymbol{I} - \mathbf{S}^H\mathbf{S}\right)\mathbf{V}^+ = 0$ **For all V$^+$**

- Therefore: $\boldsymbol{I} - \mathbf{S}^H\mathbf{S} = 0$ $\Rightarrow \mathbf{S}^H\mathbf{S} = \boldsymbol{I}$

Matched, Lossless, Reciprocal Devices (contd.)

$\Rightarrow \mathbf{S}^H \mathbf{S} = \boldsymbol{I}$ → a special kind of matrix known as a **unitary matrix**

If a network is **lossless**, then its scattering matrix **S** is **unitary**

- **How to recognize a unitary matrix?**

The **columns** of a unitary matrix form an **orthonormal set**!

Example:

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

each **column** of the scattering matrix will have a **magnitude equal to one**

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \qquad \text{For all } n$$

inner product (i.e., dot product) of **dissimilar columns** must be **zero**

$$\sum_{m=1}^{N} S_{mi} S_{mj}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \ldots + S_{Ni} S_{Nj}^* = 0 \qquad \text{For all } i \neq j$$

dissimilar columns are orthogonal

Matched, Lossless, Reciprocal Devices (contd.)

- For example, for a lossless **three-port** device: say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{\left|V_1^+\right|^2}{2Z_0}$$

- and the power **exiting** the device at each port is:

$$P_m^- = \frac{\left|V_m^-\right|^2}{2Z_0} = \frac{\left|S_{m1}V_1^+\right|^2}{2Z_0} = \left|S_{m1}\right|^2 P_1^+$$

- The **total** power exiting the device is therefore:

$$P^- = P_1^- + P_2^- + P_3^- = \left|S_{11}\right|^2 P_1^+ + \left|S_{21}\right|^2 P_1^+ + \left|S_{31}\right|^2 P_1^+$$

$$\Rightarrow P^- = \left(\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2\right) P_1^+$$

Matched, Lossless, Reciprocal Devices (contd.)

$$\Rightarrow P^{-} = \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 \right) P_1^{+}$$

- Since this device is **lossless**, then the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^{-} = P_1^{+}$). This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1$$

Of course, this will be true if the incident wave is placed on **any** of the **other** ports of this lossless device

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1 \qquad |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1$$

- We can state in general then that: $\sum_{m=1}^{N} |S_{mn}|^2 = 1$ For all n
- In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

Matched, Lossless, Reciprocal Devices (contd.)

- An **example** of a (unitary) scattering matrix for a 4-port **lossless** device is:

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1/2 & j\sqrt{3}/2 & 0 \\ 1/2 & 0 & 0 & j\sqrt{3}/2 \\ j\sqrt{3}/2 & 0 & 0 & 1/2 \\ 0 & j\sqrt{3}/2 & 1/2 & 0 \end{bmatrix}$$

Reciprocal Device

- Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.
- For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

Matched, Lossless, Reciprocal Devices (contd.)

- For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$S^T = S$$

where T indicates transpose.

- An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\mathbf{S} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10-j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Example – 3

- A **lossless**, **reciprocal** 3-port device has S-parameters of $S_{11} = {}^{1}/_{2}$, $S_{31} = {}^{1}/_{\sqrt{2}}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.

→ Find the remaining **6** scattering parameters.

Q: This problem is clearly **impossible**—you have not provided us with sufficient **information**!

A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

Example – 3 (contd.)

- Start with what we **currently** know:

$$\mathbf{S}=\begin{bmatrix} 1/2 & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$$

- As the device is **reciprocal**, we then also know:

$$S_{12} = S_{21} \qquad S_{13} = S_{31} = 1/\sqrt{2} \qquad S_{32} = S_{23}$$

- And therefore:

$$\mathbf{S}=\begin{bmatrix} 1/2 & S_{21} & 1/\sqrt{2} \\ S_{21} & S_{22} & S_{32} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$$

- Now, since the device is **lossless**, we know that:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1 \Rightarrow (1/2)^2 + |S_{21}|^2 + (1/\sqrt{2})^2 = 1$$

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1 \Rightarrow |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1 \Rightarrow (1/2)^2 + |S_{32}|^2 + (1/\sqrt{2})^2 = 1$$

Columns have unit magnitude

Example – 3 (contd.)

$$0 = S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* = \frac{1}{2}S_{12}^* + S_{21}S_{22}^* + \frac{1}{\sqrt{2}}S_{32}^*$$

$$0 = S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* = \frac{1}{2}\frac{1}{\sqrt{2}} + S_{21}S_{32}^* + \frac{1}{\sqrt{2}}(0)$$

$$0 = S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* = S_{21}\left(\frac{1}{\sqrt{2}}\right) + S_{22}S_{32}^* + S_{32}(0)$$

Dissimilar columns are **orthogonal**

We can simplify these expressions and can further simplify them by using the fact that the elements are all **real**, and therefore $S_{21} = S_{21}{}^*$ (etc.).

Q: I count the simplified expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is **no** solution!

Example – 3 (contd.)

A: Actually, we have **six** real equations and **six** real unknowns, since scattering element has a magnitude and phase. In this case we know the values are **real**, and thus the phase is either $0°$ or $180°$(i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

- the scattering matrix for the given **lossless**, **reciprocal** device is:

$$\mathbf{S}=\begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

A Matched, Lossless, Reciprocal 3-Port Network

- Consider a 3-port device. Such a device would have a scattering matrix :

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

- Assuming the device is passive and made of simple (isotropic) materials, the device will be **reciprocal**, so that:

$$S_{21} = S_{12} \qquad S_{31} = S_{13} \qquad S_{23} = S_{32}$$

- Similarly, if it is **matched**, we know that: $S_{11} = S_{22} = S_{33} = 0$

- As a result, a **lossless, reciprocal** device would have a scattering matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{21} & 0 & S_{32} \\ S_{31} & S_{32} & 0 \end{bmatrix}$$

A Matched, Lossless, Reciprocal 3-Port Network (contd.)

- if we wish for this network to be **lossless**, the scattering matrix must be **unitary**, and therefore:

$$|S_{21}|^2 + |S_{31}|^2 = 1 \qquad S_{31}^* S_{32} = 0$$
$$|S_{12}|^2 + |S_{32}|^2 = 1 \qquad S_{21}^* S_{32} = 0$$
$$|S_{13}|^2 + |S_{23}|^2 = 1 \qquad S_{31}^* S_{31} = 0$$

- Since each complex value S is represented by **two real numbers** (i.e., real and imaginary parts), the equations above result in **9** real equations. The problem is, the 3 complex values S_{21}, S_{31} and S_{32} are represented by only **6** real unknowns.

We have **over constrained** our problem ! There are **no unique solutions** to these equations !

A Matched, Lossless, Reciprocal 3-Port Network (contd.)

As unlikely as it might seem, this means that a matched, lossless, reciprocal **3-port** device of **any** kind is a **physical impossibility**!

You **can** make a lossless reciprocal 3-port device, **or** a matched reciprocal 3-port device, **or even** a matched, lossless (but non-reciprocal) 3-port network.

But try as you might, you **cannot** make a lossless, matched, **and** reciprocal three port component!

Matched, Lossless, Reciprocal 4-Port Network

Guess what! I have determined that—unlike a **3-port** device—a matched, lossless, reciprocal **4-port** device **is** physically possible! In fact, I've found **two** general solutions!

- The first solution is referred to as the **symmetric** solution:

$$\boldsymbol{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Matched, Lossless, Reciprocal 4-Port Network (contd.)

- Note for the symmetric solution, every row and every column of the scattering matrix has the **same** four values (i.e., α, jβ, and two zeros)!
- The second solution is referred to as the **anti-symmetric** solution:

$$\boldsymbol{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Note that for this anti-symmetric solution, **two** rows and **two** columns have the same four values (i.e., α, β, and two zeros), while the **other** two row and columns have (slightly) **different** values (α, -β, and two zeros)

- It is **quite** evident that each of these solutions are **matched** and **reciprocal**. However, to ensure that the solutions are indeed **lossless**, we must place an **additional** constraint on the values of α, β. Recall that a **necessary** condition for a lossless device is:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \qquad \textbf{For all } n$$

Matched, Lossless, Reciprocal 4-Port Network (contd.)

- For the **symmetric** case, we find:

$$|\alpha|^2 + |\beta|^2 = 1$$

- Similarly, for the **anti-symmetric** case, we find:

$$|\alpha|^2 + |\beta|^2 = 1$$

- It is evident that if the scattering matrix is **unitary** (i.e., lossless), the values α and β **cannot** be independent, but must be **related** as:

$$|\alpha|^2 + |\beta|^2 = 1$$

- **Generally** speaking, we can find that α ≥ β. Given the constraint on these two values, we can thus conclude that:

$$0 \le |\beta| \le \frac{1}{\sqrt{2}} \qquad\qquad \frac{1}{\sqrt{2}} \le |\beta| \le 1$$

Example – 4

- Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\boldsymbol{S} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0 & 0.2 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

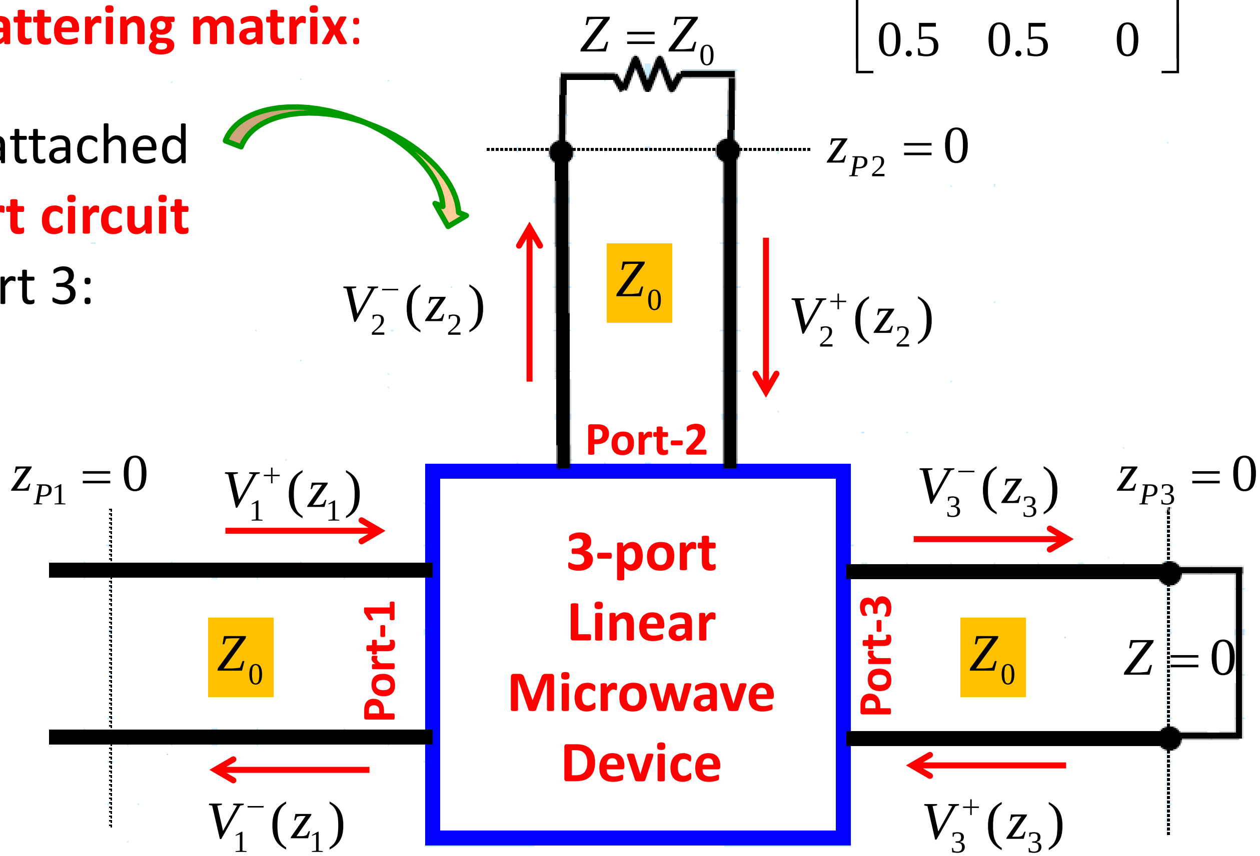
- A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:

a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 = \frac{V_1^-(z_{P1})}{V_1^+(z_{P1})}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} = \frac{V_2^-(z_{P2})}{V_1^+(z_{P1})}$$



Example – 4 (contd.)

Solution:

I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:

$$\Gamma_1 = \frac{V_1^-}{V_1^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_2^-}{V_1^+} = S_{21} = 0.5$$

NO!!! The above solution is **not correct**!

Example – 4 (contd.)

Remember, $V_1^-/V_1^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit.**

Therefore: $$\Gamma_1 = \frac{V_1^-}{V_1^+} \neq S_{11}$$

and similarly: $$T_{21} = \frac{V_2^-}{V_1^+} \neq S_{21}$$

- To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_1^- = 0.2V_2^+ + 0.5V_3^+$$

$$V_2^- = 0.5V_1^+ + 0.2V_3^+$$

$$V_3^- = 0.5V_1^+ + 0.5V_2^+$$

- **and** the **two** equations provided by the **attached loads**:

$$V_2^+ = 0$$

$$V_3^+ = -V_3^-$$

Example – 4 (contd.)

Solve those five expressions to find:

$$\Gamma_1 = \frac{V_1^-}{V_1^+} = -0.25$$

$$T_{21} = \frac{V_2^-}{V_1^+} = 0.4$$

Example – 5

- Consider a **two-port device** with Z_0 =50Ω and scattering matrix (at some specific frequency ω_0):

$$S(\omega = \omega_0) = \begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

- Say that the transmission line connected to **port 2** of this device is terminated in a **matched** load, and that the wave **incident** on **port 1** is:

$$V_1^+(z_1) = -j2e^{-j\beta z_1}$$ **where** $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$

2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$

3. the net power flowing into port 1

Example – 5 (contd.)

Solution:

1. Given the **incident** wave on port 1 is: $V_1^+(z_1) = -j2e^{-j\beta z_1}$

- we can conclude (since $z_{1P} = 0$):

$$V_1^+(z_1 = z_{1P}) = -j2e^{-j\beta z_{1P}} = -j2e^{-j\beta(0)} = -j2$$

- since port 2 is **matched** (and **only** because its matched!), we find:

$$V_1^-(z_1 = z_{1P}) = S_{11}V_1^+(z_1 = z_{1P}) = 0.1(-j2) = -j0.2$$

- The voltage at port 1 is thus:

$$V_1(z_1 = z_{1P}) = V_1^+(z_1 = z_{1P}) + V_1^-(z_1 = z_{1P}) = -j2 + (-j0.2) = -j2.2 = 2.2e^{j(-\pi/2)}$$

- Similarly, since port 2 is **matched**: $V_2^+(z_2 = z_{2P}) = 0$

- Therefore: $V_2^-(z_2 = z_{2P}) = S_{21}V_1^+(z_1 = z_{1P}) = j0.7(-j2) = 1.4$

Example – 5 (contd.)

- The voltage at port 2 is thus:

$$V_2(z_2 = z_{2P}) = V_2^+(z_2 = z_{2P}) + V_2^-(z_2 = z_{2P}) = 0 + 1.4 = 1.4 = 1.4e^{-j0}$$

2. **The port currents** can be easily determined from the results of the previous section

$$I_1(z_1 = z_{1P}) = I_1^+(z_1 = z_{1P}) - I_1^-(z_1 = z_{1P}) = \frac{V_1^+(z_1 = z_{1P})}{Z_0} - \frac{V_1^-(z_1 = z_{1P})}{Z_0}$$

$$\Rightarrow I_1(z_1 = z_{1P}) = -j\frac{2.0}{50} + j\frac{0.2}{50} = -j\frac{1.8}{50} = -j0.036 = 0.036e^{-j\pi/2}$$

$$I_2(z_2 = z_{2P}) = I_2^+(z_2 = z_{2P}) - I_2^-(z_2 = z_{2P}) = \frac{V_2^+(z_2 = z_{2P})}{Z_0} - \frac{V_2^-(z_2 = z_{2P})}{Z_0}$$

$$\Rightarrow I_2(z_2 = z_{2P}) = \frac{0}{50} - \frac{1.4}{50} = -\frac{1.4}{50} = -0.028 = 0.028e^{+j\pi}$$

Example – 5 (contd.)

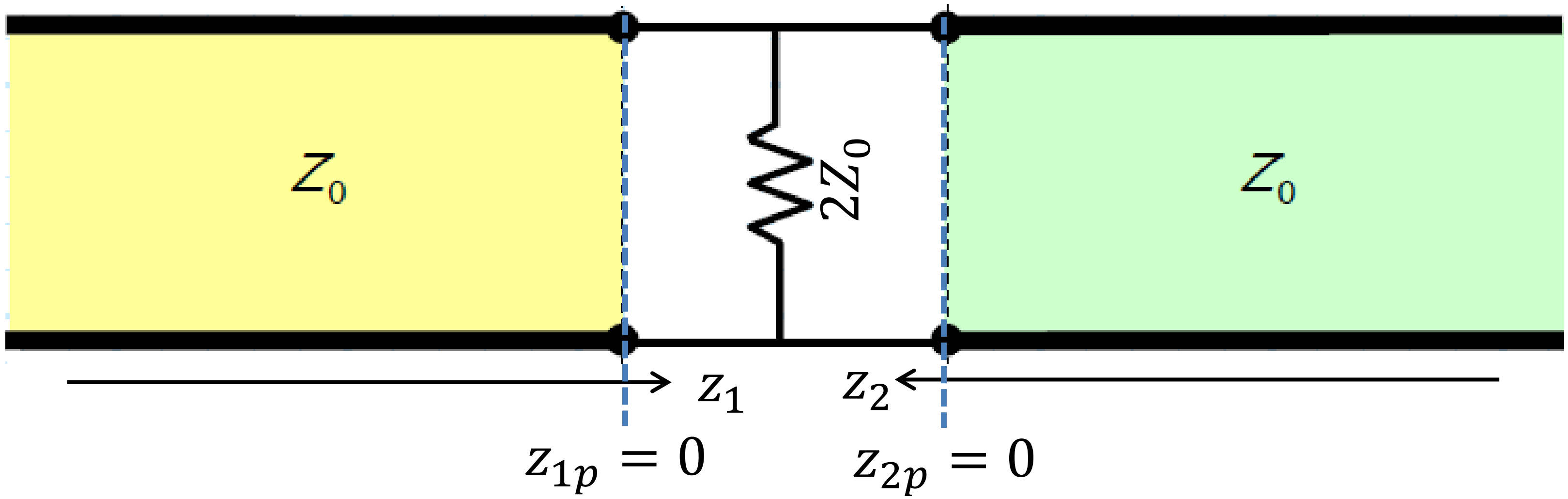
3. The **net power** flowing into port 1 is:

$$\Delta P_1 = P_1^+ - P_1^-$$

$$\Rightarrow \Delta P_1 = \frac{\left|V_1^+\right|^2}{2Z_0} - \frac{\left|V_1^-\right|^2}{2Z_0} \qquad \Rightarrow \Delta P_1 = \frac{(2)^2 - (0.2)^2}{2(50)} = 0.0396 \text{ Watts}$$

Example – 6

- determine the **scattering matrix** of this two-port device:



HA # 2