Lecture – 12

Date: 16.02.2015

- Biot-Savart Law
- Ampere's Circuital Law
- Applications of Ampere's Law
- Magnetic Flux Density

Maxwell's Equations for Magnetostatics

- From the **point form** of Maxwell's equations, we find that the **static** case reduces to another (in addition to electrostatics) pair of **decoupled differential equations** involving magnetic flux density $\vec{B}(\bar{r})$ and current density $\vec{J}(\bar{r})$:

$$\nabla . \vec{B}(\bar{r}) = 0$$

$$\nabla \times \vec{B}(\bar{r}) = \mu_0 \vec{J}(\bar{r})$$

The Integral Form of Magnetostatics

- Say, we evaluate the **surface integral** of the point form of **Ampere's Law** over some arbitrary surface S.

$$\iint_S \nabla \times \vec{B}(\bar{r}).\overline{ds} = \mu_0 \iint_S \vec{J}(\bar{r}).\overline{ds}$$

- Using **Stoke's Theorem**, we can write the **left** side of the above equation as:

$$\iint_S \nabla \times \vec{B}(\bar{r}).\overline{ds} = \oint_C \vec{B}(\bar{r}).\overline{dl}$$

- We also recognize that the **right** side of the equation is:

$$\mu_0 \iint_S \vec{J}(\bar{r}).\overline{ds} = \mu_0 I$$

  - where I is the **current** flowing through surface S.

- Therefore, combing these two results, we find the integral form of **Ampere's Law** (Note the **direction** of I is defined by the **right-hand rule**):

$$\oint_C \vec{B}(\bar{r}).\overline{dl} = \mu_0 I$$

- Amperes law states that the **line integral** of $\vec{B}(\bar{r})$ around a **closed contour** C is proportional to the **total current** I flowing **through** this closed contour ($\vec{B}(\bar{r})$ is **not** conservative!).

The Integral Form of Magnetostatics (contd.)

- Likewise, we can take a **volume integral** over both sides of the magnetostatic equation $\nabla.\vec{B}(\bar{r}) = 0$:

$$\iiint_v \nabla.\vec{B}(\bar{r})dv = 0$$

- But wait! The left side can be rewritten using the **Divergence Theorem**

$$\iiint_v \nabla.\vec{B}(\bar{r})dv = \oiint_S \vec{B}(\bar{r}).\overline{ds}$$

where S is the **closed surface** that **surrounds** volume V.

- Therefore, we can write the integral form of $\nabla.\vec{B}(\bar{r}) = 0$ as:

$$\oiint_S \vec{B}(\bar{r}).\overline{ds} = 0$$

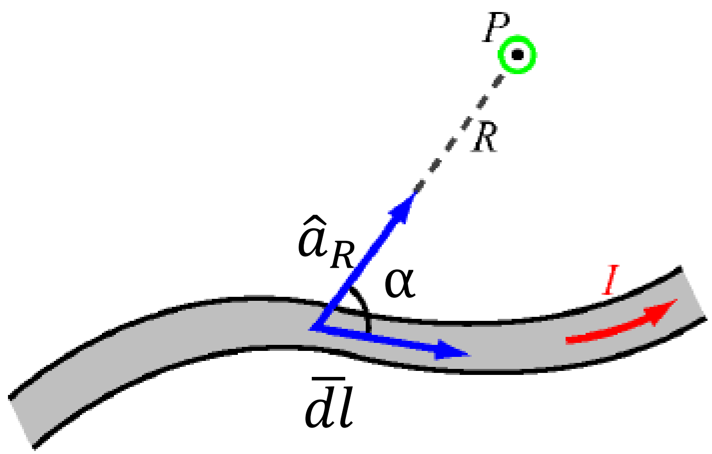
- **Summarizing,** the **integral form** of the magnetostatic equations are:

$$\oiint_S \vec{B}(\bar{r}).\overline{ds} = 0$$

$$\oint_C \vec{B}(\bar{r}).\overline{dl} = \mu_0 I$$

Biot-Savart's Law

- It states that: differential magnetic field intensity $\overrightarrow{dH}(\bar{r})$ produced at point P, shown in figure, by the differential current element $I\bar{dl}$ is related as:



$$dH(\bar{r}) \propto Idl\sin\alpha$$

where, α is the angle between the current element and the line joining the point P

$$dH \propto \frac{1}{R^2}$$

where, R is the distance between the current element and the the point P

- Combining them together results into:

$$dH(\bar{r}) = \frac{k}{R^2} Idl\sin\alpha$$

In SI units

$$dH(\bar{r}) = \frac{1}{4\pi R^2} Idl\sin\alpha$$

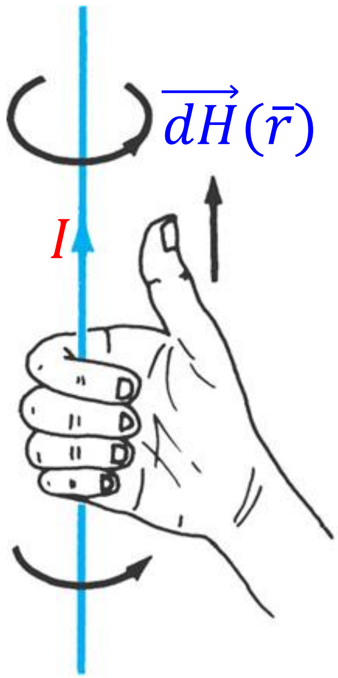
Biot-Savart's Law (contd.)

- From the definition of cross product, we can transform the equation in vector form as:

$$\overrightarrow{dH}(\bar{r}) = \frac{I\,\overline{dl} \times \hat{a}_R}{4\pi R^2} \Rightarrow \overrightarrow{dH}(\bar{r}) = \frac{I\,\overline{dl} \times \vec{R}}{4\pi R^3}$$

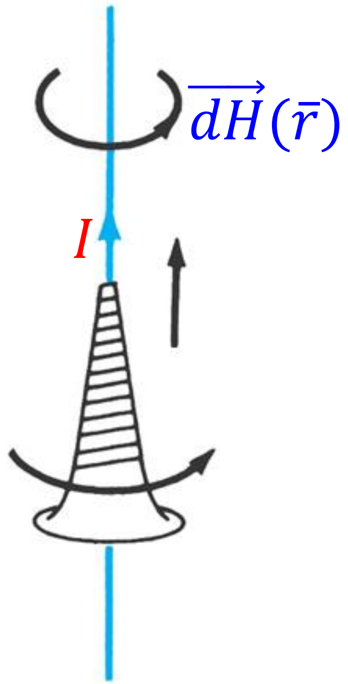
$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|}$$

- This direction of $\overrightarrow{dH}(\bar{r})$ can be obtained from right-hand rule: **right-hand thumb points in the direction of current** **<u>and</u>** **the right hand fingers encircle the wire in the direction of $\overrightarrow{dH}(\bar{r})$.**



Biot-Savart's Law (contd.)

- **Alternatively,** we can use the right-handed-screw rule to determine the direction of $\overrightarrow{dH}(\bar{r})$.
  - The screw is placed along the wire and pointed in the direction of current flow.
  - The direction of the advance of the screw is the direction of $\overrightarrow{dH}(\bar{r})$.

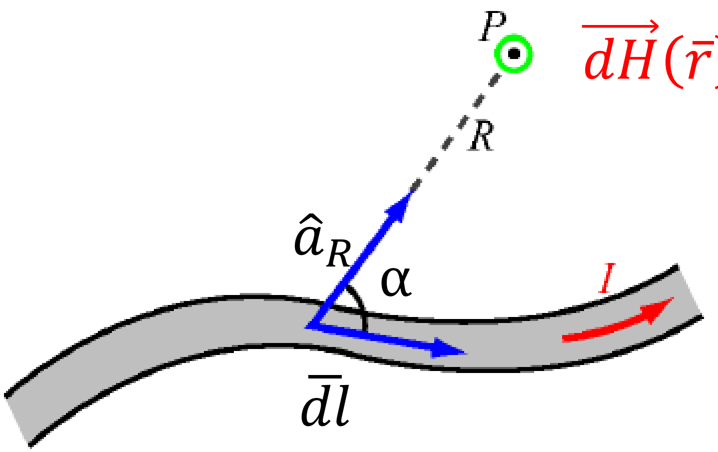


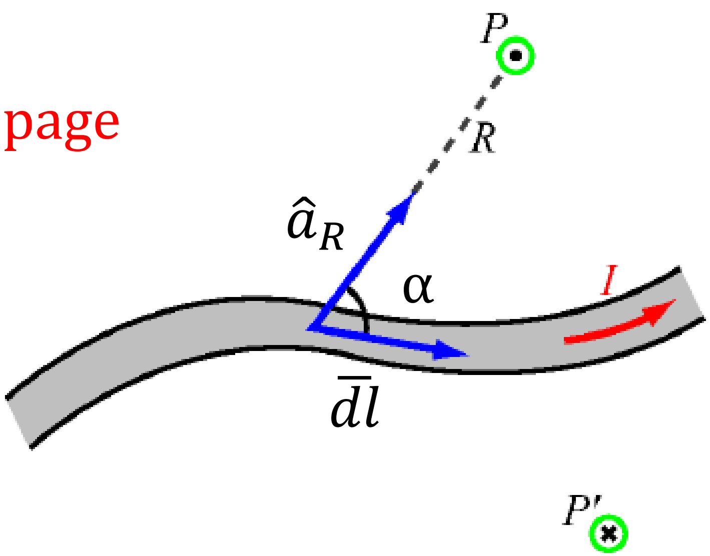
Biot-Savart's Law (contd.)

- It is a standard practice to represent the direction of magnetic field intensity $\vec{H}(\bar{r})$ (or current I) by a small circle with a dot or cross depending on whether $\vec{H}(\bar{r})$ (or I) is out of or into the page.

$\vec{H}(\bar{r})$ (or I) is out

$\vec{H}(\bar{r})$ (or I) is in





Biot-Savart's Law (contd.)

- To determine the total magnetic field $\vec{H}(\bar{r})$ due to a finite sized conductor, we need to sum up the contributions due to all the current elements making up the conductor.

- Therefore the Biot-Savart law becomes:

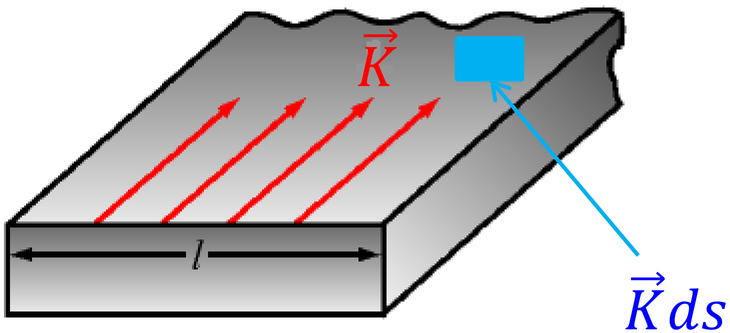
$$\vec{H}(\bar{r}) = \frac{I}{4\pi}\int_L \frac{\overline{dl}\times\hat{a}_R}{R^2}$$

Magnetic field due to line current

where L is the line path along which I exists

Biot-Savart's Law (contd.)

- If we define $\vec{K}$ as the surface current density in ampere/metre then the total magnetic field $\vec{H}(\vec{r})$ can be expressed as:

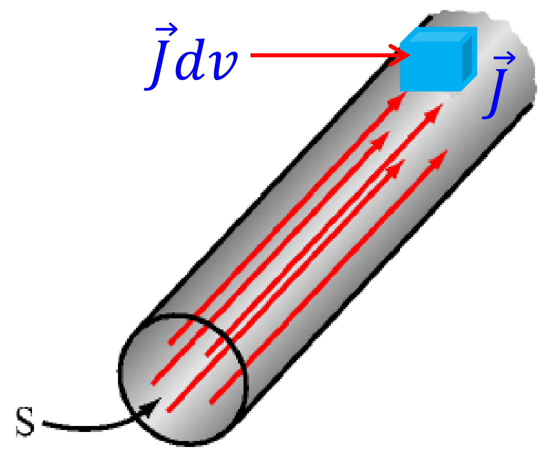


$$\vec{H}(\bar{r}) = \frac{1}{4\pi}\int_L \frac{\vec{K}\times\hat{a}_R}{R^2}ds$$

$$\vec{H}(\bar{r}) = \frac{1}{4\pi}\int_L \frac{K\overline{ds}\times\hat{a}_R}{R^2}$$

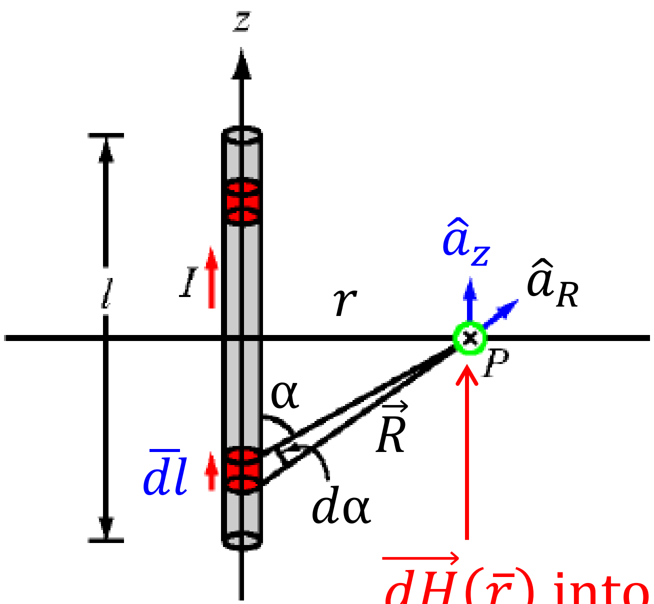
- Similarly, we can express the magnetic field $\vec{H}(\vec{r})$ due to volume current $\vec{J}$ (ampere/m^2) as:

$$\vec{H}(\bar{r}) = \frac{1}{4\pi}\int_L \frac{\vec{J}\times\hat{a}_R}{R^2}dv$$



Example – 1

- A free-standing linear conductor of length l carries a current I along z-axis as shown in Figure. Determine the magnetic field intensity at point P located at a distance r in the x-y plane.



- It is apparent that:

$$\bar{d}l = dz\hat{a}_z \Rightarrow \bar{d}l \times \hat{a}_R = dz\sin\alpha\,\hat{a}_\phi$$

where α is the angle between $\bar{d}l$ and $\hat{a}_R$

Example – 1 (contd.)

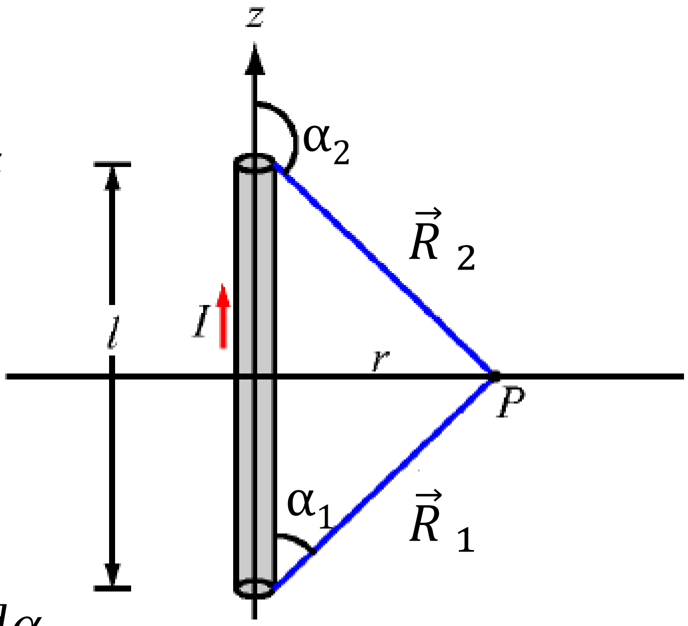
- From Biot-Savart Law: $\vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi} \int_{z=-\frac{l}{2}}^{z=\frac{l}{2}} \frac{\sin\alpha}{R^2} dz$

- Here, both α and R are dependent on the integration variable z, but the radial distance r is not.

- Lets use the following transformation:

$$R = r(\cos ec\alpha) \qquad z = -r(\cot\alpha) \qquad dz = r(\text{cosec}^2\,\alpha)\,d\alpha$$

- Therefore: $\vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\sin\alpha}{R^2} dz$



Where α_1 and α_2 are the limiting angles at $z = -\frac{l}{2}$ and $z = \frac{l}{2}$ respectively.

$$\therefore \vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi r}(\cos\alpha_1 - \cos\alpha_2)$$

Example – 1 (contd.)

- This expression is usually valid for any straight filamentary conductor of finite length.
- The conductor need not lie on the z-axis but it must be straight.
- It is evident that $\vec{H}$ is always along the unit vector $\hat{a}_\phi$ (i.e, along concentric circular paths) irrespective of the length of the wire or the point of interest P.
- As a special case: when the conductor is semi-finite (with respect to P) so that its bottom end is at the origin (i.e., 0, 0, 0) while the top end is at (0, 0, ∞) then,

$$\therefore \vec{H}(\bar{r}) = \frac{I}{4\pi r}\hat{a}_\phi$$

$$\alpha_1 = 90^o \text{ and } \alpha_2 = 180^o$$

- Another special case: when the conductor is infinite (with respect to P) so that its bottom end is at (i.e., 0, 0, -∞) while the top end is at (0, 0, ∞) then,

$$\therefore \vec{H}(\bar{r}) = \frac{I}{2\pi r}\hat{a}_\phi$$

$$\alpha_1 = 0^o \text{ and } \alpha_2 = 180^o$$

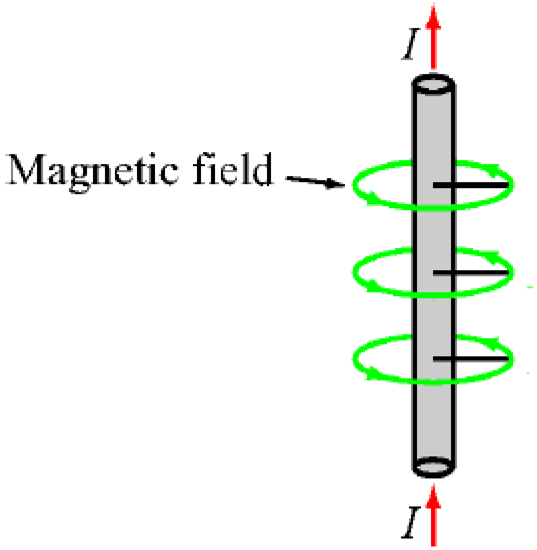
Example – 1 (contd.)

$$\vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi r}(\cos\alpha_1 - \cos\alpha_2)$$

- Its not always easy to find the unit vector $\hat{a}_\phi$.
- A simple approach is to determine $\hat{a}_\phi$ from: $\hat{a}_\phi = \hat{a}_l \times \hat{a}_R$

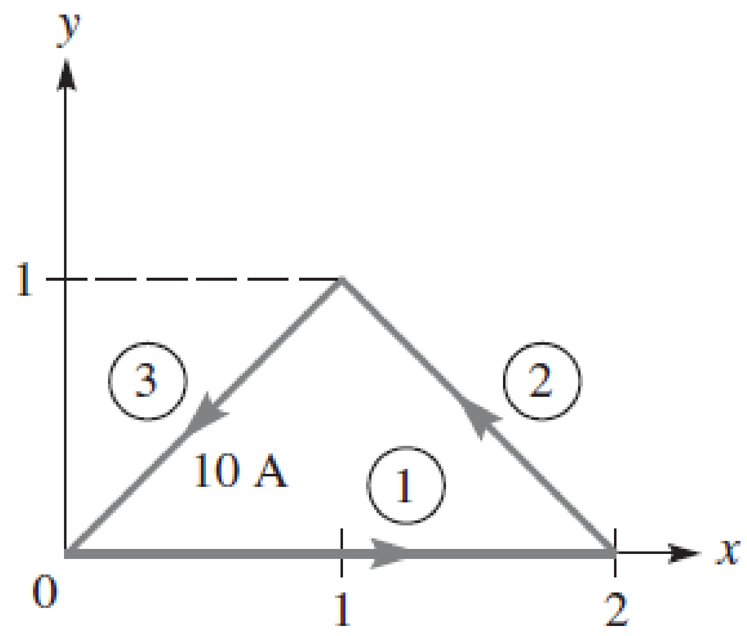
where, $\hat{a}_l$ is the unit vector along the line current and $\hat{a}_R$ is a unit vector along the perpendicular line from the line current to the field point.

- This result is very useful expression to memorize. It states that in the neighbourhood of a linear conductor carrying a current I, the induced magnetic field forms concentric circles around the wire and its intensity is directly proportional to I and inversely proportional to distance r.



Example – 2

- The conducting triangular loop in the figure carries a current of 10A. Find $\vec{H}$ at (0, 0, 5) due to side 1 of the loop.



Example – 2 (contd.)

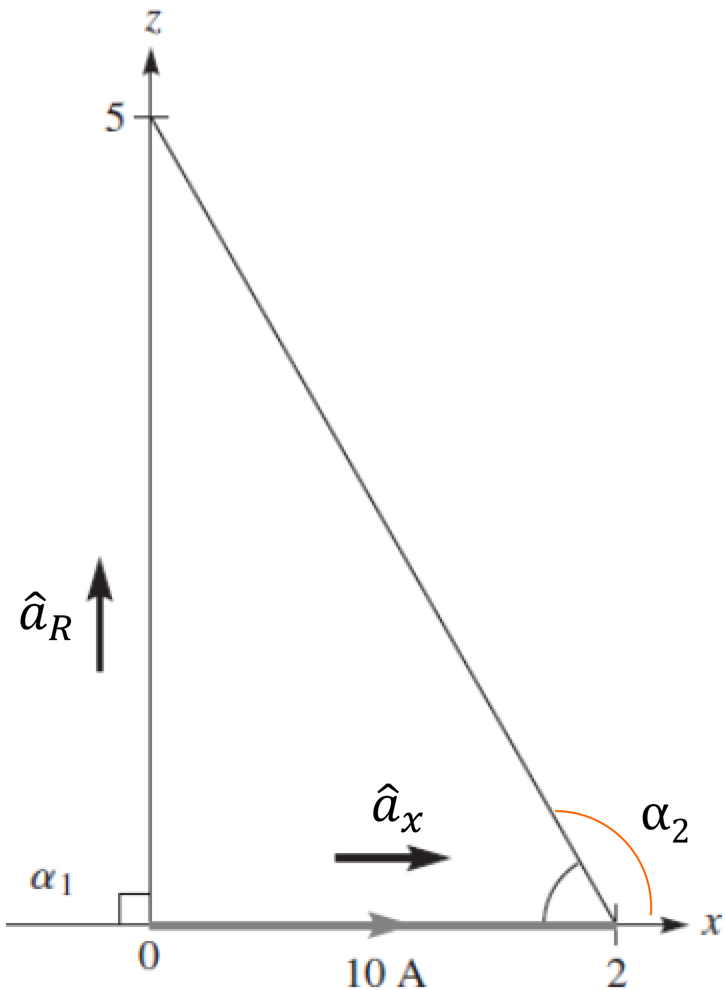
Here: $\hat{a}_l = \hat{a}_x$ $\hat{a}_R = \hat{a}_z$

$$\therefore \hat{a}_\phi = \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

$\cos\alpha_1 = 0$ $\cos\alpha_2 = -\frac{2}{\sqrt{29}}$

$r = 5$

$$\Rightarrow \vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi r}\left(\cos\alpha_1 - \cos\alpha_2\right) = -59.1\hat{a}_y mA/m$$



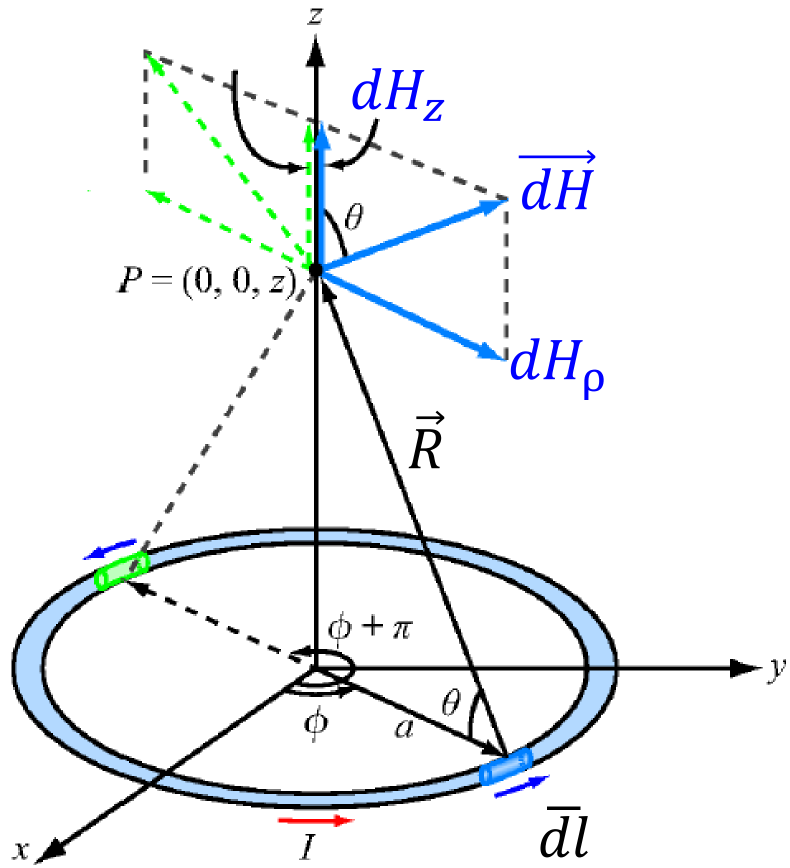
Example – 3

- A circular loop of radius a carries a steady current I. Determine the magnetic field $\vec{H}$ at a point on the axis of the loop.

- Let us place the loop in the xy-plane as shown.
- We want to obtain expression for $\vec{H}$ at (0, 0, z).
- Let us take an element $\overline{dl}$ at (x, y, 0)
- The magnetic field $\overrightarrow{dH}$ due to this element is:

$$\overrightarrow{dH} = \frac{I\overline{dl} \times \hat{a}_R}{4\pi R^2}$$

$$\vec{R} = (0,0,z) - (x,y,0) = -a\hat{a}_\rho + z\hat{a}_z$$

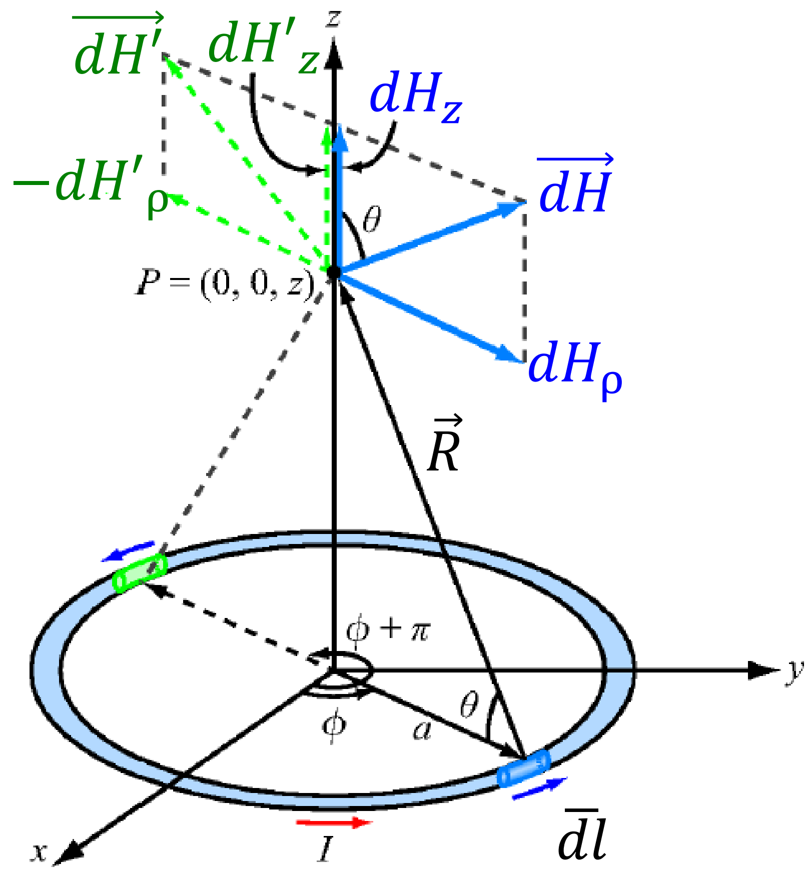


Clearly indicates two components of $\vec{H}$

Example – 3 (contd.)

- If we consider element $\bar{dl}'$ located diametrically opposite to $\bar{dl}$ then we observe that the z-components of the magnetic fields due to $\bar{dl}'$ and $\bar{dl}$ add because they are in the same direction, but their ρ-components cancel because they are in opposite directions.
- Hence the net magnetic field is along z-axis only.
- We have:

$$\left|\overrightarrow{dH}\right| = \left|\frac{I\,\bar{dl} \times \hat{a}_R}{4\pi R^2}\right| = \frac{Idl}{4\pi(a^2+z^2)}$$



- Therefore: $$\overrightarrow{dH} = \hat{a}_z dH_z = \hat{a}_z dH\cos\theta = \hat{a}_z \frac{I(\cos\theta)}{4\pi(a^2+z^2)}dl$$

Example – 3 (contd.)

- For a fixed point $P(0,0,z)$ on the axis of the loop, all quantities in the above expression are constant except for $\overline{dl}$, therefore:

$$\vec{H} = \hat{a}_z \frac{I(\cos\theta)}{4\pi(a^2+z^2)} \oint dl$$

- Thus: $$\vec{H} = \hat{a}_z \frac{I(\cos\theta)}{4\pi(a^2+z^2)} (2\pi a)$$

- We can also derive:

$$\cos\theta = \frac{a}{\sqrt{a^2+z^2}}$$

$$\therefore \vec{H} = \hat{a}_z \frac{Ia^2}{2(a^2+z^2)^{3/2}}$$

- At the center of the loop $(z = 0)$: $$\therefore \vec{H} = \hat{a}_z \frac{I}{2a}$$

- At a point far away from the loop $(|z| \gg a)$: $$\therefore \vec{H} = \hat{a}_z \frac{Ia^2}{2|z|^3}$$

Example – 4

- A solenoid, lying along z-axis, of length l and radius a consists of N turns of wire carrying current I. show that at point P along its axis:

$$\vec{H} = \hat{a}_z \frac{NI}{2l}\left(\cos\theta_2 - \cos\theta_1\right)$$

where, θ_1 and θ_2 are the angle subtended at P by the end turns.

- Alo show that if $l \gg a$, at the center of the solenoid:

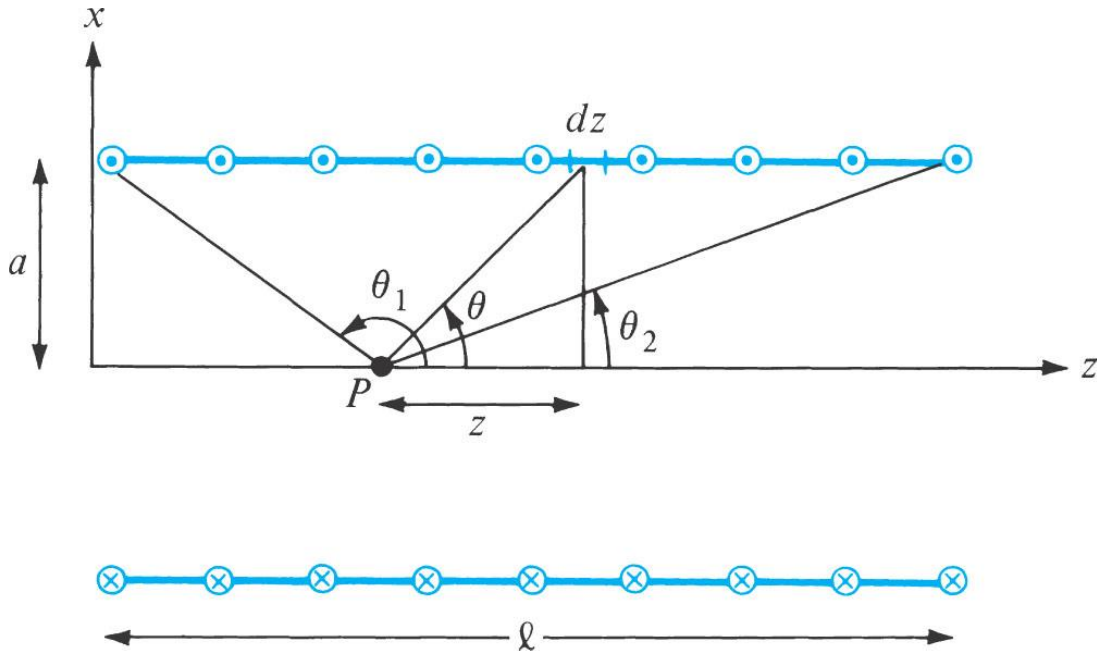
$$\vec{H} = \hat{a}_z \frac{NI}{l}$$

Example – 4 (contd.)

- An important structure in electrical and computer engineering is the **solenoid**.
- A solenoid is a **tube of current**. However, it is different from the hollow cylinder, in that the current flows **around** the tube, rather than down the tube:

Example – 4 (contd.)

- Let us consider the cross section of solenoid as shown below.



Make use of example-3

- The magnetic field at P due to length dz is:

$$dH_z = \frac{NIa^2}{2l(a^2+z^2)^{3/2}}dz$$

where

$$\frac{N}{l}dz = dl$$

- **From figure:** $\tan\theta = \frac{a}{z}$ ⇒ $dz = -a\,\text{cosec}^2\theta\, d\theta$ ⇒ $dz = -a\frac{\left(z^2+a^2\right)^{3/2}}{a^2}\sin\theta\, d\theta$

Example – 4 (contd.)

- Therefore:

$$dH_z = -\frac{NI}{2l}\sin\theta d\theta$$

$$H_z = -\frac{NI}{2l}\int_{\theta_1}^{\theta_2}\sin\theta d\theta$$

$$\therefore \vec{H} = \frac{NI}{2l}\left(\cos\theta_2 - \cos\theta_1\right)\hat{a}_z$$

- At the center of the Solenoid:

$$\cos\theta_2 = \frac{l/2}{\left[a^2 + \frac{l^2}{4}\right]^{1/2}} = -\cos\theta_1$$

- Thus:

$$\therefore \vec{H} = \frac{NI}{2\left[a^2 + \frac{l^2}{4}\right]^{1/2}}\hat{a}_z$$

- If $l \gg a$, then:

$$\therefore \vec{H} = \frac{NI}{l}\hat{a}_z$$

Ampere Circuital Law

- Earlier we learnt that the electrostatic field is conservative, meaning its line integral along a closed contour always vanishes.

- This property was expressed as:

$$\nabla \times \vec{E} = 0 \quad \Longleftrightarrow \quad \oint_C \vec{E}.\overline{dl} = 0$$

- **The magnetostatic counterpart known as Ampere's Law is:**

$$\nabla \times \vec{H} = \vec{J} \quad \Longleftrightarrow \quad \oint_C \vec{H}.\overline{dl} = I_{encl}$$

- The sign convention for the direction of contour path C in Ampere's law is taken so that I and $\vec{H}$ satisfy the right-hand rule defined earlier in connection with Biot-Savart law→ If the direction of I is aligned with the direction of the thumb then the direction of the contour C should be chosen along that of the other four fingers.

Ampere Circuital Law (contd.)

- In words, Ampere's circuital law states that the line integral of $\vec{H}$ around a closed path is equal to the current traversing the surface bounded by that path.

$$\oint_C \vec{H}.\overline{dl} = I_{encl}$$

We know:

$$I_{encl} = \int_S \vec{J}.\overline{ds}$$

From Stoke's Theorem

$$\oint_C \vec{H}.\overline{dl} = \int_S (\nabla \times \vec{H}).\overline{ds}$$

- Therefore: $$\nabla \times \vec{H} = \vec{J}$$

Maxwell's Equation for Magnetostatics

Magnetostatic field is not conservative

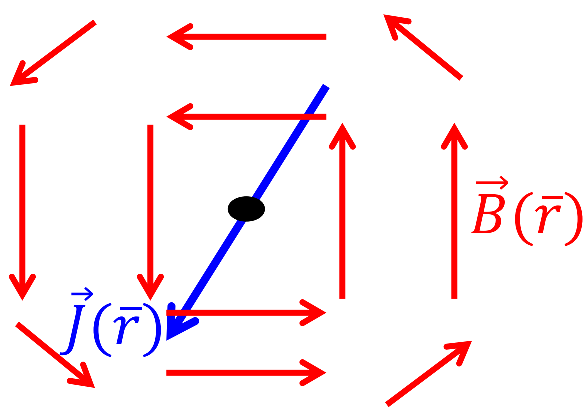
Ampere Circuital Law (contd.)

- This Maxwell's equation for magnetostatic equation is referred to as **Ampere's Circuital Law**:

$$\nabla \times \vec{H}(\bar{r}) = \vec{J}(\bar{r})$$

Ampere's Circuital Law

This equation indicates that the magnetic flux density $\vec{H}(\bar{r})$ **rotates around** current density $\vec{J}(\bar{r})$ --the **source** of magnetic field intensity is current!.



Applications of Ampere's Law

$$\oint_C \vec{H}.\overline{dl} = I_{encl}$$

This equation holds regardless of whether the current distribution is symmetrical or otherwise

But $\vec{H}$ can be determined using this expression only if the symmetrical current distribution exists

Examples include: an infinite line current, an infinite sheet of current, and an infinitely long coaxial transmission line

In each case, we apply $\oint_C \vec{H}.\overline{dl} = I_{enc}$. For symmetrical current distribution, $\vec{H}$ is either parallel or perpendicular to $\overline{dl}$. When $\vec{H}$ is parallel to $\overline{dl}$, $|\vec{H}| = constant$.