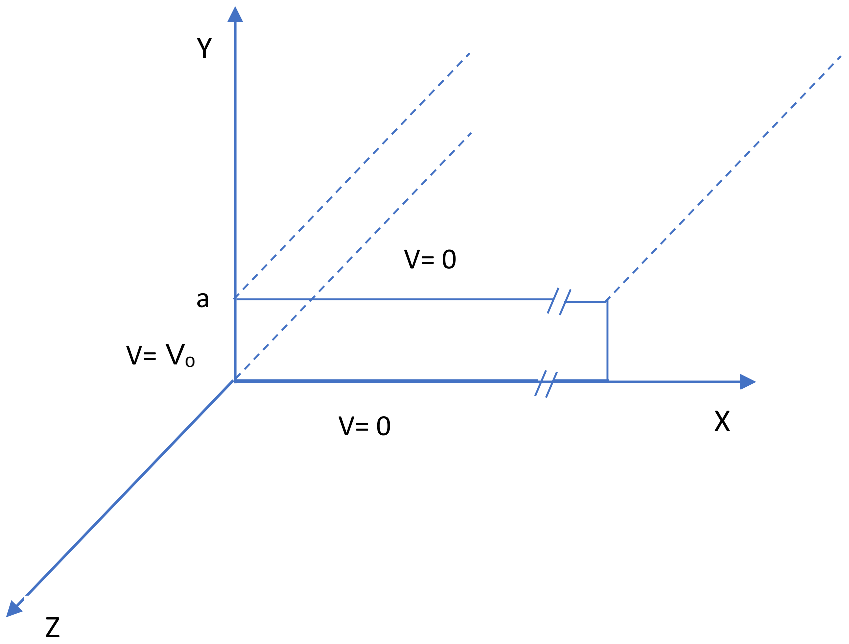
TUTORIAL 3

Q1. Consider the slot shown, the slot is infinite along x –direction. The faces parallel to xz plane are grounded. The face parallel to yz plane are maintained at constant potential V_o. What is this potential distribution V(x,y) in the slot ?



Boundary conditions:

1. V = 0 at y = 0 -------------1
2. V = 0 at y = a -------------2
3. V = V_o at x = 0-------------3
4. V -> 0 at x -> ∞-----------4

Separation of variables:

Assume $V(x,y) = X(x)Y(y)$ ---------------6

Laplace equation: $\nabla^2 V = 0$

⇨ $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$

From 6,

⇨ $Y\frac{d^2X}{dx^2} + X\,\frac{d^2Y}{dy^2} = 0$

⇨ $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\,\frac{d^2Y}{dy^2} = 0$ ----------------7

Each term must be a constant

Let, $\frac{1}{X}\frac{d^2X}{dx^2} = c_1$ and $\frac{1}{Y}\frac{d^2Y}{dy^2} = c_2$

From 7 , $c_1 + c_2 = 0$

Choose $c_1 > 0$ $(c_1 = k^2, where\ k > 0)$

$\frac{d^2X}{dx^2} = k^2X => X = Ae^{kx} + Be^{-kx}$ -------------------8

$Y = C\sin ky + D\cos ky$ --------------------9

Apply boundary conditions:

V -> 0 as x -> ∞

- ⇨ $A = 0 => X = Be^{-kx}$

Also, V = 0 at y = 0

- ⇨ $D = 0 => Y = C\sin ky$

Finally, $V(x, y) = Be^{-kx}C\sin ky = Pe^{-kx}\sin ky$ ------------------- 10

But V = 0 at y = a

- ⇨ $\sin ka = 0$
- ⇨ $ka = n\pi => k = \frac{n\pi}{a}$, $where\ n \neq 0$

$V_n(x, y) = P_n e^{\frac{-n\pi x}{a}} \sin\frac{n\pi y}{a}$, n = 1,2,3 …….

In general,

$$V(x, y) = \sum_{n=1}^{\infty} V_n(x, y)$$

- ⇨ $V(x, y) = \sum_{n=1}^{\infty} P_n e^{\frac{-n\pi x}{a}} \sin\frac{n\pi y}{a}$

One boundary condition was unused, until now!

$V = V_o(y) = V_o$ at x = 0

⇨ $V_o = \sum_{n=1}^{\infty} P_n \sin\frac{n\pi y}{a}$

⇨ $\int_0^a V_o \sin\frac{m\pi y}{a}\, dy = \sum_{n=1}^{\infty} P_n \int_0^a \sin\frac{m\pi y}{a} \sin\frac{n\pi y}{a}\, dy$

Note:

$$\int_0^a \sin\frac{m\pi y}{a} \sin\frac{n\pi y}{a}\, dy \quad = 0 \text{ , if m } \neq \text{ n}$$

$$= \frac{a}{2}, if\ m = n$$

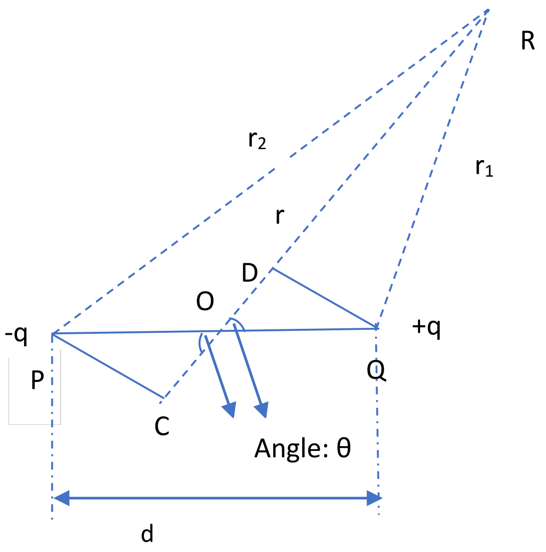
⇨ $P_m = \frac{2V_o}{a} \int_0^a \sin\frac{m\pi y}{a}\, dy = \frac{2V_o}{a}\frac{a}{m\pi}\left[\cos\frac{m\pi y}{a}\right]_a^0 = \frac{2V_o}{m\pi}[1 - \cos m\pi] = 0\ , m\ even$

⇨ $= \frac{4V_o}{m\pi}\ , m\ odd$

$$V(x, y) = \sum_{m=1}^{\infty} \frac{4V_o}{m\pi} e^{\frac{-m\pi x}{a}} \sin\frac{m\pi y}{a}$$

Q2) A (physical) electric dipole consists of two equal and opposite charges (±q) separated by a distance d. Find the approximate potential at a point far from the dipole.

Sol.



Let charge -q is placed at point P and charge +q is placed at point Q.

Let R be a point at a distance from the midpoint O of dipole and r_1 and r_2 are the distance of charge +q and -q from point R respectively.

Now, Electric potential due to +q charge is –

$$V_1 = \frac{1}{4\pi\epsilon 0}\ \frac{q}{r_1} \qquad \text{----------1}$$

And due to -q charge is –

$$V_2 = \frac{1}{4\pi\epsilon 0}\ \frac{(-)q}{r_2} \qquad \text{----------2}$$

We know that electric potential obeys superposition principle. Therefore, Potential due to dipole as a whole is –

$$V = \frac{1}{4\pi\epsilon 0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

$$V = \frac{q}{4\pi\epsilon 0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \text{----------3}$$

Drawing line PC and QD perpendicular to RO and considering

triangle POC, we get

$$cos\theta = \frac{OC}{OP} = \frac{OC}{d/2}$$

$$\therefore\ OC = \frac{d}{2}\ cos\theta$$

Similarly, OD = $\frac{d}{2}\ cos\theta$ (considering ΔQOD)

Now, r_1 = QR $\cong\ RD\ = OR - OD\ = r\ - \frac{d}{2}\ cos\theta$

$$r_2\ = PR\ \cong\ RC\ = OR + OC\ = r\ + \frac{d}{2}\ cos\theta$$

Now, substituting r1 and r2 in equation 3

$$V = \frac{q}{4\pi\in 0}\left(\frac{1}{\left(r-\frac{d}{2}cos\theta\right)} - \frac{1}{\left(r+\frac{d}{2}cos\theta\right)}\right)$$

$$V = \frac{q}{4\pi\in 0}\ (\frac{r + \frac{d}{2}\ cos\theta - r + \frac{d}{2}\ cos\theta}{r^2 - \frac{d^2}{4}\ cos^2\theta})$$

$$V\ = \frac{q}{4\pi\in 0}\ (\frac{dcos\theta}{r^2 - \frac{d^2}{4}\ cos^2\theta})$$

$$V\ = \frac{1}{4\pi\in 0}\ (\frac{qdcos\theta}{r^2 - \frac{d^2}{4}\ cos^2\theta})$$

Here we are considering the case for r >> d

$$\therefore\ V\ = \frac{1}{4\pi\in 0}\ (\frac{qdcos\theta}{r^2})$$

As magnitude of dipole moment is $|\vec{p}|\ = qd$

$$\therefore\ V\ = \frac{1}{4\pi\in 0}\ (\frac{pcos\theta}{r^2})$$

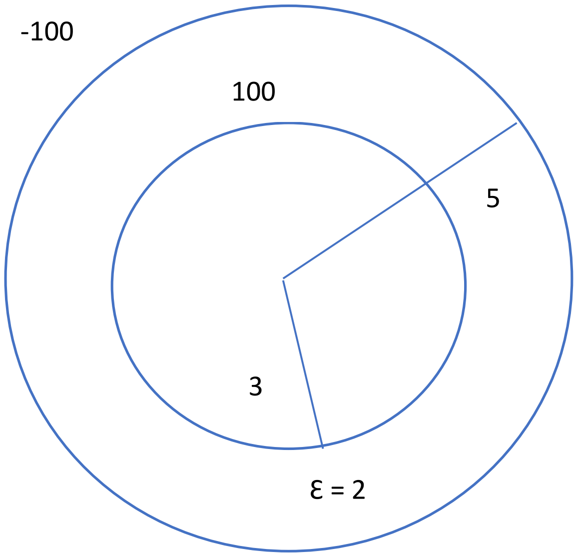
Also, $pcos\theta\ = \vec{p}.\hat{r}\ where\ \hat{r}$ is the unit vector along vector $\vec{r}$.

Therefore, $V\ = \frac{1}{4\pi\in 0}\ (\frac{\vec{p}.\hat{r}}{r^2})$

Q3) Two concentric conducting spheres having radii 3 cm and 5 cm are centered at origin. The potential on the inner sphere is 100 volt while the outer sphere is -100 volt. The region between them is filled with a homogenous dielectric having relative permittivity to 2.0. Find

1) Potential function
2) Potential midway between the conducting spheres
3) Value of r, at which V = 0, (r spherical co-ordinate)
4) Find expression for electric field

Sol.



V(r) = ?

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \varphi^2} = -\frac{\rho_v}{\epsilon}$$

As V does not depend on φ & θ.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

on integration

$$r^2\frac{\partial V}{\partial r} = c_1$$

$$\frac{\partial V}{\partial r} = \frac{c_1}{r^2}$$

$$V = -\frac{c_1}{r} + c_2$$

$$V(r) = -\frac{c_1}{r} + c_2$$

$$100 = -\frac{c_1}{3\times10^{-2}} + c_2 \qquad \text{------- } 1$$

$$-100 = -\frac{c_1}{5\times10^{-2}} + c_2 \qquad \text{-------}\ 2$$

From equation 1 & 2

$$200 = -\frac{c_1}{3\times10^{-2}} + \frac{c_1}{5\times10^{-2}}$$

$$\frac{200\times3\times5\times10^{-4}}{10^{-2}} = -5c_1 + 3c_1$$

$$c_1 = -15$$

Put the value of c_1 in equation1

$$100 = -\frac{(-15)}{3\times10^{-2}} + c_2$$

$$c_2 = 100 - \frac{5}{10^{-2}} = -400$$

1) $V(r) = \frac{15}{r} - 400$
2) $V(r = 4\times10^{-2}) = 15\times 4\times 10^{-2} - 400 = -25$
3) V(r) = 0

   $0 = \frac{15}{r} - 400$

   15 = 400 r

   r = 3.75 $\times\ 10^{-2}$ cm
4) E = $-\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{a}_\theta + \frac{1}{r sin\theta}\frac{\partial V}{\partial\varphi}\hat{a}_\varphi$

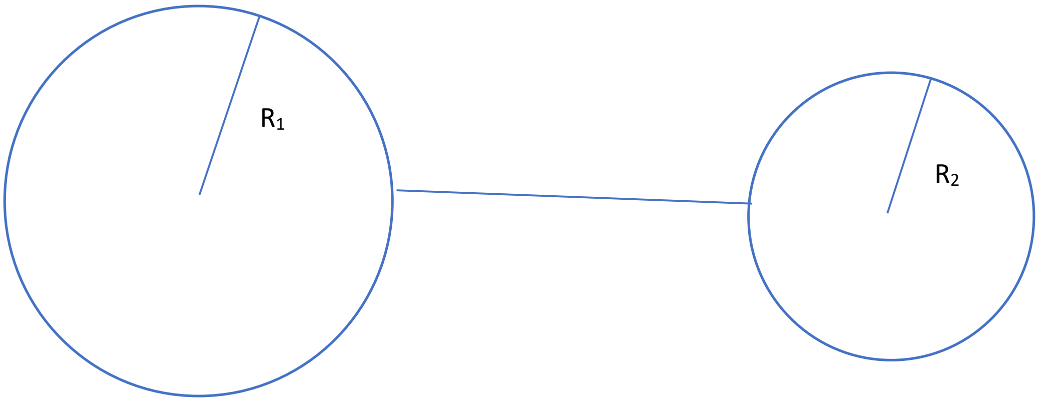
Independent

E = $\frac{15}{r^2}\hat{a}_r$

Q 4) Two conducting spheres of radius r_1, r_2 $(r_1 > r_2)$ having a charge of Q_1 Q_2 are connected by a long conducting wire then find the

1) Charge on each sphere
2) Electric field & the potential on each sphere

Sol



$$V_1 = V_2$$

$$\frac{1}{4\pi\epsilon}\frac{Q_1}{R_1} = \frac{1}{4\pi\epsilon}\frac{Q_2}{R_2}$$

$$\frac{Q_1}{Q_2} = \frac{R_1}{R_2}$$

$$Q_T = Q_1 + Q_2$$

$$Q_1 = \left(\frac{R_1}{R_1 + R_2}\right) Q_T$$

$$Q_2 = \left(\frac{R_2}{R_1 + R_2}\right) Q_T$$

$$V_1 = \frac{1}{4\pi\epsilon}\frac{Q_1}{R_1} = \frac{1}{4\pi\epsilon}\frac{1}{R_1}\left(\frac{R_1 \cdot Q_T}{R_1 + R_2}\right) = \frac{1}{4\pi\epsilon}\left(\frac{Q_T}{R_1 + R_2}\right)$$

$$V_2 = \frac{1}{4\pi\epsilon}\frac{Q_2}{R_2} = \frac{1}{4\pi\epsilon}\frac{1}{R_2}\left(\frac{R_2 \cdot Q_T}{R_1 + R_2}\right) = \frac{1}{4\pi\epsilon}\left(\frac{Q_T}{R_1 + R_2}\right)$$

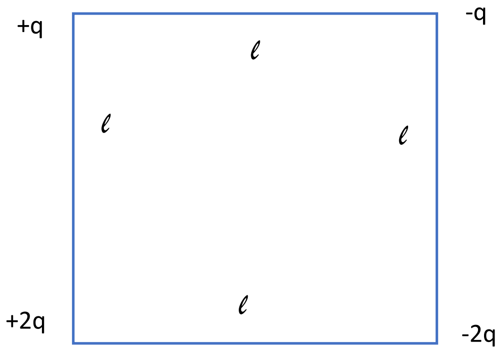
$$E_1 = \frac{1}{4\pi\epsilon}\frac{Q_1}{R_1{}^2} = \frac{1}{4\pi\epsilon}\frac{1}{R_1{}^2}\left(\frac{R_1 \cdot Q_T}{R_1 + R_2}\right) = \frac{1}{4\pi\epsilon}\frac{Q_T}{R_1\,(R_1 + R_2)}$$

$$E_2 = \frac{1}{4\pi\epsilon}\frac{Q_2}{R_2{}^2} = \frac{1}{4\pi\epsilon}\frac{1}{R_2{}^2}\left(\frac{R_2 \cdot Q_T}{R_1 + R_2}\right) = \frac{1}{4\pi\epsilon}\frac{Q_T}{R_2\,(R_1 + R_2)}$$

$$\frac{E_1}{E_2} = \frac{R_2}{R_1} \longrightarrow E \propto \frac{1}{R}$$

Q5) Work done to bring charge in this orientation is (from ∞) ?

Sol)



Bring $+q \; => \; 0 \; work$

Bring $-q \; => \; -\frac{kq^2}{l}$

$Bring \; +2q \; => \; \frac{k\,2q\;q}{l} - \frac{k\;q\;2q}{l\,\sqrt{2}}$

$Bring \; -2q \; => \; \frac{k\;q\;(-2q)}{l\,\sqrt{2}} + \frac{q\;k\;2q}{l} - \frac{2q\;k\;2q}{l}$

Total = $-\frac{kq^2}{l} + \frac{2q^2\,k}{l} - \frac{\sqrt{2}\,k\,q^2}{l} - \frac{\sqrt{2}\,k\,q^2}{l} + \frac{2q^2\,k}{l} - \frac{4kq^2}{l}$

Total = $-\frac{kq^2}{l}\,(1 + 2\sqrt{2})$ J

$-ve \; work \; done \; implies \; work \; done \; by \; field$

Q6) Find the electric field due to a dipole with dipole moment 10^{-29}cm at an angle of 45° from the dipole & 10m far away.

Sol $\quad p = 10^{-29}$cm

r = 10m, $\theta = 45^{\circ}$

$E_r = \frac{2p\cos\theta}{4\pi\epsilon 0\, r^3} = \frac{2\times10^{-29}\times\frac{1}{\sqrt{2}}}{4\times\pi\times\epsilon 0\times1000} = 9\sqrt{2} \times 10^{-23}$ v/m

$E_Q = \frac{p\sin\theta}{4\pi\epsilon 0\, r^3} = \frac{10^{-29}\times\frac{1}{\sqrt{2}}}{4\times\pi\times\epsilon 0\times1000} = \frac{1}{\sqrt{2}}\times 10^{-23}$ v/m