TUTORIAL 2

Q1) Divergence of the three- dimensional radial vector field $\vec{r}$ is ___?

Ans: $\vec{r}$ = x$\hat{\imath}$ + y$\hat{\jmath}$ + z$\hat{k}$

$\vec{\nabla}$ = ($\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$)

$\vec{\nabla}.\vec{r}$ = $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$ = 1 +1 +1 = 3

Q2) Compute $\oiint(ax^2 + \mathrm{b}y^2 + cz^2).\vec{ds}$ over the surface of the sphere $x^2 + y^2 + z^2$ = 1.

Ans: By divergence theorem $\oiint \vec{F}.\hat{n}\, ds = \iiint \vec{\nabla}.\vec{F}\, dv$

Given $\vec{F}.\hat{n}$ = $ax^2 + \mathrm{b}y^2 + cz^2$

Let $\varphi = x^2 + y^2 + z^2 - 1$

Normal vector $\vec{n}$ to the surface φ is –

$\vec{\nabla}\varphi$ = ($\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$)($x^2 + y^2 + z^2 - 1$)

= 2 (x$\hat{\imath}$ + y$\hat{\jmath}$ + z$\hat{k}$)

Unit normal vector = $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$

$= \frac{2\,(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{2\sqrt{(x^2 + y^2 + z^2)}}$

$= x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ as ($x^2 + y^2 + z^2 = 1$)

$\vec{F}.\hat{n}$ = $\vec{F}$.($x\hat{\imath} + y\hat{\jmath} + z\hat{k}$) = $ax^2 + \mathrm{b}y^2 + cz^2$

= ($ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}$).($x\hat{\imath} + y\hat{\jmath} + z\hat{k}$)

i.e, $\vec{F}$ = $ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}$

Now, $\vec{\nabla}.\vec{F}$ = $\frac{\partial}{\partial x}ax + \frac{\partial}{\partial y}by + \frac{\partial}{\partial z}cz$ = a + b + c

By Gauss Divergence Theorem,

$\oiint(ax^2 + \mathrm{b}y^2 + cz^2)\; ds = \iiint(a + b + c)\; dv$

= (a+b+c) V

Now, V is volume of sphere of unit radius

V = $\frac{4\pi}{3}(1)^3 = \frac{4\pi}{3}$

$\oiint(ax^2 + \mathrm{b}y^2 + cz^2)\; ds = \frac{4\pi}{3}$ (a+b+c)

Q3) $\vec{V}$ = x $cos^2y\ \hat{\imath}$ + $x^2e^z\ \hat{\jmath}$ + z $sin^2y\ \hat{k}$. Then the value of $\oiint \vec{V}.\hat{n}$ ds over the unit cube is?

Ans: we know $\overrightarrow{ds}$ = ds.$\hat{n}$ where ds is magnitude and $\hat{n}$ is direction.

$\oiint \vec{V}.\vec{d}s = \iiint \vec{\nabla}.\vec{V}\ dv$(From divergence theorem)

$\vec{\nabla}.\vec{V} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}$

$\vec{\nabla}.\vec{V} = cos^2y$ + 0 + sin^2y

$\vec{\nabla}.\vec{V}$ = 1

$\oiint \vec{V}.\vec{d}s = \iiint 1.dv$ = volume = $a^3|_{a=1}$ = 1

Q4) The value of $\oiint 5\vec{r}.\vec{d}\,s$ where $\vec{r}$ = position vector and s is the closed surface having volume = V.

a) 5 V
b) 10 V
c) 15 V
d) 20 V

Ans:

$\vec{r}$ = x$\hat{\imath}$ + y$\hat{\jmath}$ + z$\hat{k}$

$\vec{\nabla}.\vec{r}$ = 3

$$\oiint 5\vec{r}.\vec{d}\,s = 5\oiint \vec{r}.\vec{d}\,s = 5\iiint \vec{\nabla}.\vec{r}\,dV = 5\iiint 3\,dV = 15\iiint dV = 15V$$

Q 5) If the vector

$$\overrightarrow{F} = (3y - k_1 z)\hat{a}_x + (k_2 x - 2z)\hat{a}_y - (k_3 y + z)\hat{a}_z$$

is irrotational value of k_1, k_2, k_3 are

Sol .

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y - k_1 z & k_2 x - 2z & -k_3 y - z \end{vmatrix} = \hat{a}_x(-k_3 + 2) - \hat{a}_y(+k_1) + \hat{a}_z(k_2 - 3) = 0$$

$$k_1 = 0, \quad k_2 = 3, \quad k_3 = 2$$

Q6) Find the curl of

$$\vec{V} = x^2 y\, \hat{x} + yz\, \hat{y} + x\, \hat{z}$$

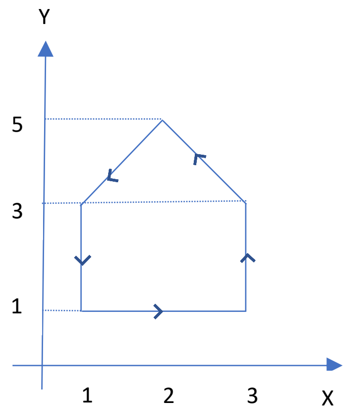
Sol.

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & yz & x \end{vmatrix} = \hat{x}(0 - y) - \hat{y}(1 - 0) + \hat{z}(0 - x^2) = -y\hat{x} - \hat{y} - x^2\hat{z}$$

Q7)

$$\vec{F} = (x - y)\hat{x} + (y - z)\hat{y} + (z - x)\hat{z}$$

Find $\oint \vec{F}.\vec{dl}$.



Sol. $\oint \vec{F}.\vec{dl} = \iint (\vec{\nabla} \times \vec{F}).\vec{ds}$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y & y - z & z - x \end{vmatrix} = \hat{x}(+1) - \hat{y}(-1) + \hat{z}(+1)$$

$$= \hat{x} + \hat{y} + \hat{z}$$

Also, $\vec{ds} = ds\,\hat{z}$

$$\iint (\vec{\nabla} \times \vec{F}).\vec{ds} = \iint ds = 4 + \frac{1}{2} \times 2 \times 2 = 6$$