TUTORIAL 1 SOLUTIONS

Q1. Find the gradiant of the following.

(i) F(x,y,z) = $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ where a, b, c are non-zero constants.

(ii) F(x,y,z) = $x^3 y^3 z^6$

(iii) F(x,y,z) = $e^x log_e x log_e y$

Sol.

(i) $$\vec{\nabla}F = \left(\frac{\partial}{\partial x}\ \hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\ \right)F$$

$$= \left(\frac{\partial F}{\partial x}\ \hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}\ \right)$$

$$= \frac{2x}{a^2}\hat{x} + \frac{2y}{b^2}\hat{y} + \frac{2z}{c^2}\hat{z}$$

(ii) $$\vec{\nabla}F = \left(\frac{\partial}{\partial x}\ \hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\ \right)F$$

$$= \left(\frac{\partial F}{\partial x}\ \hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}\ \right)$$

$$= 3x^2y^3z^6\ \hat{x} + 3x^3y^2z^6\ \hat{y} + 6x^3y^3z^5\ \hat{z}$$

(iii) $$\vec{\nabla}F = \left(\frac{\partial}{\partial x}\ \hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\ \right)F$$

$$= \left(\frac{\partial F}{\partial x}\ \hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}\ \right)$$

$$= log_e y\left(\frac{e^x}{x} + \ log_e x\ e^x\right)\ \hat{x} + \frac{e^x log_e x}{y}\ \hat{y}$$

$$= \frac{log_e y\ e^x}{x}(1 + \ x\ log_e x)\ \hat{x} + \frac{e^x log_e x}{y}\ \hat{y}$$

Q2. Let a function be defined in 3D state by –

$$T = xy + yz + zx$$

Calculate $\int_a^b \vec{\nabla}T.\vec{dl}$ where a = (1,1,1) and b = (3,2,4) for the following paths:

a. (1,1,1) to (3,1,1) to (3,2,1) to (3,2,4)
b. (1,1,1) to (3,2,1) to (3,2,4)
c. (1,1,1) to (3,2,4)

Also, what do you observe?

Sol. => $\vec{\nabla}T = (y+z)\hat{x} + (x+z)\,\hat{y} + (x+y)\hat{z}$

a. (1,1,1) to (3,1,1)

$=> \vec{dl} = dx\,\hat{x}$

$=> \vec{\nabla}T.\vec{dl} = (y+z)dx$

$$=> \int_1^3 \vec{\nabla}T.\vec{dl} = \int_1^3 (y+z)dx = (1+1)(2) = 4$$

(3,1,1) to (3,2,1)

$=> \vec{\mathrm{d}}l = dy\,\hat{y}$

$\vec{\nabla}T.\vec{dl} = (x+z)dy$

$$\int_1^2 \vec{\nabla}T.\vec{dl} = \int_1^2 (x+z)dy = (3+1)(1) = 4$$

(3,2,1) to (3,2,4)

$\vec{dl} = dz\,\hat{z}$

$\vec{\nabla}T.\vec{dl} = (x+y)dz$

$$\int_1^4 \vec{\nabla}T.\vec{dl} = \int_1^4 (x+y)dz = (3+2)(3) = 15$$

Total = 4 + 4 + 15 = 23

b. (1,1,1) to (3,2,1)

$=> y = \frac{x+1}{2}$ -> $dy = \frac{dx}{2}$

$=> \vec{d}l = dx\,\hat{x} + dy\,\hat{y} = dx\,\hat{x} + \frac{dx}{2}\hat{y}$

$$=> \vec{\nabla}T.\vec{d}l = (x+z)\frac{dx}{2} + (y+z)\,dx = \left(\frac{x}{2} + y + \frac{3z}{2}\right)dx$$

$$=> \int_{(1,1,1)}^{(3,2,1)} \vec{\nabla}T.\vec{d}l = \int_{(1,1,1)}^{(3,2,1)} \left(\frac{x}{2} + y + \frac{3z}{2}\right)dx = \left(\frac{3}{2}\right)(2) + \int_{1}^{3} \left(\frac{x}{2} + \frac{x+1}{2}\right)dx$$

$$= 3 + \frac{1}{2}[x^2]_1^3 + \frac{1}{2}(2) = 3 + 4 + 1 = 8$$

(3,2,1) to (3,2,4)

$=> \vec{d}l = dz\,\hat{z}$

$$=> \int_{1}^{3} \vec{\nabla}T.\vec{d}l = \int_{1}^{4} (x+y)dz = (3+2)(3) = 15$$

=>Total = 8 + 15 = 23

c. Direct path (1,1,1) to (3,2,4)

$=> \frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{3}$ -> $\frac{dx}{2} = dy = \frac{dz}{3}$

$=> \vec{d}l = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} = 2dy\,\hat{x} + dy\,\hat{y} + 3dy\,\hat{z}$

$$=> \int_{(1,1,1)}^{(3,2,4)} \vec{\nabla}T.\vec{d}l = \int_{(1,1,1)}^{(3,2,4)} (2(y+z) + (x+z) + 3(x+y))dy$$

$$= \int_{(1,1,1)}^{(3,2,4)} (5y + 4x + 3z)dy$$

$$= \int_{y=1}^{y=2} (5y + 4(2y-1) + 3(3y-2))dy = \int_{y=1}^{y=2} (22y - 10)dy$$

$$= \frac{22}{2}(4-1) - 10(1) = 33 - 10 = 23$$

Observation : Line Integral is path independent

Q3) $T = x^2 y e^z$ (it is function defined in 3D)

You are at (2,1,2). What is the unit Vector in the direction in which you should go for maximum increase in value of the function?

Ans: $\vec{\nabla} T = 2xye^z\ \hat{x} + x^2e^z\ \hat{y} + x^2ye^z\ \hat{z}$

$\vec{\nabla} T$ at (2,1,2) = $4e^2\ \hat{x} + 4e^2\ \hat{y} + 4e^2\ \hat{z} = 4e^2(\hat{x} + \hat{y} + \hat{z})$

$\hat{x} + \hat{y} + \hat{z}$ => direction of most increase

$\frac{1}{\sqrt{3}}\hat{x} + \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{z}$ => unit vector along $\vec{\nabla} T$

Q4) The divergence of the vector field $\vec{V} = e^x(\cos y\ \hat{\imath} + \sin y\ \hat{\jmath})$ is _____?

Ans: $\vec{\nabla}.\vec{V} = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}) . (V_x\ \hat{\imath} + V_y\ \hat{\jmath} + V_z\ \hat{k})$

$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$= \frac{\partial(e^x cosy)}{\partial x} + \frac{\partial(e^x siny)}{\partial y} + \frac{\partial(0)}{\partial z}$

$= e^x cosy + e^x cosy = 2e^x cosy$

Q5) The divergence of the vector field ($3xz\ \hat{\imath} + 2xy\ \hat{\jmath} - yz^2\ \hat{k}$) at a point (1,1,1) is ___?

Ans: Let $E = 3xz\ \hat{\imath} + 2xy\ \hat{\jmath} - yz^2\ \hat{k}$

$\vec{\nabla}.\vec{E} = \frac{\partial(3xz)}{\partial x} + \frac{\partial(2xy)}{\partial y} - \frac{\partial(yz^2)}{\partial z}$

$= 3z + 2x - 2yz$

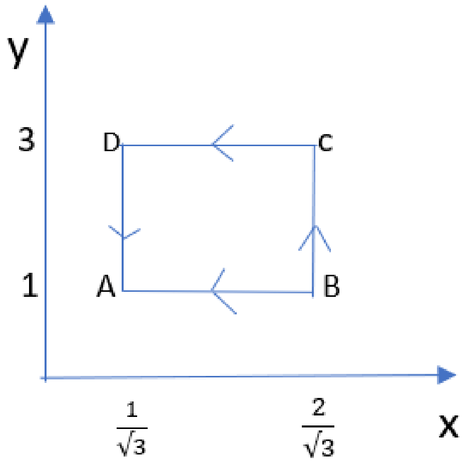
$\vec{\nabla}.\vec{E}|_{(1,1,1)} = 3(1) + 2(1) - 2(1)(1)$

$= 3 + 2 - 2 = 3$

Q6) if $\vec{A}$ = xy $\widehat{a_x}$ + x^2 $\widehat{a_y}$. Find $\oint \vec{A}.\mathrm{dl}$ along the path shown below:

Ans:

1) C along A -> B
   y = 1, dy = 0, x = $\frac{1}{\sqrt{3}}$ to $\frac{2}{\sqrt{3}}$



2) C along B-> C
   x = $\frac{2}{\sqrt{3}}$, dx = 0, y = 1 to 3

3) C along C -> D
   y= 3, dy = 0, x = $\frac{2}{\sqrt{3}}$ to $\frac{1}{\sqrt{3}}$

4) C along D-> A
   x = $\frac{1}{\sqrt{3}}$, dx = 0, y = 3 to 1

To calculate $\oint \vec{A}.\vec{dl} = \oint((xy)\,\widehat{a_x} + x^2\,\widehat{a_y}$) (dx $\widehat{a_x}$ + dy $\widehat{a_y}$)

$\int x\,dx\ \ (along\ c1) = \int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} x\,dx\ = \frac{1}{2}$

$\int x^2 dy\ \ (along\ c2) = \int_1^3 x^2 dy = \frac{8}{3}$

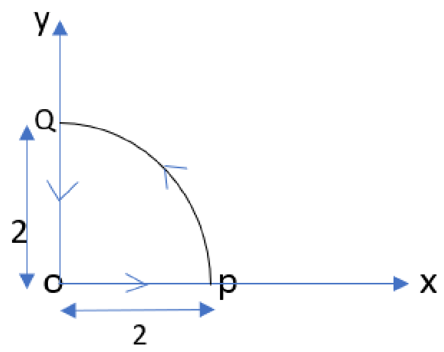
$\int 3x\,dx\ \ (along\ c3) = \int_{\frac{2}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} 3x\,dx\ = -\ \frac{3}{2}$

$\int x^2 dy\ \ (along\ c4) = \int_3^1 \frac{1}{3} dy = -\frac{2}{3}$

$\oint \vec{A}.\vec{dl} = \frac{8}{3} - \frac{2}{3} + \frac{1}{2} - \frac{3}{2} = \frac{6}{3} - 1$ = 2-1 = 1

Q7) $\vec{A} = \widehat{a_\rho} + \widehat{a_\varphi} + \widehat{a_z}$. Find the value of $\oint \vec{A}.\vec{dl}$ along the path shown below:

Sol. We know



$x^2+y^2 = 4$ for $0 \le t \le \frac{\pi}{2}$
Question is in cylindrical coordinate system.

The path shown can be divided into three parts to solve the line integral

1. O -> P where $\rho = 0\ to\ 2$
   $\vec{dl} = d\rho\ \hat{a}_\rho$

   $\vec{A}.\vec{dl} = d\rho$

   $$\int_O^P \vec{A}.\vec{dl} = \int_{\rho=0}^{\rho=2} d\rho = 2$$

2. P -> Q where $\rho = 2\ and\ \emptyset = 0\ to\ \frac{\pi}{2}$
   $\vec{dl} = \rho d\varphi\ \hat{a}_\varphi$

   $\vec{A}.\vec{dl} = \rho d\varphi$

   $$\int_P^Q \vec{A}.\vec{dl} = \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} 2\varphi d\varphi = \pi$$

3. Q -> O where $\rho = 2\ to\ 0$
   $\vec{dl} = d\rho\ \hat{a}_\rho$

   $\vec{A}.\vec{dl} = d\rho$

   $$\int_Q^O \vec{A}.\vec{dl} = \int_{\rho=2}^{\rho=0} d\rho = -2$$

Total = $2 + \pi - 2 = \pi$