ECE230: Fields & Waves (Lect. 4, Winter 2021)

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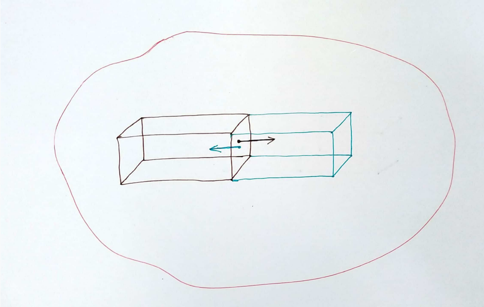
Divergence theorem



- Any arbitrary volume can be chopped up into tiny boxes. Two such sie-by-side boxes have been shown above.
- What happens to the outward flux at the surface shared by the boxes?
- Add up the outward fluxes from all the tiny boxes spanning the macroscopic volume: $\oint\limits_{S_{big}} \vec{V}.\vec{ds} = \int\limits_{V_{big}} (\vec{\nabla}.\vec{V})dv$

Combining divergence and gradient:Laplacian

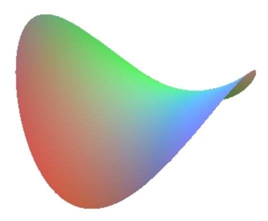
- $\vec{\nabla}.\left(\vec{\nabla}U\right) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right).\left(\hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}\right) = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$
- Shorthand: $\vec{\nabla}.\left(\vec{\nabla}U\right) = \nabla^2 U$
- $\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$, known as Laplacian operator

Earnshaw's theorem

- A scalar field $\phi(x, y, z)$ that has $\nabla^2\phi = 0$ over a region in space, cannot have a local maxima or minima in that region.
- Proof: assume that the field ϕ has a local minima. So, each of $\frac{\partial^2\phi}{\partial x^2}$, $\frac{\partial^2\phi}{\partial y^2}$ and $\frac{\partial^2\phi}{\partial z^2}$ must be positive and would add up to a positive number.
- At best, we can have a saddle point (say, $\frac{\partial^2\phi}{\partial x^2}$ and $\frac{\partial^2\phi}{\partial y^2} > 0$ and $\frac{\partial^2\phi}{\partial z^2} < 0$ yielding $\nabla^2\phi = 0$)

You eat it often!

Hyperbolic paraboloid and Paul trap



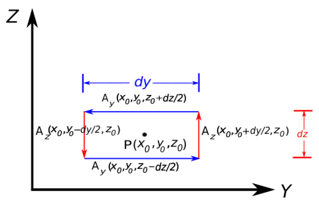
- $z - \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$
- Assume there's a roof of this shape and you need to stabilize a football at the saddle point. Can you do that?
- Wolfgang Paul: Nobel prize 1989 for Paul trap (Demo, Demo)

Curl operator

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} =$$

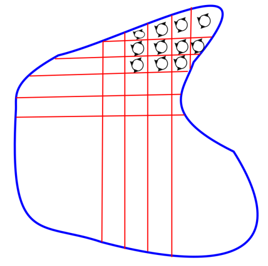
$$\hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Definition of curl: line integral along an infinitesimal loop



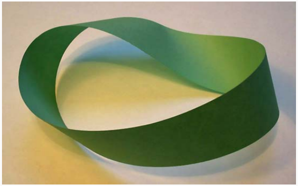
- $\oint \vec{A}.\vec{dl} = \left\{A_y(x_0,y_0,z_0) - \frac{dz}{2}\frac{\partial A_y}{\partial Z}\right\}dy - \left\{A_y(x_0,y_0,z_0) + \frac{dz}{2}\frac{\partial A_y}{\partial Z}\right\}dy + \left\{A_z(x_0,y_0,z_0) + \frac{dy}{2}\frac{\partial A_z}{\partial y}\right\}dz - \left\{A_z(x_0,y_0,z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y}\right\}dz$
- $\oint \vec{A}.\vec{dl} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial Z}\right)dydz$
- $\oint \vec{A}.\vec{dl} = \underset{dS\to 0}{Lt}\left\{\left(\vec{\nabla}\times\vec{A}\right).\hat{x}\right\}dS = \underset{dS\to 0}{Lt}\left(\vec{\nabla}\times\vec{A}\right).\vec{dS}$

Finite loop: Stoke's theorem



- Divide the finite blue loop into infinitesimal small
- $\int\limits_{Blue} \left(\vec{\nabla} \times \vec{A}\right).\vec{dS} = \sum\limits_{Reds} \left(\vec{\nabla} \times \vec{A}\right).\vec{dS} = \sum \oint\limits_{Reds} \vec{A}.\vec{dl} \Rightarrow$
  $\int\limits_{Blue} \left(\vec{\nabla} \times \vec{A}\right).\vec{dS} = \oint\limits_{Blue} \vec{A}.\vec{dl}$
- When is Stoke's theorem not applicable?

Mobius Strip: the one-sided surface



- Which way does the area vector point?

Questions so far?