ECE230: Fields & Waves (Lect. 3, Winter 2021)

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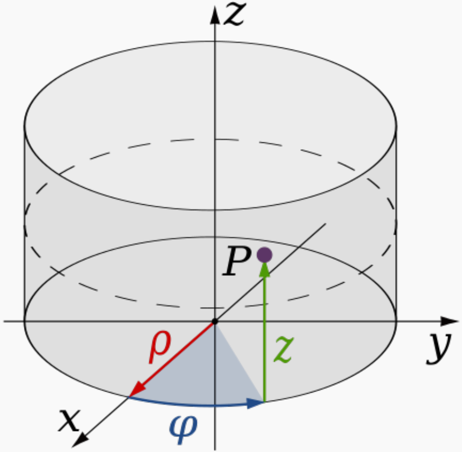
Gradient operator

- $d\phi = \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right) = (\vec{\nabla}\phi).\vec{dr}$
- $(\vec{\nabla}\phi)$ points to the direction of rate of maximum increase of the scalar field ϕ (for a level surface, this direction is along surface normal)

Potential

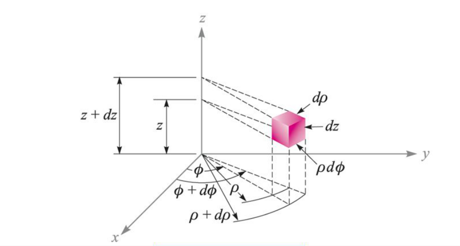
- In one dimension, you have learnt: $\vec{E} = -\frac{d\phi}{dx}\hat{x}$
- Generalize this to $3D$: $\vec{E} = -\left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right) = -\vec{\nabla}\phi$

Cylindrical coordinate system



- $0 \leq \rho < \infty$
- $0 \leq \phi < 2\pi$
- $-\infty < z < \infty$

Cylindrical coordinate system



- Differential length: $\vec{dl} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$
- Differential areas: $\vec{ds_1} = \rho d\phi dz\hat{\rho}$, $\vec{ds_2} = d\rho dz\hat{\phi}$, $\vec{ds_3} = \rho d\phi d\rho\hat{z}$
- Differential volume: $dv = \rho d\phi d\rho dz$

Spherical coordinate system

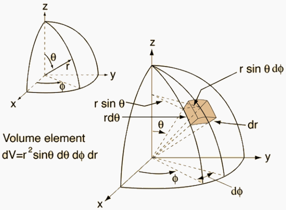
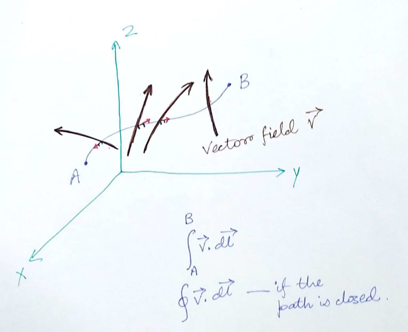


Figure 1: Image source: Hyperphysics

- $0 \leq r < \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$
- Differential length: $\vec{dl} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$
- Differential areas: $\vec{ds_1} = r^2\sin\theta d\theta d\phi\hat{r}$, $\vec{ds_2} = r\sin\theta drd\phi\hat{\theta}$, $\vec{ds_3} = rd\theta dr\hat{\phi}$
- Differential solid angle: $d\Omega = \frac{ds_1}{r^2} = \sin\theta d\theta d\phi$

Line integral



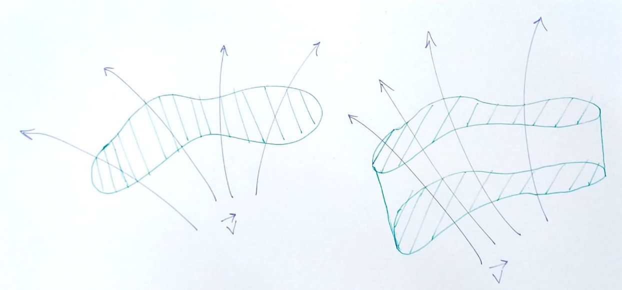
- Few examples you encountered already: Work=$\int_A^B \vec{F}.\vec{d}l$, electrostatic potential $(\phi_{AB}) = -\int_A^B \vec{E}.\vec{d}l$
- In general, depends on the path taken.

Conservative field

- Assume, $\vec{V} = \vec{\nabla} U$
- $\int_A^B \vec{V}.\vec{dl} = \int_A^B (\vec{\nabla} U).\vec{dl} = \int_A^B dU = U_B - U_A$ ——— Only depends on the end points, path independent!
- Such a field that can be written as a gradient of a scalar is called conservative field.
- What about $\oint_A^B \vec{V}.\vec{dl}$? This is simply,

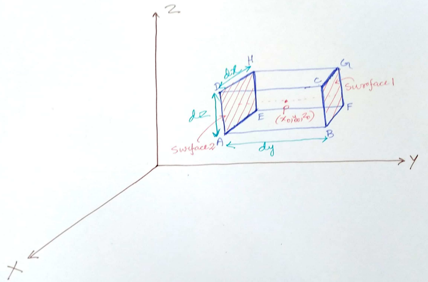
  $\int_A^B dU + \int_B^A dU = U_B - U_A + U_A - U_B = 0$
- Gravitational and electric fields are conservative. Each can be written as gradient of a scalar quantity (potential). Work done in moving a mass in a G-field/ charge in an E-field along a closed path is zero. Also, potential difference between two points doesn't depend on the path taken to move the mass/charge.

Flux of a vector field



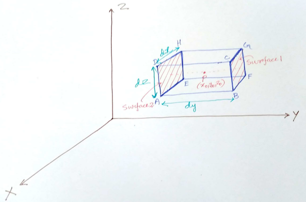
- Open surface: $\int\limits_{S_{open}} \vec{V}.\vec{ds}$
- Closed surface: $\oint\limits_{S_{closed}} \vec{V}.\vec{ds}$

Divergence



- If you add up the outgoing fluxes though all the surfaces, $\oint_{S_{closed}} \vec{V}.\vec{ds} = \underset{dv \to 0}{Lt} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dv$
- Let's introduce the divergence operator $\vec{\nabla}. \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right)$.

Divergence



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- Let's introduce the divergence operator $\vec{\nabla}. \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right).$ (note the 'dot' at the end of this operator)
- Total outgoing flux: $\oint_{S_{closed}} \vec{V}.\vec{ds} =$ $\underset{dv \to 0}{Lt} \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right).(\hat{x}V_x + \hat{y}V_y + \hat{z}V_z)\, dv = \underset{dv \to 0}{Lt} (\vec{\nabla}.\vec{V}) dv$
- $\vec{\nabla}.\vec{V} = \underset{dv \to 0}{Lt} \frac{\oint_{S_{closed}} \vec{V}.\vec{ds}}{dv}$ ——— Definition of divergence