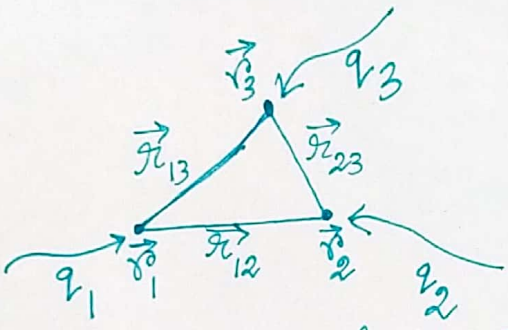
$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

$$\Rightarrow \int \vec{\nabla}\cdot(f\vec{A})\,d\tau = \int f(\vec{\nabla}\cdot\vec{A})\,d\tau + \int \vec{A}\cdot(\vec{\nabla}f)\,d\tau$$

$$\Rightarrow \boxed{\int f(\vec{\nabla}\cdot\vec{A})\,d\tau = -\int \vec{A}\cdot(\vec{\nabla}f)\,d\tau + \oint (f\vec{A})\cdot d\vec{S}}$$

Energy of a point charge distribution:



Work done to bring q_1 to $\vec{r}_1 = 0$

[Because, there was no electric field initially]

Once, q_1 is in place, it 'll create an electrostatic field. So, now when you try to bring q_2 to $\vec{r}_2$, you need to do work against this field. Similarly when you are bringing q_3 to $\vec{r}_3$, you need to work against the combined field created by q_1 and q_2.

Work done to bring q_2 to $\vec{r}_2 = W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{12}}\right)$

" " " " q_3 to $\vec{r}_3 = W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$

" " " " q_4 to $\vec{r}_4 = W_4 = \frac{1}{4\pi\varepsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$

$$\Rightarrow W_4 = \frac{1}{8\pi\varepsilon_0}\left[\frac{q_4 q_1}{r_{14}} + \frac{q_4 q_2}{r_{24}} + \frac{q_4 q_3}{r_{34}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right]$$

In general, $W_i = \frac{1}{8\pi\varepsilon_0} q_i \left(\frac{q_j}{r_{ij}}\right)$

Total work done to assemble n charges $= W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ (i\neq j)}}^{n} \frac{q_i q_j}{r_{ij}}$ ——— (1)

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}} \right)$$

$$\Rightarrow \boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i)} \quad \text{——————} (2)$$

$V(r_i)$ = potential at $\vec{r_i}$ due to all charges other than q_i.

Energy of a continuous charge distribution:

$$W = \frac{1}{2}\int_v \rho V \, dv \quad \text{——————} (3)$$

$\rho = \varepsilon_0(\vec{\nabla}\cdot\vec{E})$. From (3),

$$W = \varepsilon_0/2 \int_v (\vec{\nabla}\cdot\vec{E}) V \, dv$$

$$= \varepsilon_0/2 \left[-\int_v \vec{E}\cdot(\vec{\nabla}V)\, dv + \oint_S V\vec{E}\cdot d\vec{s} \right]$$

[We have used:

$$\int f(\vec{\nabla}\cdot\vec{A})\, dv = -\int \vec{A}\cdot(\vec{\nabla}f)\, dv + \oint f\vec{A}\cdot d\vec{s} \,]$$

$$\therefore W = \varepsilon_0/2 \left[\int_v E^2 \, dv + \oint_S V\vec{E}\cdot d\vec{s} \right] \quad \text{——————} (4)$$

But what volume and surface are we talking about? The volume v must enclose all the charges. However, from eq.(3), if we include a larger volume, $\rho = 0$ ensures that there's no contribution to W beyond v.

Q. Does W diverge if we increase the volume?

E^2 is positive. So, the first term in eq.(4) increases. But the second term must decrease in order to leave the sum intact.

But how to understand this?

$V \sim \frac{1}{r}$ (at least), $E \sim \frac{1}{r^2}$ (at least).

So, $EV \sim \frac{1}{r^3}$. but, $S \sim r^2$.

$\therefore EV\,dS \sim \frac{1}{r}$.

As $r \to \infty$, $\oint \vec{E} \cdot d\vec{S} \longrightarrow 0$.

$$\therefore W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 \, dv$$

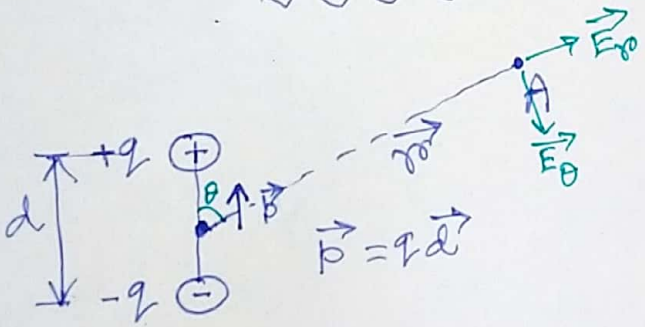
For any physical field, $E \to 0$ as $r \to \infty$.

$\therefore W = \varepsilon_0/2 \int E^2 dv$ doesn't diverge!

~~E dv~~

Energy per unit volume $= \frac{1}{2}\varepsilon_0 E^2$

Electrostatic field inside a material



$\vec{p} = q\vec{d}$

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2}$$

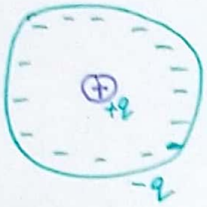
$$= \left(\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}\right) \quad \text{---(5)}$$

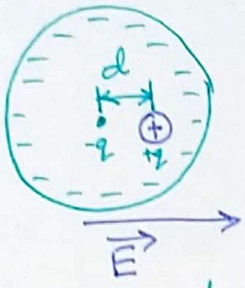
$$\vec{E} = -\vec{\nabla} V(r,\theta) = -\left(\frac{\partial V}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \hat{r}\,\frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} + \hat{\theta}\,\frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}, \quad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

Induced dipole:





$\vec{p} \propto \vec{E}$

$\Rightarrow \vec{p} = \alpha \vec{E}$

(α is called polarizability)

This is what happens to an atom when you place it in an E-field.

Same thing happens to molecules as well, but the situation may be slightly complicated. Consider, CO_2:

If $\vec{E}$ is along the axis (represented by $E^{\parallel}$), polarizability is more than if we had applied $\vec{E}$ perpendicular to the axis! (i.e. $\alpha^{\parallel} > \alpha^{\perp}$)

$$\vec{p} = \alpha^{\parallel} \vec{E}^{\parallel} + \alpha^{\perp} \vec{E}^{\perp}$$

In general,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

↖ polarizability "matrix" (tensor, to be formal)

$$\therefore p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

$$p_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

$$p_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$

$$\vec{p} = [\bar{\alpha}] \vec{E}$$

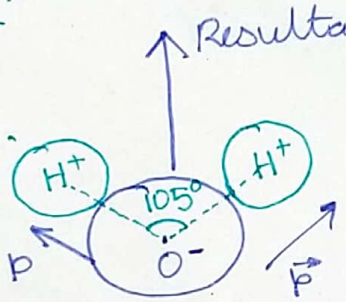
In general, $\vec{p}$ and $\vec{E}$ are not along the same direction.
$\vec{p}$ and $\vec{E}$ are ~~only~~ along the same direction only if:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \leftarrow \text{Isotropic}$$

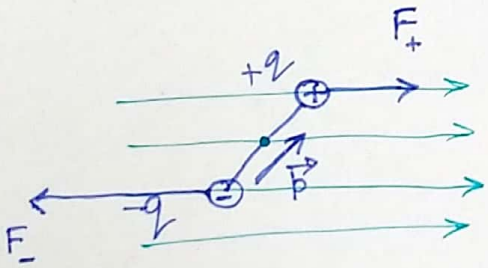
i.e. $p_x = \alpha E_x$, $p_y = \alpha E_y$, $p_z = \alpha E_z$

$$\therefore \vec{p} = \hat{x} p_x + \hat{y} p_y + \hat{z} p_z = \alpha [\hat{x} E_x + \hat{y} E_y + \hat{z} E_z]$$
$$= \alpha \vec{E}$$

Polar molecule:



What happens when such tiny dipoles are placed in an E-field?



Torque (τ)

$$= (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$

$$= \left(\frac{\vec{d}}{2} \times q\vec{E}\right) + \left\{\left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})\right\}$$

$$= (q\vec{d} \times \vec{E}) = (\vec{p} \times \vec{E})$$

This torque 'll tend to align the dipole along the direction of $\vec{E}$.