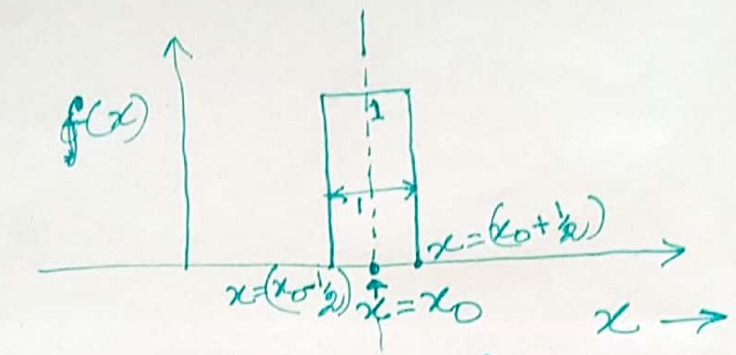
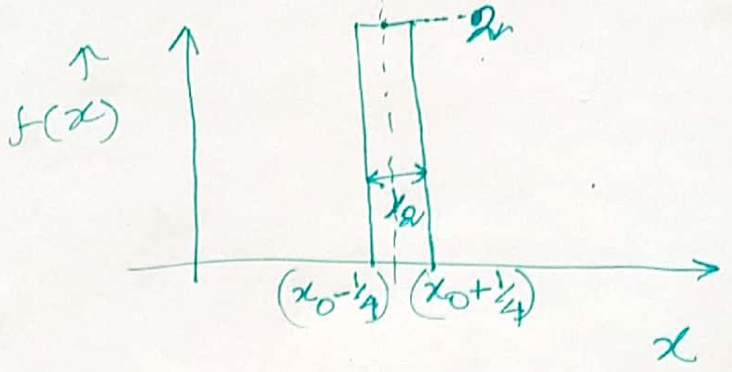
Dirac Delta function:





$$\boxed{\begin{aligned} &\delta(x-x_0) = \begin{cases} 0 & \text{if } x \neq x_0 \\ \infty & \text{if } x = x_0 \end{cases} \\ &\text{and } \int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1 \end{aligned}}$$

- You can easily show that:

$$\int_a^b \delta(x-x_0)\,dx = 1 \text{ if } x_0 \text{ falls within } [a,b]$$

$$= 0 \text{ if } x_0 \text{ is outside } [a,b].$$

- How much is $f(x)\delta(x-x_0)$?

$$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$$

- How much is $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx$?

$$I = \int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = \int_{-\infty}^{\infty} f(x_0)\delta(x-x_0)\,dx$$

$$= f(x_0)\int_{-\infty}^{\infty} \delta(x-x_0)\,dx$$

$$= f(x_0)$$

• Try to show that $\delta(kx) = \frac{1}{|k|}\delta(x)$

3-dimensional delta function:

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

— Dirac delta around origin.

$$\delta^3(\vec{r}-\vec{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$[\, r_0 \equiv (x_0, y_0, z_0)\,]$$

Definition:

$$\int_{\substack{\text{entire}\\ \text{space}}} \delta^3(\vec{r})\, dv = 1 \quad \text{and} \quad \delta(\vec{r}) = \begin{cases} 0 & \text{if } r \neq 0 \\ \infty & \text{if } r = 0. \end{cases}$$

Divergence in spherical coordinate:

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\left(\frac{\partial F_\phi}{\partial\phi}\right)$$

A paradox

Calculate $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = 0 \quad \text{—— (1)}$$

Alternate method; apply divergence theorem.

Let, $\vec{F} = (\hat{r}/r^2)$

$$\int (\vec{\nabla}\cdot\vec{F})\, dv = \oint \vec{F}\cdot d\vec{s} = \oint \left(\frac{\hat{r}}{r^2}\right)\cdot(r^2 \sin\theta\, d\theta . d\phi\; \hat{r})$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta . d\theta = (4\pi)$$

But according to eq.(1), $\int(\vec{\nabla}\cdot\vec{F})\,dv$ should have been 0!!

We have divided by $\frac{1}{r^2}$ in eq. (1). Also, at $r=0$, $\vec{F}$ blows up.

Thus, $\vec{\nabla}\cdot\vec{F}=0$ everywhere except for $r=0$.

If origin is included, $\vec{\nabla}\cdot\vec{F}=4\pi$. (The entire contribution comes from origin!)

$$\boxed{(\vec{\nabla}\cdot\vec{F}) = 4\pi\,\delta^3(r)}$$

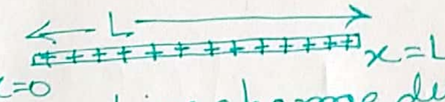
where $\vec{F}=\left(\frac{\hat{r}}{r^2}\right)$

Note that: $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

i.e. $\vec{\nabla}\cdot\left\{\vec{\nabla}\left(\frac{1}{r}\right)\right\} = -4\pi\,\delta^3(r)$

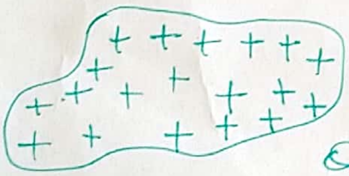
$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(r)$

Charge density:



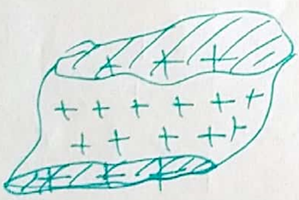
Line charge density λ (coulomb/m)

Total charge $Q = \int_0^L \lambda(x)\,dx$



surface charge density σ (coulomb/m^2)

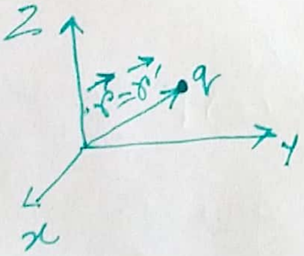
$$Q_{Total} = \int_{Surface} \sigma(x,y)\,dx\,dy = \int_{Surface} \sigma(x,y)\,ds$$



volume charge density ρ (coulomb/m^3)

$$Q_{Total} = \int_V \rho(x,y,z)\,dv = \int_V \rho(r,\theta,\phi)\,dv.$$

• Suppose, there is a point charge q located at $\vec{r}=\vec{r}'$. How much is volume charge density (ρ)?



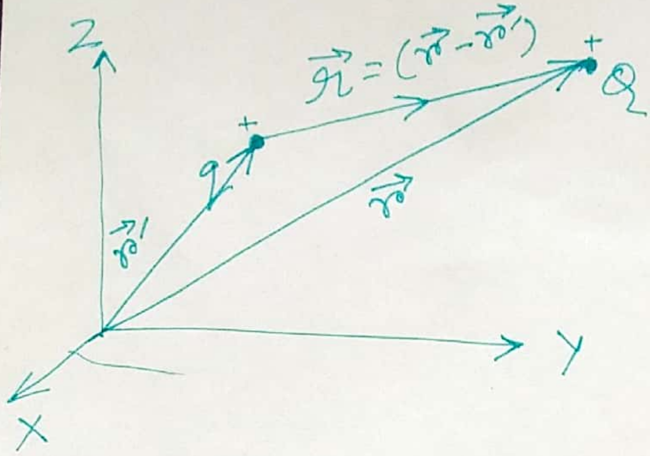
$$\rho = q\,\delta^3(\vec{r}-\vec{r}')$$

check: Total charge $= \int_{All\,space} \rho(r')\,d^3r'$

$= q\int_{All\,space} \delta^3(\vec{r}-\vec{r}')\,d^3r'$

$= q.$

Coulomb's law:



$$\cancel{\vec{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{r}}$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\hat{r}}{r^2}$$

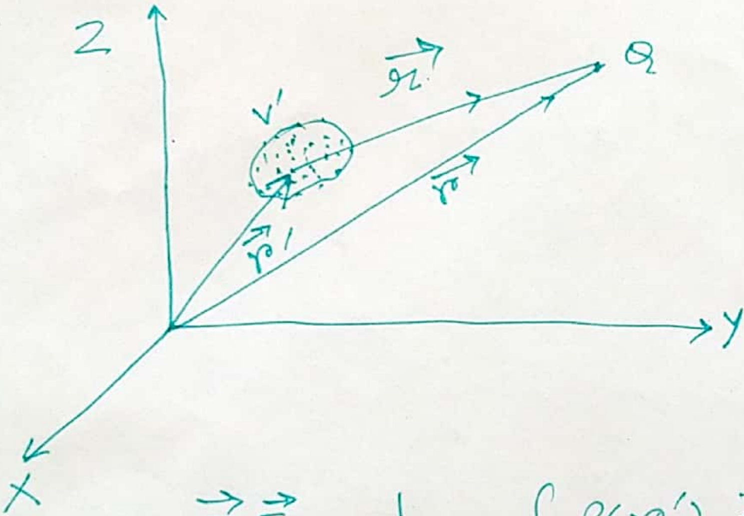
Define potential as: $\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}\right)$

$$\vec{\nabla}\phi = -\frac{q}{4\pi\varepsilon_0}\left(\frac{\hat{r}}{r^2}\right)$$

$$= -\vec{E}$$

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla}\phi}$$

For a continuous charge distribution;



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}')\, dv' \left(\frac{\hat{r}}{r^2}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \rho(\vec{r}')\left(\frac{\hat{r}}{r^2}\right) dv'$$

$$\therefore \vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) dv'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\, 4\pi\delta^3(\vec{r})\, dv'$$

$$= \frac{\cancel{4\pi}}{\cancel{4\pi}\varepsilon_0}\int \rho(\vec{r}')\,\delta^3(\vec{r} - \vec{r}')\, dv'$$

$$\Rightarrow \boxed{(\vec{\nabla}\cdot\vec{E}) = \frac{\rho(\vec{r})}{\varepsilon_0}}$$

← Gauss's law in differential form for free space. —— (2)

Potential of a continuous charge distribution:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\vec{r}')\, dv'}{r}$$

Take volume integral of (2) [bounding surface encloses all the charges]

$$\int (\vec{\nabla}\cdot\vec{E})\, dv = \frac{1}{\varepsilon_0}\int \rho(\vec{r})\, dv \Rightarrow \boxed{\oint \vec{E}\cdot d\vec{S} = \left(\frac{Q_{enc}}{\varepsilon_0}\right)}$$

Gauss's law (integral form in free space)