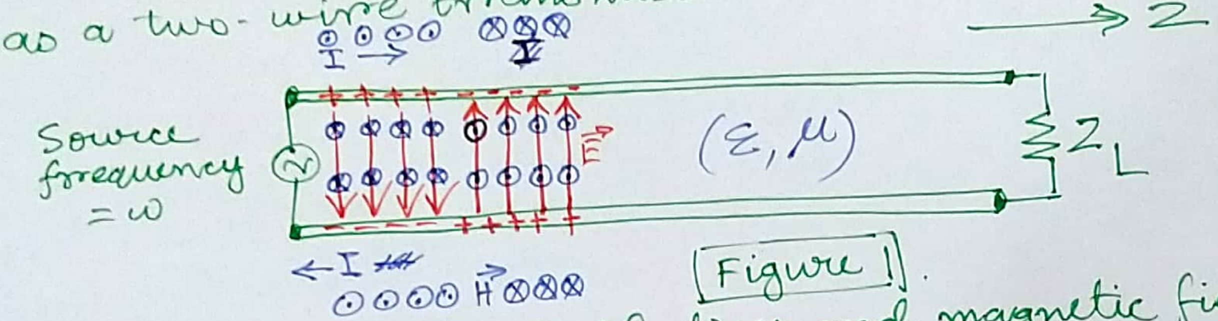
Transmission lines

So far, we have considered wave propagation in an unbounded medium (except for the last lecture where we considered an interface between two media). However, in all these cases the energy were free to extend anywhere over the space. In these two lectures, we are going to consider guided wave propagation. i.e. there would be some mechanism in place ~~two~~ that would try to confine the energy within a finite spatial extent while allowing the energy to flow along the direction perpendicular to the confining directions.

Let's start with a simple example: this is known as a two-wire transmission line.



Here, the wires are conductors and the space between them is filled up by a dielectric.

Figure 1.

Here, the electric field lines and magnetic field lines are more intense within the region between the two conductors and directed in a way so that the wave propagates along the positive z-direction. After every half-wavelength (determined by the source frequency and the speed of the guided wave), both $\vec{E}$ and $\vec{H}$ flips the directions to give rise to a propagation of energy from source to load. This is how energy flows in A.C. circuits! [Just a note on the side: the ~~velocity~~ speed of the guided wave is different from and less ~~than~~ than the free space wave]

At this point you can say, well! This is simply AC circuit then. Why are we making a big deal out of it? We can simply apply KCL, KVL we learnt in circuit theory!

That's where it becomes tricky. Here goes the reasons:

① Consider a transmission line that is very very long but without ~~an~~ any other component between the source and load. So according to KCL, current leaving the ~~load~~ source is exactly the current entering the load. Also, if V_g is source/generator voltage and V_L is voltage across load, according to KVL, $V_g = V_L$. But if the distance between the source ~~to~~ and load is very long, how could the load immediately got to know what was the source current and voltage? Nothing can move faster than light!

Thus, there'll always be a delay between V_g and V_L ~~(as~~ (also between I_g and I_L). You cannot apply KCL and KVL in long transmission line.

Now you may ask, how 'long'? The answer is when the length of your transmission line is comparable to the voltage/current wave's wavelength. (we'll prove in a bit that there's a voltage and current wave).

If the length of the AC circuit is much smaller than the wavelength, you can safely say that the signal took so less time to travel from source to load, you can effectively neglect the delay and go on applying KCL and KVL.

What looks simple, isn't that simple always.

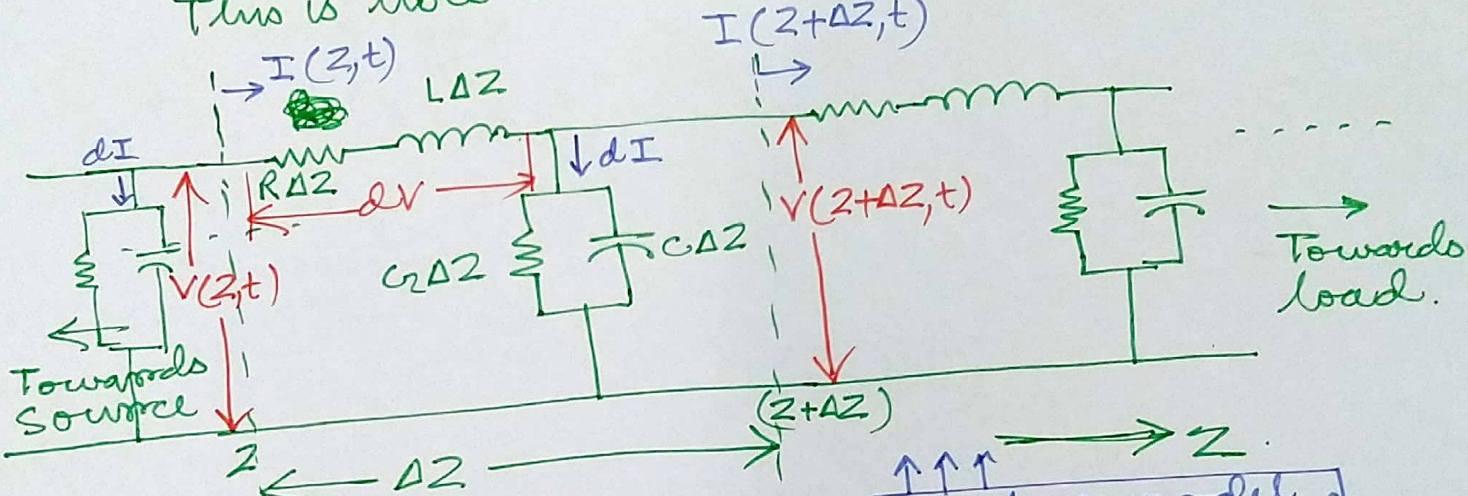
② The conductor may have finite conductivity. This'll give rise to a series resistance R_s. The dielectric between the two wires can have non-zero conductivity. So, there's a shunt resistance of a large value between the two lines, denoted by R_{sh}.

The magnetic field is always changing and you can imagine the entire circuit along with the source and load as a big loop. Thus, there's a self-inductance (L) associated with the circuit.

There's a dielectric between the wires through which electric field lines are passing. So, there's a capacitance in shunt!

More importantly, these R_S, R_{Sh}, L and C are present at every point of the circuit! They are distributed throughout and not lumped at a particular point. So, we must express them as "per unit length" parameters i.e. Ohm/m, Farad/m or, Henry/m.

This is how the actual circuit looks like:



The above model represents any type of transmission line. (Not only two wire line)!

$$\boxed{R = R_S \text{ and } G = \frac{1}{R_{sh}}}$$

When $\Delta z \to 0$, we can apply KCL and KVL over this small section (replace Δz by dz)

$$\therefore dV = V(z+dz,t) - V(z,t) = -I(z,t)\,Z\,dz$$

[where, $Z = (R + j\omega L)$: series impedance/unit length].

$$\Rightarrow \frac{\partial V}{\partial z} = -IZ \quad \text{———} \quad (1)$$

Applying KCL,

$$-dI = I(z+dz,t) - I(z,t) = +VY\,dz$$

[$Y = (G + j\omega C)$: admittance/unit length]

$$\Rightarrow \boxed{\frac{\partial I}{\partial z} = -VY} \text{ ——— (2)}$$

$$\boxed{\frac{\partial V}{\partial z} = -IZ} \text{ ——— (1)}$$

Differentiating (2),

$$\frac{\partial^2 I}{\partial z^2} = -\frac{\partial V}{\partial z} \cdot Y$$

$$= ZY \cdot I$$

[using value of $\frac{\partial V}{\partial z}$ from 1]

$$\Rightarrow \frac{\partial^2 I}{\partial z^2} - \gamma^2 I = 0 \text{ ——— (3)}$$

Differentiating (1),

$$\frac{\partial^2 V}{\partial z^2} = -Z \cdot \frac{\partial I}{\partial z}$$

$$= YZ\,V$$

[using value of $\frac{\partial I}{\partial z}$ from (2)].

$$\Rightarrow \frac{\partial^2 V}{\partial z^2} - \gamma^2 V = 0 \text{ ——— (4)}$$

Both I and V satisfy Helmholtz eqn. So, the solution must be waves.

wavelength (λ) = $2\pi/\beta$

$$\gamma = \sqrt{YZ} = \sqrt{(G + j\omega C)(R + j\omega L)} = (\alpha + j\beta)$$

A lossless transmission line is one where there is no resistive loss, i.e. the conductors are perfect conductors and dielectrics are perfect dielectrics.
$(\sigma \to \infty)$ $(\sigma = 0)$

$\therefore R = G = 0$ for a lossless transmission line.

$$\therefore \gamma = \sqrt{j^2\omega^2 LC} = j\omega\sqrt{LC}.$$

$\therefore$ Eqn. (3) and (4) becomes;

$$\frac{\partial^2 I}{\partial z^2} + \omega^2 LC\,I = 0 \text{ ——— (5)}$$

and $$\frac{\partial^2 V}{\partial z^2} + \omega^2 LC\,V = 0 \text{ ——— (6)}$$

In general, the solutions of eqn (3) and (4) are:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$= I_0^+ e^{-\alpha z} \cdot e^{-i\beta z} + I_0^- e^{\alpha z} e^{i\beta z} \quad \text{——— (7)}$$

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$= \underbrace{V_0^+ e^{-\alpha z} e^{-i\beta z}}_{+z \text{ travelling wave}} + \underbrace{V_0^- e^{\alpha z} e^{i\beta z}}_{-z \text{ travelling wave}} \quad \text{——— (8)}$$

$$v(z,t) = V_0^+ e^{-\alpha z} e^{i(\omega t - \beta z)} + V_0^- e^{\alpha z} \cdot e^{i(\omega t + \beta z)}$$

- If there is no loss (i.e. loss-less transmission line), $\alpha = 0$.

$$\therefore \quad I(z) = I_0^+ e^{-i\beta z} + I_0^- e^{i\beta z}$$

$$V(z) = V_0^+ e^{-i\beta z} + V_0^- e^{i\beta z}$$

and $\beta = \omega\sqrt{LC} \Rightarrow \omega/\beta = \dfrac{1}{\sqrt{LC}}$

$\Rightarrow v_0$ = propagation vel. $= \dfrac{1}{\sqrt{LC}}$.

- If there is no reflected wave (i.e. $I_0^- = V_0^- = 0$)

$$\left.\begin{aligned} I(z) &= I_0^+ e^{-\alpha z} e^{-i\beta z} = I_0^+ e^{-\gamma z} \\ V(z) &= V_0^+ e^{-\alpha z} e^{-i\beta z} = V_0^+ e^{-\gamma z} \end{aligned}\right\} (9a)$$

$$\therefore Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+} \overset{(5)}{=} Z_0 \quad \text{——— (9)}$$

Z_0 is known as the characteristic impedance of the transmission line.

(Denotes the ratio of ~~forward~~ voltage to current when no reflected wave is ~~pr~~ present).

Analogous to $\eta = E/H = \sqrt{\mu/\varepsilon}$

Substituting the value of $V(Z)$ and $I(Z)$ from eqn(9a) into eqn(

$$\frac{\partial (V_0^+ e^{-\gamma Z})}{\partial Z} = -IZ$$

$$\Rightarrow -\gamma V_0^+ e^{-\gamma Z} = -IZ = -I_0^+ e^{-\gamma Z} Z$$

$$\Rightarrow -\gamma V_0^+ e^{-\gamma Z} = -I_0^+ e^{-\gamma Z} Z$$

$$\Rightarrow \frac{V_0^+ e^{-\gamma Z}}{I_0^+ e^{-\gamma Z}} = \left(\frac{Z}{\gamma}\right)$$

$$\Rightarrow \frac{V_0^+}{I_0^+} = \frac{Z}{\sqrt{YZ}} = \sqrt{Z/Y}$$

$$\Rightarrow \boxed{Z_0 = \sqrt{Z/Y} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = (R_0 + jX_0)}$$

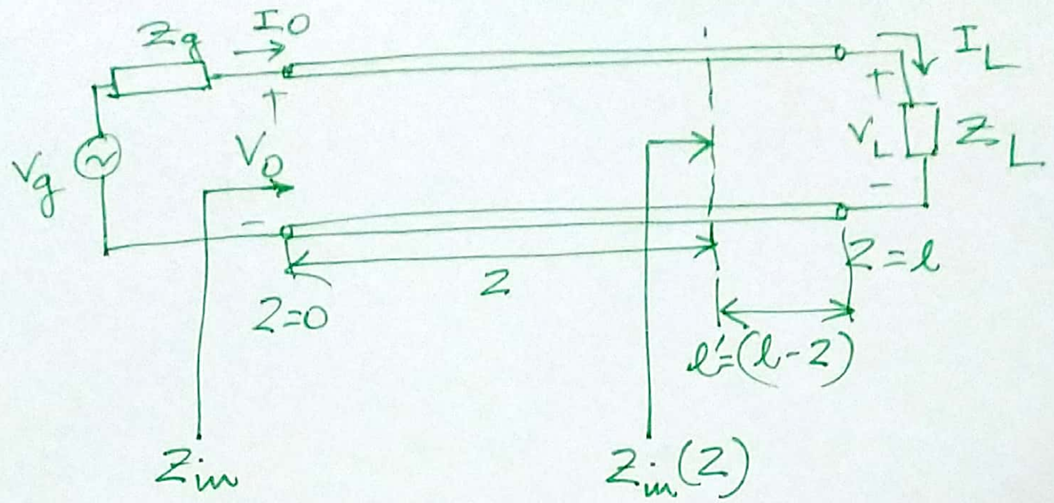
For a lossless transmission line, (i.e. $R = G = 0$)

$$Z_0 = R_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}.$$

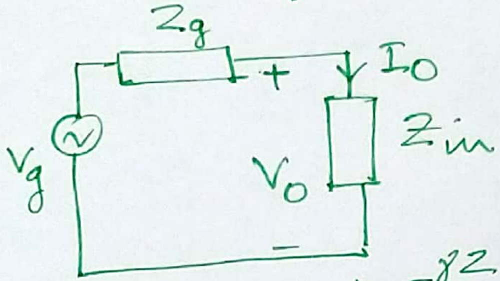
Using same line of attack, you can also show that $Z_0 = -V_0^- / I_0^-$

$$\therefore \boxed{\boxed{I_0^+ = Z_0 V_0^+} \text{ and } \boxed{I_0^- = -Z_0 V_0^-}} \quad \text{——— (10)}$$

Input impedance:



$\Downarrow$ We want to find an equivalent circuit



$$V_0 = \frac{V_g Z_{in}}{(Z_{in}+Z_g)}$$

$$I_0 = \frac{V_g}{(Z_{in}+Z_g)} \quad \text{—— (11)}$$

We have, $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \quad \text{—— (12)}$$

The boundary conditions are:

$$V(z=0) = V_0 \quad \text{and} \quad I(z=0) = I_0$$

$$\Rightarrow V_0 = (V_0^+ + V_0^-) \quad \text{and} \quad I_0 = \frac{(V_0^+ - V_0^-)}{Z_0}$$

(13) (14)

From (14), $V_0^+ - V_0^- = I_0 Z_0$ —— (15)

$\therefore$ Solving (13) and (15), $V_0^+ = \frac{1}{2}(V_0 + I_0 Z_0)$ —— (16)

$$V_0^- = \frac{1}{2}(V_0 - I_0 Z_0) \quad \text{—— (17)}$$

Boundary conditions at the load are:

$$V(z=l) = V_L \,, \quad I(z=l) = I_L$$

$$\therefore V_0^+ = \frac{1}{2}(V_L + Z_0 I_L)e^{\gamma l} \qquad \text{——— (18)}$$

$$V_0^- = \frac{1}{2}(V_L - Z_0 I_L)e^{-\gamma l} \qquad \text{——— (19)}$$

$$\therefore Z_{in} = Z_{in}(z=0)$$

$$= \frac{V(z=0)}{I(z=0)} = \frac{Z_0(V_0^+ + V_0^-)}{(V_0^+ - V_0^-)} \quad \text{[from eqn. (13) and (14)]}$$

$$= Z_0 \left[\frac{\frac{1}{2}(V_L + Z_0 I_L)e^{\gamma l} + \frac{1}{2}(V_L - Z_0 I_L)e^{-\gamma l}}{\frac{1}{2}(V_L + Z_0 I_L)e^{\gamma l} - \frac{1}{2}(V_L - Z_0 I_L)e^{-\gamma l}} \right]$$

$$= Z_0 \left[\frac{V_L\left(\frac{e^{\gamma l} + e^{-\gamma l}}{2}\right) + Z_0 I_L \left(\frac{e^{\gamma l} - e^{-\gamma l}}{2}\right)}{V_L\left(\frac{e^{\gamma l} - e^{-\gamma l}}{2}\right) + Z_0 I_L \left(\frac{e^{\gamma l} + e^{-\gamma l}}{2}\right)} \right]$$

$$= Z_0 \left[\frac{Z_L I_L \cosh(\gamma l) + Z_0 I_L \sinh(\gamma l)}{Z_L I_L \sinh(\gamma l) + Z_0 I_L \cosh(\gamma l)} \right]$$

$$\Rightarrow \boxed{Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \right]} \qquad \text{——— (20)}$$

For a lossless line ($R = G = 0$), ~~$\gamma = j$~~ $\gamma = i\beta$ and $\tan(\gamma l) = i\tan(\beta l)$

$$\therefore \boxed{Z_{in} = Z_0\left[\frac{Z_L + iZ_0\tan(\beta l)}{Z_0 + iZ_L\tan(\beta l)}\right]} \qquad \text{——— (21)}$$

In analogy to reflection coefficient of EM waves, we can define a voltage reflection coefficient at the load as: $\Gamma_L = \dfrac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}}$ (i.e. $\dfrac{\text{reflected voltage at load}}{\text{Incident voltage at load}}$)

$$= \frac{(V_L - Z_0 I_L)}{(V_L + Z_0 I_L)} = \boxed{\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) = \Gamma_L}$$

Voltage reflection coefficient at any point on the line

$$\Gamma(z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \frac{V_0^-}{V_0^+} e^{2\gamma z}$$

$$= \left(\frac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}}\right)\frac{e^{-\gamma l}}{e^{\gamma l}} \cdot e^{2\gamma z}$$

$$= \Gamma_L e^{-2\gamma(l-z)}$$

$$\Rightarrow \boxed{\Gamma(z) = \Gamma_L e^{-2\gamma l'}} \longleftarrow (22)$$

[$l' = l - z$ = distance from load]

For a lossless line,

$$\Gamma(z) = \Gamma_L e^{-2i\beta l'}$$

- Note the analogy between the reflection coefficients: $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ and $\left(\dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)$
- $\Gamma_L = 0$ when $Z_L = Z_0$ (this is called a matched load)

~~Using the same method we used to derive eqns (20) and (21);~~

Now that we have calculated i/p impedance (located at $z=0$) when we look from source, towards the load, how can we calculate i/p impedance looking into the load from any point z on the line?

$$Z_{in}(z) = \frac{V(z)}{I(z)} = \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{\frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}}$$

$$= Z_0 \left[\frac{V_0^+ + V_0^- e^{2\gamma z}}{V_0^+ - V_0^- e^{2\gamma z}} \right]$$

$$= Z_0 \left[\frac{1 + \frac{V_0^-}{V_0^+} e^{2\gamma z}}{1 - \frac{V_0^-}{V_0^+} e^{2\gamma z}} \right]$$

$$\Rightarrow \boxed{Z_{in}(z) = Z_0 \left[\frac{1+\Gamma_z}{1-\Gamma_z} \right]} \quad \text{———} (23)$$

$$\Rightarrow \boxed{\Gamma_z = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}}$$

[Note that when you are at the load and look into the load, $Z_{in} = Z_L$ and you get back Γ_L]

Eqn. (23) can be written in another form (in terms of Γ_L):

$$Z_{in}(z) = Z_0 \left[\frac{1 + \Gamma_L e^{-2\gamma l'}}{1 - \Gamma_L e^{-2\gamma l'}} \right] \quad \text{———} (24)$$

So far, we have talked only about voltage reflection coefficient. How about current reflection coefficient?

We can define the current reflection coefficient at the load to be

$$\frac{I_0^- e^{+\gamma l}}{I_0^+ e^{-\gamma l}} = -\frac{V_0^-/Z_0 \, e^{+\gamma l}}{V_0^+/Z_0 \, e^{-\gamma l}}$$

$$= -\frac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}}$$

$$= -\Gamma_L .$$

In analogy to SWR of $\vec{E}$, we can also define a quantity call voltage standing wave ratio (VSWR). This is denoted by the symbol 's' and represents the content of standing wave.

$$s = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \qquad \boxed{1 \leq s < \infty}$$

When there's no reflected wave (i.e. the transmission line is terminated with a matched load), $\Gamma_L = 0$ and $s = 1$. When $|\Gamma_L| = 1$ and $s \to \infty$.

~~i.e.~~

A natural question to ask at this point is, under what conditions $|\Gamma_L| = ?$ There are two possibilities $\Gamma_L = -1$ and $\Gamma_L = 1$ (actually there are infinite to be precise, but let's talk about these two under the scope of the present course).

(1) $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$. Suppose we terminate the transmission line by <u>short circuit</u> i.e. $Z_L = 0$. This would give $\Gamma_L = -1$.

(2) Terminate the transmission line by open circuit.

i.e. $Z_L \to \infty$.

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1 - Z_0/Z_L}{1 + Z_0/Z_L}$$

$$\therefore \lim_{Z_L \to \infty} \Gamma_L = 1.$$

So, open and short circuit terminations of transmission line are two conditions that reflect the entire incident power, producing a pure standing wave ($S \to \infty$).

Another surprise: In circuit theory, you have always read that when $Z_L = 0$, $Z_{in} = 0$ and when $Z_L \to \infty$, $Z_{in} \to \infty$. But these are only low frequency approximations!

Consider, $Z_L = 0$ (and for simplicity, a lossless line)

$\therefore$ From eqn. (21),

$$Z_{in} = Z_0 \left[\frac{\cancel{Z_L} + iZ_0 \tan(\beta l)}{Z_0 + iZ_L \tan(\beta l)}\right]$$

$$= Z_0 \cdot \frac{i\cancel{Z_0} \tan(\beta l)}{\cancel{Z_0}}$$

$$= iZ_0 \tan(\beta l)$$

For $Z_L \to \infty$,

$$Z_{in} = \lim_{Z_L \to \infty} Z_0 \left[\frac{1 + i\frac{Z_0}{Z_L} \tan(\beta l)}{\frac{Z_0}{Z_L} + i\tan(\beta l)}\right]$$

$$= Z_0 \frac{1}{i\tan(\beta l)} = -iZ_0 \cot(\beta l).$$