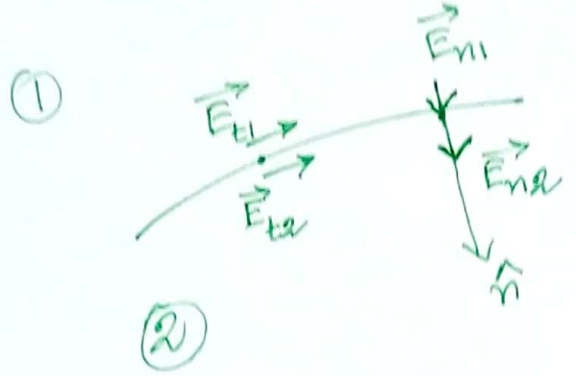
Recap of boundary conditions of $\vec{E}$ and $\vec{H}$:

- Tangential components of $\vec{E}$ is continuous across an interface between two media. $(E_{t1} = E_{t2})$



- Normal components of $\vec{D}$ are discontinuous by an amount ~~of~~ σ, the surface charge density at the interface. i.e. $(D_{n2} - D_{n1}) = \sigma_{surface}$

  If surface charge is zero,

$$D_{n2} = D_{n1}$$

$$\Rightarrow \varepsilon_2 E_{n2} = \varepsilon_1 E_{n1}$$

- Normal components of $\vec{B}$ are continuous across the interface. $B_{n2} = B_{n1}$

- Tangential components of $\vec{H}$ are discontinuous by an amount $\vec{K}$, the surface current density.

$$H_{t2} - H_{t1} = K$$

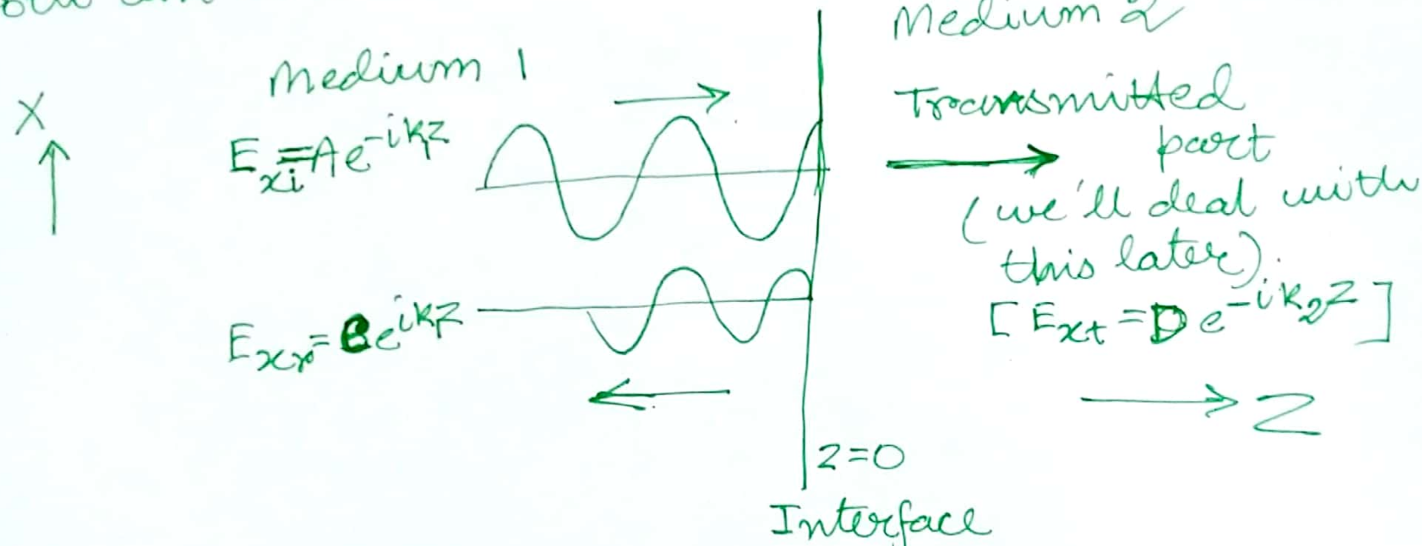
  For a perfect dielectric (conductivity $= 0$),

$$H_{t2} = H_{t1}$$

- We also learnt that electromagnetic field can not exist inside a perfect conductor. So, when an e.m. wave strikes a conducting (to be precise, perfectly conducting) surface, the entire energy is reflected back.

Travelling and standing wave:

Recall: we solved 1D Helmholtz equation and saw that two solutions are possible. One is Be^{ikz} (wave travelling along $-z$ direction and the other one is Ae^{-ikz} (wave travelling along $+z$). We considered only one of these two candidates so far. But in general, both can exist simultaneously:



(X-polarized ~~planar~~ plane wave incident at a planar interface between two dielectric media)

Total $\vec{E}$-field in medium 1:

$$E_{x1} = Ae^{-ik_1 z} + Ce^{ik_1 z} \quad \text{———} (1)$$

Total $\vec{H}$-field in medium 1:

$$H_{y1} = \frac{1}{\eta_1}\left(Ae^{ik_1 z} - Ce^{ik_1 z}\right)$$

[η_1 = intrinsic impedance of medium 1]

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

We are omitting the time dependence for ease of writing.

But always remember, the actual field is:

$$\mathcal{E}_{x1}(z,t) = E_{x1}(z)e^{i\omega t} = Ae^{i(\omega t - k_1 z)} + Ce^{i(\omega t + k_1 z)}$$

How 'll the function E_{x_1} in eq. (1) look like?

Let's try to write it in a form:

$$E_{x_1} = r e^{-i\theta}$$

$$\Rightarrow A e^{-iK_1 z} + C e^{iK_1 z} = r e^{-i\theta}$$

$$\Rightarrow (A+C)\cos(K_1 z) - i(A-C)\sin(K_1 z) = r\cos\theta - i r\sin\theta.$$

Equate real and imaginary parts;

$$r\cos\theta = (A+C)\cos(K_1 z) \quad \text{———} (3)$$

$$r\sin\theta = (A-C)\sin(K_1 z) \quad \text{———} (4)$$

$$\therefore \tan\theta = \left(\frac{A-C}{A+C}\right)\tan(K_1 z)$$

$$\Rightarrow \theta = \tan^{-1}\left[\left(\frac{A-C}{A+C}\right)\tan(K_1 z)\right] \quad \text{———} (5)$$

$$\text{and } r = \sqrt{(A+C)^2\cos^2(K_1 z) + (A-C)^2\sin^2(K_1 z)}$$

$$= \sqrt{A^2 + C^2 + 2AC\cos^2(K_1 z) - 2AC\sin^2(K_1 z)}$$

$$= \sqrt{A^2 + C^2 + 2AC\cos(2K_1 z)}$$

$$\therefore E_{x_1} = r e^{-i\theta} = \sqrt{A^2 + C^2 + 2AC\cos(2K_1 z)} \cdot e^{-i\tan^{-1}\left\{\left(\frac{A-C}{A+C}\right)\tan K_1 z\right\}}$$

$$|E_{x_1}| = \sqrt{A^2 + C^2 + 2AC\cos(2K_1 z)}$$

Known as standing wave pattern.

$$|E_{x1}|_{max} = \sqrt{A^2+C^2+2AC} = (A+C)$$

$\longrightarrow$ Occurs at $(2k_1 z) = 2n\pi$

$[n = 0, \pm 1, \pm 2 \ldots]$

$$\Rightarrow k_1 z = n\pi$$

$$\Rightarrow \frac{2\cancel{\pi}}{\lambda_1} z = n\cancel{\pi}$$

$$\Rightarrow z = n\left(\frac{\lambda_1}{2}\right)$$

$$|E_{x1}|_{min} = \sqrt{A^2+C^2-2AC} = (A-C)$$

$\longrightarrow$ Occurs at $(2k_1 z) = (2n+1)\pi$

$[n = 0, \pm 1, \pm 2, \ldots]$

$$\Rightarrow k_1 z = \left(n+\tfrac{1}{2}\right)\pi$$

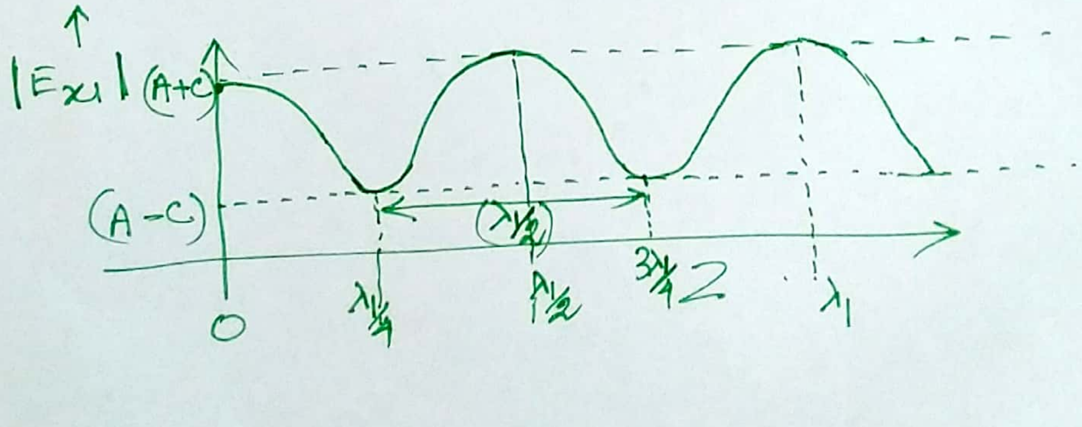
$$\Rightarrow \frac{2\cancel{\pi}}{\lambda_1} z = \left(n+\tfrac{1}{2}\right)\cancel{\pi}$$

$$\Rightarrow z = \left(n+\tfrac{1}{2}\right)\left(\frac{\lambda_1}{2}\right).$$

Also, $|E_{x1}|$ is periodic in z with a periodicity $\left(\lambda_1/2\right)$.

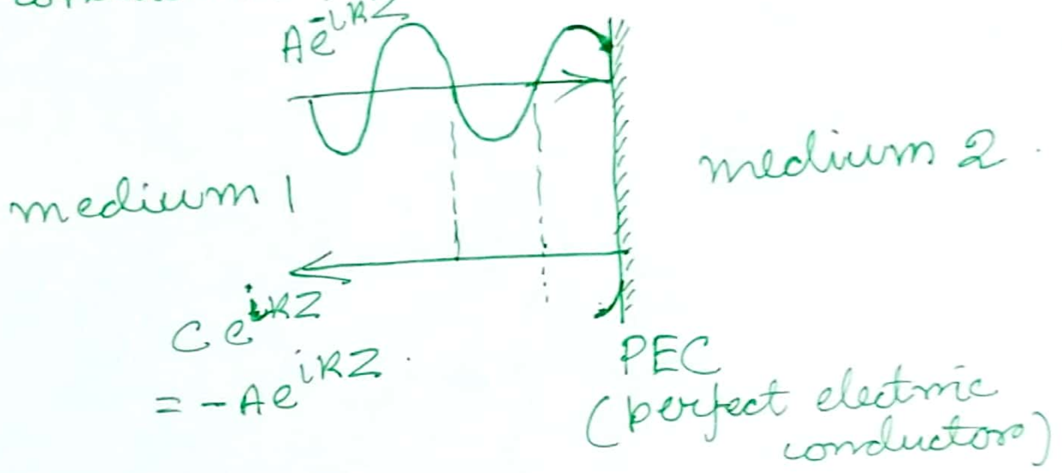
$2k_1 z =$

[can be easily seen noting that the distance between consecutive maxima and minima are $\left(\lambda_1/2\right)$]



Suppose, instead of choosing medium 2 to be a dielectric we let it be a perfect conductor. Alternate arrangement; we put a perfect conductor at the interface between two dielectric media.

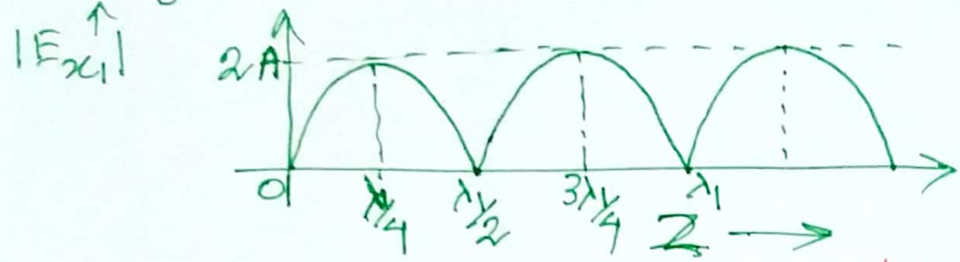
Both of these above arrangement 'll give rise to the same effect in medium 1 $\Rightarrow$ the entire incident energy must be reflected back to medium 1. Because: e.m. wave cannot penetrate perfect conductor.



$$\therefore E_{x1} = Ae^{-ik_1 z} - Ae^{ik_1 z}$$
$$= -2iA\sin(k_1 z)$$
$$\mathcal{E}_{x1}(z,t) = -2iA\sin(k_1 z)\,e^{i\omega t}$$
$$H_{y1} = \left(Ae^{-ik_1 z} + Ae^{ik_1 z}\right)\cdot \frac{1}{\eta_1}$$
$$= \frac{2A}{\eta_1}\cos(k_1 z)$$
$$\mathcal{H}_{y1}(z,t) = \frac{2A}{\eta_1}\cos(k_1 z)\,e^{i\omega t}.$$

The time averaged Poynting vector
$= \text{Real}\left\{\vec{E}_{x1} \times \vec{H}^*_{y1}\right\} = 0$ [Note the phase difference between E_{x1} and H_{y1} due to presence of i].

In other words, if $A = -C$, there is no net flow of energy averaged over time. This is the reason, the case of ~~the~~ superposition of equal amount of incident and reflected wave, is known a pure standing wave.



Standing wave pattern of a pure standing wave.

The standing wave pattern tells you how much is the content of "pure standing wave" in a given waveform: if the minima is at zero field (i.e. $|E_{x1}| = 0$) the wave is a pure standing wave (no time averaged energy flow). ~~An increasin~~ A decreased difference between field maxima and minima values would denote progressively smaller value of C, the reflected wave. Thus, the standing wave content in the waveform decreases. A flat standing wave pattern means $C = 0$. In other words difference between $|E_{x1}|_{max}$ and $|E_{x1}|_{min}$ is zero $[|E_{x1}|_{max} = |E_{x1}|_{min} = A]$. So, no reflected wave at all. Meaning that no standing wave content in the waveform.

The ratio $\dfrac{|E_{x1}|_{max}}{|E_{x1}|_{min}}$ $\cancel{\dfrac{A+C}{A-C}}$, is called standing wave ratio (SWR)

Maximum value of SWR occurs for a pure standing wave, since $|E_{x1}|_{min} = 0$. So, $SWR_{max} \to \infty$

Minimum SWR occurs when we have a pure

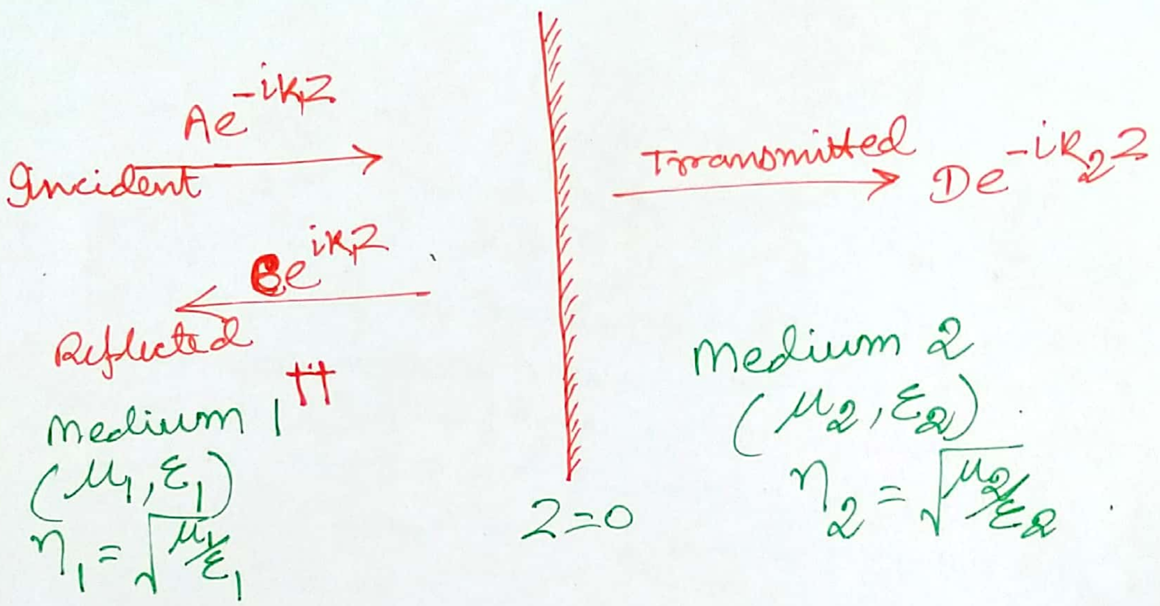
travelling wave i.e. $|E_x|_{max} = |E_x|_{min}$.

So, $(SWR)_{min} = 1$.

This is a <u>very important</u> property of SWR:

$$\boxed{1 \leq (SWR) < \infty}$$

Reflection of E. M. wave at a planar interface



$$E_{x1} = Ae^{-ik_1 z} + Ce^{ik_1 z} \quad \text{—— (6)}$$

$$E_{x2} = De^{-ik_2 z} \quad \text{—— (7)}$$

Let us define a quantity Γ_z as reflection coefficient. Γ_z is a complex number and is the ratio of the amplitude of the reflected to the incident $\vec{E}$-field.

$$\therefore \Gamma_z = \frac{Ce^{ik_1 z}}{Ae^{-ik_1 z}} = \frac{C}{A} e^{i2k_1 z}$$

$$\Gamma_z \Big|_{z=0} = \frac{C}{A} = \Gamma$$

$$\therefore \Gamma_z = \Gamma e^{i2k_1 z} \quad \text{—— (8)}$$

[Note that $\Gamma = |\Gamma_z|$]

†† Imp: These media can be conductive, the same formulation'll be applicable just by replacing ε_1 and ε_2 by effective complex ε_{1eff} and ε_{2eff}.

Similarly, we define a transmission coefficient T as the ratio of the transmitted field amplitude to the incident field amplitude at the interface.

$$\therefore \; T = D/A$$

Thus eqns. (6) and (7) can be written as:

$$E_{x1} = A\left[e^{-ik_1 z} + \Gamma e^{ik_1 z}\right] \quad \text{------} \; (9)$$

and $$E_{x2} = AT e^{-ik_2 z} \quad \text{------} \; (10)$$

The expressions for the magnetic fields will be:

$$H_{y1} = \frac{A}{\eta_1}\left[e^{-ik_1 z} - \Gamma e^{ik_1 z}\right] \quad \text{------} \; (11)$$

$$H_{y2} = \frac{AT}{\eta_2} e^{-ik_2 z} \quad \text{------} \; (12).$$

At this point, let's make an important observation: Irrespective of angle of incidence, E_x and H_y components are always tangential to the interface at $z=0$. According to the boundary conditions, ~~at a pure dielectric~~ they must be continuous across the interface. Thus, the ratio $E_x/H_y = \eta_Z$ must also be continuous across the interface.

$$\eta_1\big|_{z=0} = \eta_2\big|_{z=0}$$ [This is coming from: $$\left.\frac{E_{x1}}{H_{y1}}\right|_{z=0} = \left.\frac{E_{x2}}{H_{y2}}\right|_{z=0}$$]

$$\Rightarrow \frac{A(1+\Gamma)}{\frac{A}{\eta_1}(1-\Gamma)} = \eta_2$$

$$\Rightarrow \frac{1+\Gamma}{1-\Gamma} = \eta_2/\eta_1 \;\Rightarrow\; \boxed{\Gamma = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)} \quad \text{------} \; (13)$$

From the continuity equation of E_{x_1} and E_{x_2} at $z=0$,

$A(1+\Gamma) = AT$ [put $z=0$ in eqns (9) and (10)]

$$\Rightarrow \boxed{T = 1+\Gamma} \qquad (14)$$

$$\therefore T = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \left(\frac{2\eta_2}{\eta_1 + \eta_2}\right) \qquad (15).$$

- At this point we would like to write SWR in a nicer way:

$$SWR = \frac{|E_{x_1}|_{max}}{|E_{x_1}|_{min}} = \frac{A+C}{A-C} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

- If both media (1) and (2) are non-magnetic [i.e. $\mu_1 = \mu_2 = \mu_0$], the expressions for Γ and T can be further simplified as:

$$\Gamma = \frac{\sqrt{\mu_0/\varepsilon_2} - \sqrt{\mu_0/\varepsilon_1}}{\sqrt{\mu_0/\varepsilon_2} + \sqrt{\mu_0/\varepsilon_1}} = \left(\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right) = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)$$

$$T = \frac{2\sqrt{\mu_0/\varepsilon_2}}{\sqrt{\mu_0/\varepsilon_1} + \sqrt{\mu_0/\varepsilon_2}} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} = \frac{2n_1}{(n_1 + n_2)}$$

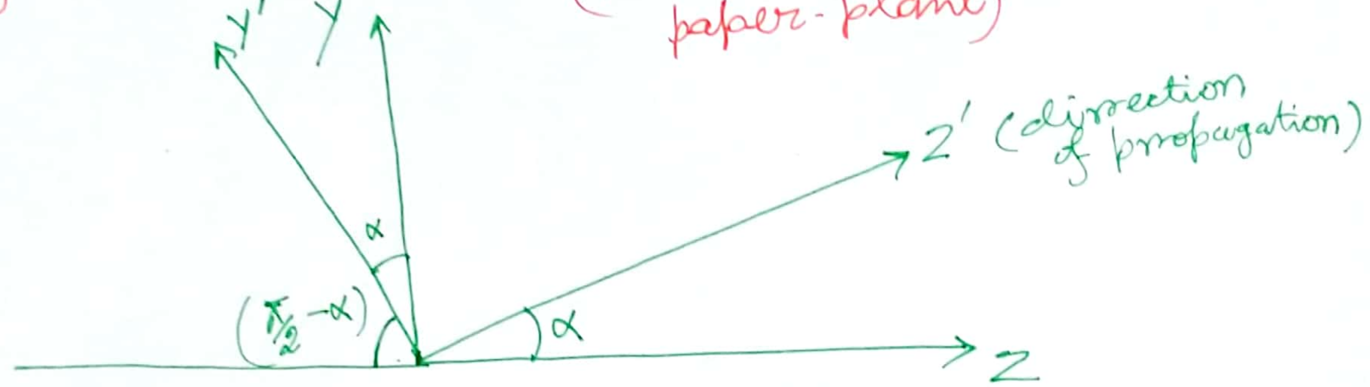
Here, $n_1 = \sqrt{\varepsilon_1/\varepsilon_0} = \sqrt{\varepsilon_{1r}}$
$n_2 = \sqrt{\varepsilon_2/\varepsilon_0} = \sqrt{\varepsilon_{2r}}$
$\Big\}$ definition of refractive indices of the media [i.e. in general, r.i. $(n) = \sqrt{\varepsilon_r} = \sqrt{\varepsilon/\varepsilon_0}$]

Uniform plane wave at an interface — Oblique incidence

Before we continue to the case of oblique incidence, we 'll need some tools in our hand. The problem statement of this introductory part is very simple:

How do we represent a ~~plane~~ unifor plane wave that travels along a direction z', where z' makes an angle α with z?

(X is directed into the paper-plane)



$$z' = (z\cos\alpha + y\sin\alpha) \text{ ——— } (16)$$

Let $\hat{y}'$ be the unit vector along Y' axis.

$$\therefore \hat{y}' = (\hat{y}\cos\alpha - \hat{z}\sin\alpha) \text{ ——— } (17)$$

Possibility 1: $\vec{E}$ is parallel to $z=0$ plane i.e. along x direction and $\vec{H}$ is along Y'.

$$E_x = Ae^{-ikz'} = Ae^{-ik(z\cos\alpha + y\sin\alpha)} \text{ ——— } (18)$$

$$\vec{H} = \hat{y}' \frac{A}{\eta} e^{-ikz'}$$

$$\Rightarrow \vec{H} = (\hat{y}\cos\alpha - \hat{z}\sin\alpha)\frac{A}{\eta} e^{-ik(z\cos\alpha + y\sin\alpha)}$$

[from eqn (17) and (18)]

——— (19)

$$\therefore H_y = \frac{A}{\left(\eta/\cos\alpha\right)} e^{-ik(z\cos\alpha + y\sin\alpha)} \text{ ——— } (20)$$

We obtain the z-directed impedance, $\eta_z = E_x/H_y$ as:

$$\boxed{\eta_z = \left(\eta/\cos\alpha\right)} \qquad \text{———} (21)$$

Possibility 2: $\vec{H}$ is parallel to $z=0$ plane i.e. along x and $\vec{E}$ is along y'.

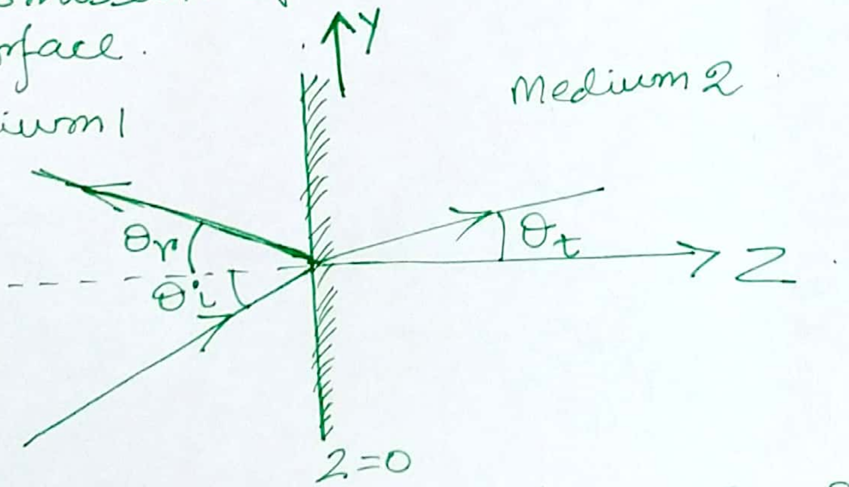
$$\cancel{\therefore H_x \neq} \qquad \therefore \vec{E} = \hat{y}' A e^{-ikz'}$$

$$= (\hat{y}\cos\alpha - \hat{z}\sin\alpha) A \cdot e^{-ik(z\cos\alpha + y\sin\alpha)}$$

$$H_x = -\frac{A}{\eta} e^{-ikz'} = -\frac{A}{\eta} e^{-ik(z\cos\alpha + y\sin\alpha)}$$

$$\therefore \boxed{\eta_z = -\frac{E_y}{H_x} = \eta\cos\alpha} \qquad \text{———} (22)$$

Now is the time for the real problem of reflection and transmission of an e.m. wave incident obliquely at an interface.



The interface is the $z=0$ plane. θ_i, θ_r and θ_t are angles of ~~reflection~~, incidence, reflection and transmission (you may call it refraction), respectively.

First, we'll consider the case I (i.e. $\vec{E}$ parallel to the interface and along X direction).

1. For the incident wave, $\alpha = \theta_i$.
2. for the reflected wave, $\alpha = (\pi - \theta_r)$
3. for the refracted wave, $\alpha = \theta_t$.

$$\therefore E_{xi} = A e^{-ik_1(z\cos\theta_i + y\sin\theta_i)} \qquad (23)$$

[from eqn (18)]

$$E_{xr} = A\Gamma e^{-ik_1(-z\cos\theta_r + y\sin\theta_r)} \qquad (24)$$

$$E_{xt} = AT e^{-ik_2(z\cos\theta_t + y\sin\theta_t)} \qquad (25)$$

Now, $E_{xi} + E_{xr} = E_{xt}\Big|_{\text{at } z=0}$, for all y.

$$\therefore A e^{-ik_1 y\sin\theta_i} + \Gamma A e^{-ik_1 y\sin\theta_r} = AT e^{-ik_2 y\sin\theta_t}.$$

for all y.

If the above equation is valid for all y,

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t.$$

$\therefore \boxed{\theta_i = \theta_r}$ — You have seen this! Law of reflection.

And,

$$k_1 \sin\theta_i = k_2 \sin\theta_t$$

$$\Rightarrow \omega\sqrt{\mu_1\varepsilon_1}\sin\theta_i = \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_t$$

$$\Rightarrow \frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}$$

If the media are non-magnetic ($\mu_1 = \mu_2 = \mu_0$)

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{\varepsilon_{2r}}{\varepsilon_{1r}}} = \frac{\sqrt{\varepsilon_{2r}}}{\sqrt{\varepsilon_{1r}}} = \frac{n_2}{n_1}$$

— Snell's law!!

Also, $H_{y1} = H_{yi} + H_{yr}$

$$= \cos\theta_i \frac{A}{\eta_1}\left(e^{-ik_1 z\cos\theta_i} - \Gamma e^{+ik_1 z\cos\theta_i}\right)$$

[we have deliberately dropped the y-variation here since ultimately its same for all 3 and ~~goin~~ going to get cancelled]

$$H_{y2} = \cos\theta_t \frac{A}{\eta_2} e^{-ik_2 z\cos\theta_t}$$

$$\left.\frac{E_{x1}}{H_{y1}}\right|_{z=0} = \left.\frac{E_{x2}}{H_{y2}}\right|_{z=0}$$

[z-directed impedances at the interface must be equal]

$$\Rightarrow \frac{A(1+\Gamma)}{A\frac{\cos\theta_i}{\eta_1}(1-\Gamma)} = \boxed{\frac{\eta_2}{\cos\theta_t}}$$

→ This part is known from eqn. (21)

$$\Rightarrow \frac{\eta_1}{\cos\theta_i}\left(\frac{1+\Gamma}{1-\Gamma}\right) = \frac{\eta_2}{\cos\theta_t}$$

$$\Rightarrow \frac{1+\Gamma}{1-\Gamma} = \frac{\left(\frac{\eta_2}{\cos\theta_t}\right)}{\left(\frac{\eta_1}{\cos\theta_i}\right)}$$

$$\Rightarrow \boxed{\Gamma = \frac{\left(\eta_2/\cos\theta_t - \eta_1/\cos\theta_i\right)}{\left(\frac{\eta_2}{\cos\theta_t} + \frac{\eta_1}{\cos\theta_i}\right)}}$$

E parallel to interface.

compare this with normal incidence:

$$\Gamma = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)$$

η_2 and η_1 have been replaced by the z-directed impedances for E parallel to interface case!

Similarly, for H parallel to the interface case, η_2 and η_1 'll be replaced by the z-directed impedance appearing in eqn. (22):

$$\boxed{\Gamma = \left(\frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}\right)}$$ — H parallel to interface.