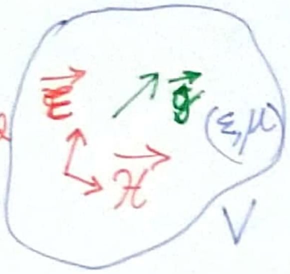
For ease of representation I'll use $\vec{E}$ instead of $\vec{\mathcal{E}}$.



$\vec{J}(t)$ = Time dependent current density.

Stored electric energy in the volume V is W_e

$$= \frac{1}{2}\varepsilon\int_V E^2 dV$$

Stored magnetic energy in the volume V is W_m

$$= \frac{1}{2}\mu\int H^2 dV$$

[Assignment 2]

Stored electric energy density $(w_e) = \frac{1}{2}\varepsilon E^2$

" magnetic " " $(w_m) = \frac{1}{2}\mu H^2$.

Total stored electromagnetic energy density

$$= w_e + w_m = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2.$$

Total stored electromagnetic energy in the volume

$$V = \int_V \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dV$$

Consider the identity:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$= \vec{H}\cdot\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) - \vec{E}\cdot\left(\vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$= -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \vec{E}\cdot\vec{J} - \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] - \vec{E}\cdot\vec{J}$$

$$\Rightarrow -\frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right]=\vec{E}\cdot\vec{J}+\vec{\nabla}\cdot(\vec{E}\times\vec{H}) \qquad (1)$$

Assume that a charge ~~particle~~ q (giving rise to the current density and field) in the volume V moves with a velocity $\vec{v}$.

$\therefore$ Work done by the electromagnetic field on the charge $=\vec{F}\cdot\vec{dl}=q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{dl}$

$$\Rightarrow dw = q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}\,dt$$

$$\Rightarrow dw = q\vec{E}\cdot\vec{v}\,dt$$

$$\Rightarrow \frac{dw}{dt}=\rho\, dV\, \vec{E}\cdot\vec{v}$$

$$=\vec{E}\cdot(\rho\vec{v})\,dV$$

$$=(\vec{E}\cdot\vec{J})\,dV$$

[Recall: coulomb's law $\vec{J}=\rho\vec{v}$]

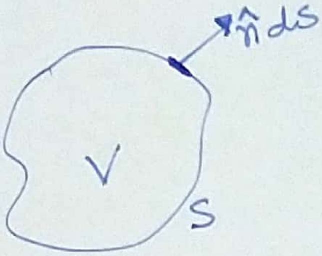
$\therefore$ Rate of work done (by the electromagnetic field) per unit volume $=(\vec{E}\cdot\vec{J})$

Let's now try to find out what does eqn. (1) mean physically: [Poynting's theorem: differential form]

$$\underbrace{-\frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right]}_{\text{Rate at which e.m. energy depletes (per unit volume)}}=\underbrace{\vec{E}\cdot\vec{J}}_{\text{rate of work done by the field (per unit volume)}}+\underbrace{\vec{\nabla}\cdot(\vec{E}\times\vec{H})}_{\text{From law of conservation of energy, this must represent the time-rate at which e.m. energy escapes out (per unit volume)}} \qquad (2)$$

Integrating (over the volume V):

$$-\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right] dV = \int_V (\vec{E}\cdot\vec{J})\,dV+\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV$$



$$= \int (\vec{E}\cdot\vec{J})\,dV + \oint_S (\vec{E}\times\vec{H})\cdot\vec{dS} \qquad \text{--- (3)}$$

While eq. (2) establishes Poynting theorem for unit volume, the above relation is restatement of the same over any arbitrary volume.

The last term, integrand of the $(\vec{E}\times\vec{H})$ is rate of outward flow of energy per unit area.

$\oint_S (\vec{E}\times\vec{H})\cdot\vec{dS}$ = rate of outward flow of energy from the entire volume V (equivalently, through the entire surface enclosing the volume V).

$\boxed{\text{Poynting vector} = \vec{S} = (\vec{\mathcal{E}}\times\vec{\mathcal{H}})}$ $\therefore |\vec{S}| = |\vec{\mathcal{E}}||\vec{\mathcal{H}}| = \frac{|\vec{\mathcal{E}}|^2}{\eta}$

EM wave in materials:

Starting with a microscopic picture, we have obtained the general expression of permittivity of a medium;

$$\varepsilon(\omega) = \varepsilon_0(1+\chi_e(\omega)) = \varepsilon_0\left[1+\frac{Nq^2}{m\varepsilon_0}\sum_j \frac{f_j}{(\omega_j^2-\omega^2+i\omega\gamma_j)}\right]$$

Claim: The imaginary part of the $\varepsilon(\omega)$ is the origin of conductivity.

[To be justified shortly].

In the last lecture, we studied 1D uniform plane wave propagation: $\vec{\mathcal{E}}(z,t) = \vec{E_0}\,e^{i(kz-\omega t)}$

To obtain this solution, we started with Helmholtz eqn. which again came from $(\vec{\nabla}\times\vec{H}) = i\omega\varepsilon\vec{E}$. --- (4)

In contrast, for a material with conductivity σ, the same eqn. gets modified to $\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$

$$= (\sigma + i\omega\varepsilon)\vec{E} \quad \text{---} (5)$$

How can we then have a solution for $\vec{E}(z)$ in a medium with conductivity σ with minimal effort?

Trick: Put eqn. (5) in a form similar to eqn (4). In other words, define an equivalent ε_{eqv} that subsumes the effect of σ. That way, we could have $k = \omega\sqrt{\mu\varepsilon_{eqv}}$ and write the solution as $\vec{\mathcal{E}}(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$

So, we want $\vec{\nabla} \times \vec{H} = i\omega\varepsilon_{eqv}\vec{E}$ ——— (6) and (5)

Comparing with eq. (4) $[\vec{\nabla} \times \vec{H} = i\omega\varepsilon\vec{E}]$, you can easily see that ε_{eqv} is going to be complex.

Let's rearrange eqn (5),

$$\vec{\nabla} \times \vec{H} = i\omega\varepsilon\left(1 + \frac{\sigma}{i\omega\varepsilon}\right)\vec{E}$$

$$= i\omega\varepsilon\left(1 - i\frac{\sigma}{\omega\varepsilon}\right)\vec{E}.$$

Compare the above equation with eqn (6);

$$\boxed{\varepsilon_{eqv} = \varepsilon\left(1 - i\frac{\sigma}{\omega\varepsilon}\right)} \quad \text{———} (7)$$

Remember we claimed that σ arises from imaginary part of $\varepsilon(\omega)$?

If $\sigma = 0$, $\varepsilon_{eqv.}$ is real.

Now, we can write the $\vec{\mathcal{E}}(z,t)$ of a propagating EM wave in a medium with σ as:

$$\vec{\mathcal{E}}(z,t) = \vec{E_0}\, e^{-i(kz-\omega t)}$$ ← Forward propagating wave.

where

$$k = \omega\sqrt{\mu\varepsilon_{equ}}.$$

Here, k is also complex: $\boxed{k = k' - ik''}$, $k''>0$

Corresponding Helmholtz eqn: $\nabla^2\vec{E}(z) + \omega^2\mu\varepsilon_{equ}\vec{E}(z) = 0.$

<u>Imp</u>: Why is imaginary part of k written as $-ik''$ and not $+ik''$?

Suppose, we allow $k = (k' + ik'')$

$$\therefore \vec{\mathcal{E}}(z,t) = \vec{E_0}\, e^{-i(k'+ik'')z} \cdot e^{i\omega t}$$

$$= e^{-i^2k''z} \cdot e^{-ik'z} \cdot e^{i\omega t}$$

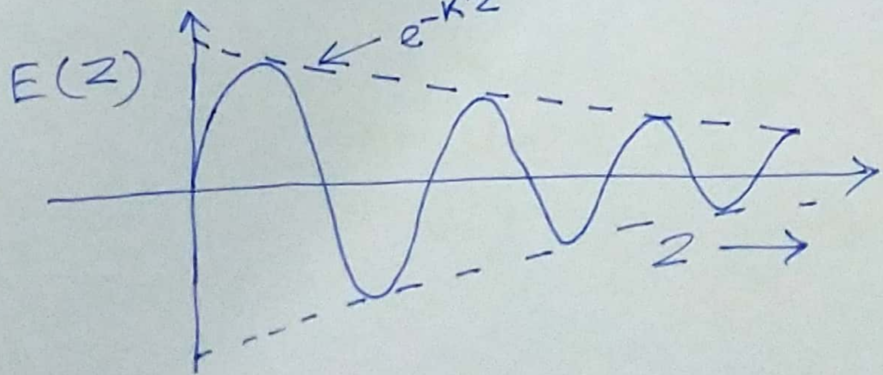
$$= \vec{E_0}\underline{\underline{e^{k''z}}}\, e^{-i(k'z-\omega t)}$$

↑ increases exponentially as the wave propagates forward. This cannot be physical!!

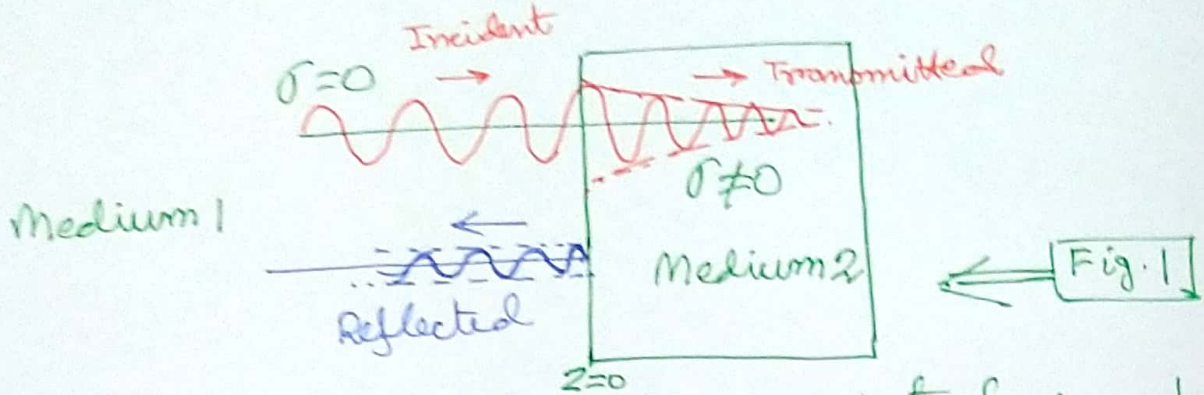
So, $k = (k' - ik'')$, $k''>0$.

This would give rise to a physical solution:

$$\vec{\mathcal{E}}(z,t) = \vec{E_0}\, e^{-k''z}\, e^{-i(k'z-\omega t)}$$



Imp: conductivity causes attenuation of EM signal in a medium



When EM wave is incident on an interface, a part is reflected back and the other part is transmitted to the second medium.

If the second medium has non-zero conductivity, the transmitted wave 'll get attenuated according to $e^{-K''z}$ as it propagates.

Q. Where does this lost energy go?

Ans: It is dissipated as heat.

Imp: what happens if the medium 2 is a perfect conductor (i.e. $\sigma \to \infty$)?

Time varying $\vec{E}$ and $\vec{B}$ creates each other. Now, $\vec{E}=0$ inside a perfect conductor. So, $\vec{B}$ must be zero in medium 2. So, EM energy can not penetrate/exist in a perfect conductor.

The entire EM energy incident on the interface must be reflected back. Transmitted EM signal in a medium with $\sigma \to \infty$ must be zero!

However, for a finite σ, the EM wave 'll penetrate to the medium. But 'll be attenuated (shown in fig. 1). The extent of attenuation/attenuation constant is K''.

If E_0 is the amplitude of the incident wave at $z=0$, the transmitted wave amplitude reduces to $\frac{1}{e}E_0$ after penetrating a depth $1/K''$:

$$E_0 e^{-K''z}\Big|_{z=1/K''} = e^{-1}\cdot E_0$$

$\frac{1}{k''}$ is known as the skin depth (denoted by δ)

$$\boxed{\delta = \frac{1}{k''} \quad \text{The length/depth after which the amplitude of the incident wave reduces to 36.79\% of its initial value}}$$

Evaluation of k' and k'':

$$k^2 = (k' - ik'')^2 = \omega^2 \mu \varepsilon_{eqv}. \qquad \boxed{\text{Imp: } k'' > 0}$$

$$\Rightarrow (k'^2 - k''^2) - 2ik'k'' = \omega^2 \mu \varepsilon \left(1 - i\frac{\sigma}{\omega\varepsilon}\right)$$

$$= \omega^2 \mu \varepsilon - i\omega\mu\sigma$$

Equating real and imaginary parts:

$$(k'^2 - k''^2) = \omega^2 \mu \varepsilon \quad \text{——— (8)}$$

$$2k'k'' = \omega\mu\sigma \quad \text{——— (9)}$$

$$\therefore (k'^2 + k''^2)^2 = (k'^2 - k''^2)^2 + 4(k'k'')^2$$

$$= \omega^4 \mu^2 \varepsilon^2 + \omega^2 \mu^2 \sigma^2$$

$$= \omega^4 \mu^2 \varepsilon^2 \left(1 + \frac{\omega^2 \mu^2 \sigma^2}{\omega^4 \mu^2 \varepsilon^2}\right)$$

$$= \omega^4 \mu^2 \varepsilon^2 \left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}\right)$$

$$\Rightarrow k'^2 + k''^2 = \omega^2 \mu \varepsilon \sqrt{1 + \sigma^2/\omega^2\varepsilon^2} \quad \text{——— (10)}$$

Adding (8) and (10),

$$k'^2 = \frac{\omega^2 \mu \varepsilon}{\cancel{\varepsilon}\, 2} \left[\sqrt{1 + \sigma^2/\omega^2\varepsilon^2} + 1\right]$$

$$\Rightarrow \boxed{k' = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1 + \sigma^2/\omega^2\varepsilon^2} + 1\right]^{1/2}}$$

Similarly, subtracting eq (8) from eq (10), we can ~~obtain~~ show:

$$\boxed{k'' = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1 + \sigma^2/\omega^2\varepsilon^2} - 1\right]^{1/2}}$$

Another important observation:
What happens to the phase between E and H? Are they in same phase? Like what happens in a circuit carrying a current I due to an applied voltage V?

Circuit quantity

Voltage (V)

Current I (A)

Impedance $= Z = V/I$

(unit Ω)

If Z is real (resistance), V and I are in same phase.

$Z = R \pm JX$ (Inductive or capacitive X)

EM wave quantities

$\vec{E}$ [V/m]

$\vec{H}$ [A/m]

wave impedance

$= \eta = E/H = \sqrt{\mu/\varepsilon}$

(unit Ω)

If η is real, E and H are in same phase.
[In other words, ε is real
i.e. no conductivity ($\sigma = 0$)
i.e. pure dielectric].

$$\eta = \sqrt{\frac{\mu}{\varepsilon_{eqv}}}$$

$$= \sqrt{\frac{\mu}{\varepsilon(1 - i\sigma/\omega\varepsilon)}}$$

$$\Rightarrow E/H = \sqrt{\frac{\mu}{\varepsilon - i\sigma/\omega}}$$

So, for a ~~conducting~~ medium with non-zero conductivity, E and H are not in same phase.