Recap from last lecture:

Helmholtz eqn / Wave eqn:

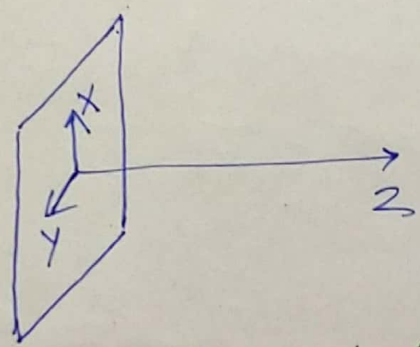
$$\nabla^2 \vec{E}(r) + \frac{\omega^2}{c^2}\vec{E}(r) = 0$$ ← Eigenvalue eqn.

Also, we saw that the spatial and time dependence of the fields can be written as:

$$\vec{\mathcal{E}}(r,t) = \vec{E}(r)e^{i\omega t}$$ — For a single frequency.

$$\vec{\mathcal{E}}(r,t) = \int_{-\infty}^{\infty} \vec{E}(r,\omega)e^{i\omega t}\,d\omega$$ — For multiple frequencies

Waves in one-dimension (Uniform plane wave):



Assumption: the fields are independent of x and y, depends only on z (this is why we have used the term 'uniform').

Wave eqn. becomes:

$$\frac{d^2\vec{E}}{dz^2} + k^2\vec{E} = 0 \quad \left[k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda},\right.$$

known as wave-number]

$$E = Ae^{ikz} + Be^{-ikz}$$

$$\therefore \mathcal{E}(z,t) = E(z)e^{i\omega t}$$

$$= \underbrace{Ae^{i(kz+\omega t)}}_{\text{wave travelling along } -z} + \underbrace{Be^{i(\omega t - kz)}}_{\text{Wave travelling along } +z}$$

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Consider the second term:

$$\mathcal{E}(z,t) = Be^{i(\omega t - kz)}$$ ← A purely forward travelling wave.

$$= Be^{-i(kz - \omega t)}$$

$$= Be^{-ik(z - \frac{\omega}{k} t)}$$

$$= \underbrace{Be^{-ik(z - ct)}}_{\text{of the form } f(z - vt)}$$

$$\mathcal{E}(z,t) = \underbrace{B}_{\text{Amplitude part}} e^{i\boxed{(\omega t - kz)}}$$ ← Phase part.

We want to find out the condition such that the phase is constant at a given time t_1.

$$\therefore \quad \omega t_1 - kz = \text{constant} = \theta_1$$

$$\Rightarrow kz = \omega t_1 - \theta_1$$

$$\Rightarrow z = \left(\frac{\omega t_1 - \theta_1}{k}\right) = \text{constant}$$ ← Plane wave.

So, constant phase surfaces are planes parallel to XY-plane. In other words, phase fronts/wave fronts are plane. This is why we call it a uniform plane wave.

↑ we mentioned earlier the meaning of uniform.

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EM wave equation in 3D.

$$\nabla^2\vec{E}+k^2\vec{E}=0, \quad k=\left(\frac{\omega}{c}\right)$$

$$\Rightarrow \frac{\partial^2\vec{E}}{\partial x^2}+\frac{\partial^2\vec{E}}{\partial y^2}+\frac{\partial^2\vec{E}}{\partial z^2}+k^2\vec{E}=0.$$

Consider the components one by one. Let's consider E_x:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}+k^2E_x=0 \qquad (1)$$

We want to solve this by method of separation of variables:

$$E_x(x,y,z)=E_1(x)\,E_2(y)\,E_3(z)$$

Substitute this into eqn. (1),

$$E_2E_3\frac{d^2E_1}{dx^2}+E_1E_3\frac{d^2E_2}{dy^2}+E_1E_2\frac{d^2E_3}{dz^2}+k^2E_1E_2E_3=0$$

$$\Rightarrow \underbrace{\frac{1}{E_1}\frac{d^2E_1}{dx^2}}_{\text{Depends only on } x}+\underbrace{\frac{1}{E_2}\frac{d^2E_2}{dy^2}}_{\text{Depends only on } y}+\underbrace{\frac{1}{E_3}\frac{d^2E_3}{dz^2}}_{\text{Depends only on } z}+\underbrace{k^2}_{\text{constant.}}=0. \qquad (2)$$

This is possible for all x, y, z only when each term is constant.

$$\frac{1}{E_1}\frac{d^2E_1}{dx^2}=-k_x^2 \qquad (3)$$

$$\frac{1}{E_2}\frac{d^2E_2}{dy^2}=-k_y^2 \qquad (4)$$

$$\frac{1}{E_3}\frac{d^2E_3}{dz^2}=-k_z^2 \qquad (5)$$

$$k_x^2+k_y^2+k_z^2=\left(\frac{\omega}{c}\right)^2=k^2$$

Represents a circle in k_x, k_y, k_z ~~pt~~ coordinate system.

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Imp: You can choose only 2 of the 3 constants k_x, k_y, k_z independently.

For example, select k_x and k_y independently. k_z is already fixed! Because, $k_z^2 = k^2 - k_x^2 - k_y^2$

$$= \left(\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right)$$

From eq. (3),

$$\frac{d^2 E_1}{dx^2} + k_x^2 E_1 = 0$$

If we had chosen $+k_x^2$ instead of $-k_x^2$, we would have solutions of the form $e^{\pm k_x x}$

Not a wave! ($e^{i\omega t} \cdot e^{\pm k_x x}$)

$$\therefore E_1(x) \sim e^{\pm i k_x x}$$

wave propagating along $\pm x$

Similarly, $E_2(y) \sim e^{\pm i k_y y}$, $E_3(z) \sim e^{\pm i k_z z}$

$$\therefore E_x(x,y,z) = A e^{\pm i(k_x x + k_y y + k_z z)}$$

$$= A e^{\pm i \vec{k} \cdot \vec{r}}$$

Propagates along $\pm \vec{k}$

$[\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}]$

Known as wave-vector

Direction of $\vec{k}$ [in other words, the choice of any two of (k_x, k_y, k_z)] determines the propagation direction.

① Also, $|\vec{k}| = \frac{2\pi}{\lambda} = \left(\frac{\omega}{c}\right) \Rightarrow \omega = c|k| = \frac{|k|}{\sqrt{\mu\varepsilon}}$

② $\vec{k} = |\vec{k}|\hat{k}$, $\hat{k}$ is the unit vector along $\vec{k}$

— represents the propagation direction

(Each $\vec{k}$ represents a plane wave propagating along a particular direction.)

In a similar manner,

$$E_y(x,y,z) = B e^{\pm i \vec{k} \cdot \vec{r}}$$

$$E_z(x,y,z) = C e^{\pm i \vec{k} \cdot \vec{r}}$$

$$\therefore \vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$
$$= e^{\pm i\vec{k}\cdot\vec{r}}(\hat{x}A + \hat{y}B + \hat{z}C)$$
$$= \vec{E}_0 e^{\pm i\vec{k}\cdot\vec{r}}$$
$$\therefore \boxed{\vec{\mathcal{E}}(r,t) = \vec{E}_0\, e^{\pm i(\vec{k}\cdot\vec{r} + \omega t)}}$$

Similar treatment on $\vec{B}$ can be performed to show that $\vec{\mathcal{B}}(r,t) = \vec{B}_0\, e^{\pm i(\vec{k}\cdot\vec{r} + \omega t)}$

Maxwell's equations: another form

propagating wave:

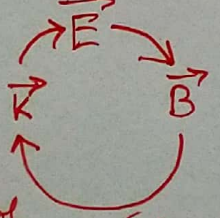
Consider E_x component of a ~~forward~~ propagating wave:

$$E_x = E_{0x} e^{-i(k_x x + k_y y + k_z z)}$$

$$\frac{\partial E_x}{\partial x} = E_{0x}\cdot(-ik_x)\, e^{-i(k_x x + k_y y + k_z z)}$$
$$= -ik_x E_x$$

$\therefore$ The operator $\frac{\partial}{\partial x}$ is equivalent to multiplication by $-ik_x$ for plane waves:

$$\frac{\partial}{\partial x} \equiv -ik_x$$
$$\frac{\partial}{\partial y} \equiv -ik_y$$
$$\frac{\partial}{\partial z} \equiv -ik_z$$



Direction of propagation $\equiv (\vec{E}\times\vec{B})$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow -i\vec{k}\cdot\vec{E} = 0 \Rightarrow \boxed{\vec{k}\cdot\vec{E} = 0}$

Similarly, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{k}\cdot\vec{B} = 0}$

Imp: $\vec{E}$ and $\vec{B}$ are both perpendicular to the direction of propagation for a plane wave.

$\vec{\nabla}\times\vec{E} = -i\omega\vec{B}$

$\Rightarrow -i\vec{k}\times\vec{E} = -i\omega\vec{B}$

$\Rightarrow \boxed{\vec{k}\times\vec{E} = \vec{B}\omega}$

$\Downarrow$

$$\frac{|\vec{E}|}{|\vec{B}|} = \frac{\omega}{|k|} = c \Rightarrow \frac{E}{H} = \mu\frac{1}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$\vec{\nabla}\times\vec{B} = i\omega\mu\varepsilon\vec{E}$

$\Rightarrow -i\vec{k}\times\vec{B} = i\omega\mu\varepsilon\vec{E}$

$\Rightarrow \boxed{\vec{k}\times\vec{B} = -\omega\mu\varepsilon\vec{E}}$

Let's have a closer look at the quantity $\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$

E has a unit : ~~A~~ V/m

H has a unit : A/m.

So, $\frac{E}{H}$ has a unit of $\frac{V}{A} \equiv$ ohm!!

So, $\frac{E}{H}$ is known as impedance of a medium.

This is given by $\sqrt{\frac{\mu}{\varepsilon}}$ (in Ω).

For free space, impedance is $= \sqrt{\frac{\mu_0}{\varepsilon_0}} \sim 377\,\Omega$