Lecture 17 (ECE230)

Recall: $\mathcal{F}\left\{\frac{dx}{dt}\right\} = i\omega X(\omega)$

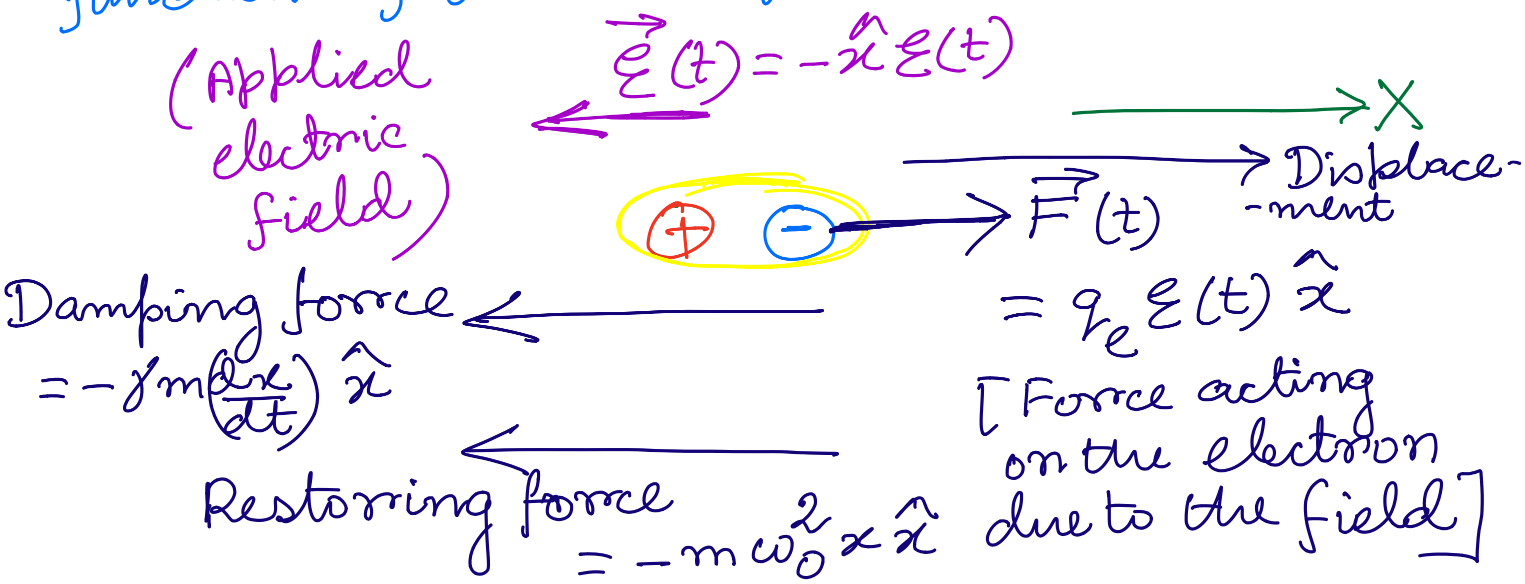
Dielectric constant of dispersive materials:

Materials with frequency dependent dielectric constants are known as dispersive materials. In this section we obtain the frequency dependence of dielectric constant.

Some useful quantities/relations:

- Polarization, $\vec{P}(\omega)$: dipole moment per unit volume.
- $\vec{P}(\omega) = \varepsilon_0 \chi_e(\omega)\, \vec{E}(\omega)$
  where, $\chi_e(\omega)$ is known as electrical susceptibility.
- Relative permittivity $= \varepsilon_r(\omega) = 1 + \chi_e(\omega)$

Now, we aim to show that $\vec{P}$ is indeed a function of frequency!



$$q_e \mathcal{E}(t) - \gamma m \frac{dx}{dt} - m\omega_0^2 x = m\left(\frac{d^2x}{dt^2}\right)$$

Taking Fourier transform,

$$q_e E(\omega) - i\omega\gamma m X(\omega) - m\omega_0^2 X(\omega) = -\omega^2 X(\omega)$$

$$\Rightarrow \frac{q_e E(\omega)}{m} = X(\omega)\left[\omega_0^2 - \omega^2 + i\omega\gamma\right]$$

$$\Rightarrow X(\omega) = \frac{q_e E(\omega)}{m} \frac{1}{\left[\omega_0^2 - \omega^2 + i\omega\gamma\right]} \quad \text{——— (1)}$$

$\therefore$ Dipole moment of a molecule with single electron $(p) = q_e X(\omega)$

$$= \frac{q_e^2 E(\omega)}{m} \frac{1}{\left[\omega_0^2 - \omega^2 + i\omega\gamma\right]} \quad \text{——— (2)}$$

suppose, a molecule has Z electrons. Out of these Z, f_j electrons have natural frequency ω_j and damping constant γ_j.

Also, $\sum_j f_j = Z$

Dipole moment of such a molecule would be a slight modification of eqn (2):

$$p(\omega) = \frac{q_e^2 E(\omega)}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

Now, suppose there are N molecules per unit volume.

So, the polarization P i.e. dipole moment per unit volume will be:

$$P(\omega) = N p(\omega) = \frac{N q_e^2 E(\omega)}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)} \qquad (3)$$

Comparing the above with $P(\omega) = \varepsilon_0 \chi_e(\omega) E(\omega)$,

$$\varepsilon_0 \chi_e(\omega) = \frac{N q_e^2 E(\omega)}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

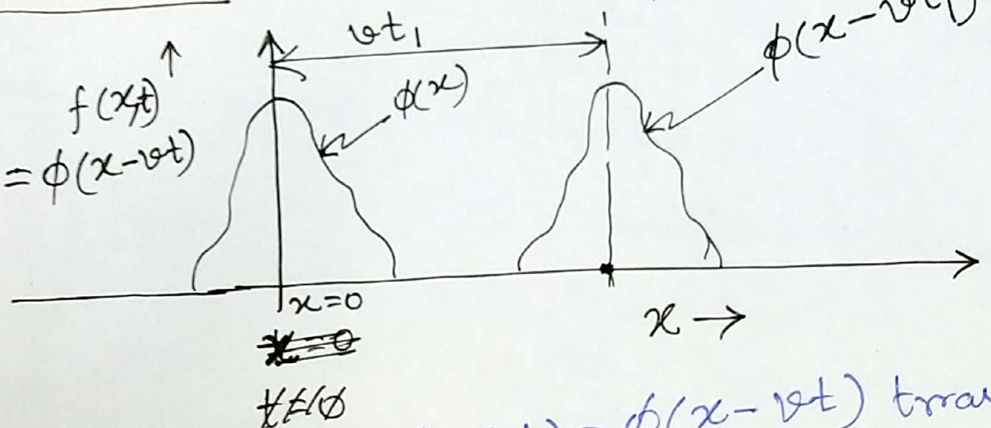
$$\Rightarrow \chi_e(\omega) = \frac{N q_e^2 E(\omega)}{m\varepsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

So, relative permittivity can be written as:

$$\varepsilon_r(\omega) = 1 + \frac{N q_e^2 E(\omega)}{m\varepsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

[redacted]

Wave equation:



The function $f(x,t) = \phi(x-vt)$ travels a distance vt_1 in time t_1. So, v represents the wave velocity.

Any function of the form $\phi(x-vt)$ is a wave travelling along $+x$ direction whereas, $\phi(x+vt)$ would represent a $-x$ travelling wave.

With this definition, let us derive a differential equation for the wave (we'll use $\phi(x-vt)$ but if we consider $\phi(x+vt)$, we can ~~do~~ obtain the same equation)

Let's define $\phi'(x-vt) = \dfrac{\partial \phi}{\partial(x-vt)}$.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial(x-vt)} \cdot \frac{\partial(x-vt)}{\partial x} = \phi'(x-vt).$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi'}{\partial(x-vt)} \cdot \frac{\partial(x-vt)}{\partial x}$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = \phi''(x-vt) \quad \text{———} (3)$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial(x-vt)} \cdot \frac{\partial(x-vt)}{\partial t} = -v.\phi'(x-vt)$$

$$\therefore \frac{\partial^2\phi}{\partial t^2} = -v.\frac{\partial\phi'(x-vt)}{\partial t} = -v.\frac{\partial\phi'}{\partial(x-vt)} \cdot \frac{\partial(x-vt)}{\partial t}.$$

$$= v^2\phi''(x-vt).$$

$$\Rightarrow \phi''(x-vt) = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} \quad \text{———} (4).$$

Substituting this into eq.(3),

$$\boxed{\frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}} \quad \text{———} (5)$$

One-dimensional scalar wave equation.

This can be generalized to 3D:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}} \quad \text{———} (6)$$

Let's now consider source-free Maxwell's equations:
($\rho=0$, $\vec{J}=0$)

$$\vec{\nabla}.\vec{E} = 0 \quad \text{———} (7)$$

$$\vec{\nabla}.\vec{B} = 0 \quad \text{———} (8)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{———} (9)$$

$$\vec{\nabla}\times\vec{B} = \mu\varepsilon\frac{\partial\vec{E}}{\partial t} \quad \text{———} (10)$$

Recall:
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E}$$

Take ($\vec{\nabla}\times$) on both sides of eq.(9),

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{\nabla}(\cancelto{0}{\vec{\nabla}.\vec{E}}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{E} = \frac{1}{c^2}\left(\frac{\partial^2 \vec{E}}{\partial t^2}\right), \quad \text{where } c = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{——— (7)}$$

[compare (6) and (7): velocity of the wave would be $c = \frac{1}{\sqrt{\mu\varepsilon}}$]

Similarly, one can show,

$$\nabla^2 \vec{B} = \frac{1}{c^2}\left(\frac{\partial^2 \vec{B}}{\partial t^2}\right) \qquad \text{——— (8)}$$

[redacted]

In fact, eqn. (7) actually represents 3 equations;

$$\cancel{\frac{\partial^2 \vec{E}}{\partial x^2} + \partial^2 \vec{E}}$$

$$\nabla^2(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) = \frac{1}{c^2}\left[\hat{x}\frac{\partial^2 E_x}{\partial t^2} + \hat{y}\frac{\partial^2 E_y}{\partial t^2} + \hat{z}\frac{\partial^2 E_z}{\partial t^2}\right]$$

$$\therefore \left.\begin{aligned} \nabla^2 E_x &= \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} \\ \nabla^2 E_y &= \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \\ \nabla^2 E_z &= \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \end{aligned}\right\}$$

Separation of space and time in wave-equation (Time independent form / frequency domain representation):

$$\nabla^2 \vec{\mathcal{E}}(r,t) = \frac{1}{c^2}\frac{\partial^2 \vec{\mathcal{E}}(r,t)}{\partial t^2} \qquad \text{——— (9)}$$

Method 1: Take a Fourier transform

$$\nabla^2 \vec{E}(r,\omega) = \frac{1}{c^2}(i^2\omega^2)\vec{E}(r,\omega)$$

$$= -\frac{\omega^2}{c^2}\vec{E}(r,\omega)$$

$$\Rightarrow \nabla^2 \vec{E}(r,\omega) + \frac{\omega^2}{c^2}\vec{E}(r,\omega) = 0 \qquad \boxed{\vec{\mathcal{E}}(r,t) = \int_{-\infty}^{\infty} \vec{E}(r,\omega)e^{i\omega t}\,d\omega}$$

For a fixed ω,

$$\boxed{\nabla^2 \vec{E}(r) + \frac{\omega^2}{c^2}\vec{E}(r) = 0}$$

← Helmholtz equation

Method 2: Separation of variables.

$$\vec{\mathcal{E}}(r,t) = \vec{E}(r)\psi(t)$$

Eqn. (9) becomes:

$$\psi(t)\nabla^2\vec{E}(r) = \frac{\vec{E}(r)}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\Rightarrow c^2\frac{\nabla^2\vec{E}(r)}{\vec{E}(r)} = \frac{1}{\psi(t)}\cdot\frac{\partial^2\psi}{\partial t^2} = -\omega^2$$

[ω is real]

$$\therefore \nabla^2\vec{E}(r) = -\frac{\omega^2}{c^2}\vec{E}(r)$$

$$\Rightarrow \boxed{\nabla^2\vec{E}(r) + \frac{\omega^2}{c^2}\vec{E}(r) = 0}$$

$$\frac{1}{\psi(t)}\frac{d^2\psi}{dt^2} = -\omega^2$$

$$\Rightarrow \frac{d^2\psi(t)}{dt^2} + \omega^2\psi(t) = 0.$$

$$\therefore \boxed{\psi(t) = Ae^{i\omega t} + Be^{-i\omega t}}$$

$$\therefore \boxed{\vec{\mathcal{E}}(r,t) = \vec{E}(r)\left[Ae^{i\omega t} + Be^{-i\omega t}\right]}$$