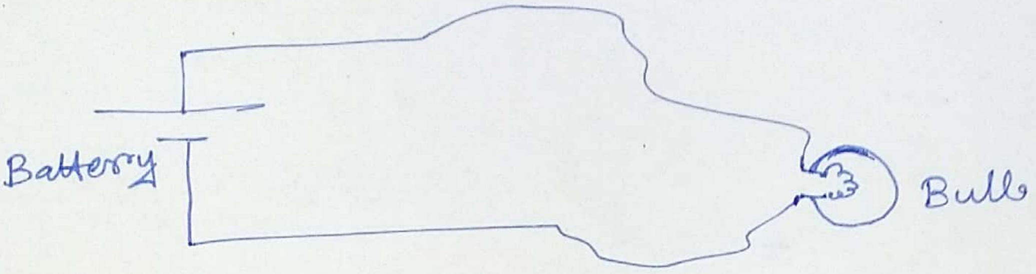
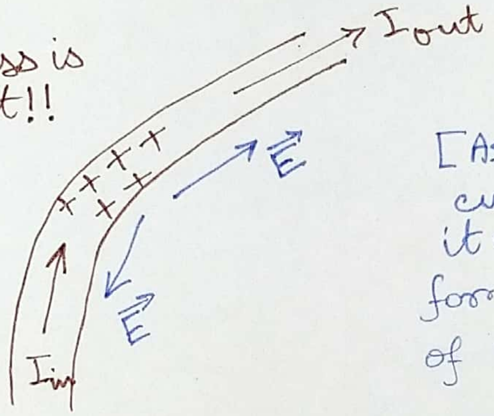
Electromotive force



Battery (or the source) drives the current in nearby parts to the battery. Think of a circuit that is very very long and a light-bulb is connected at the end. What drives the current near the bulb? Concentrate on a particular part (let's say a bend) of the circuit:

Note: this process is very very fast!!



Assume that this bend has an accumulation of positive charge.

[As the battery drives current in its neighbourhood, it creates such accumulation for a very very short period of time].

The electric field (more ~~pre~~ precisely, electrostatic field) created due to such accumulation will impede the incoming current (the reason of the imbalance) and help the I_{out}, so that the charge imbalance gets evened out. In effect, the electrostatic field drives the current in the subsequent parts of the circuit.

Thus, there are two forces involved:

$$\vec{f} = \vec{f_s} + \vec{E} \qquad (1)$$

$\vec{E}$: Drives current through the rest of the circuit.

$\vec{f_s}$: Effect is confined in the neighbourhood of the source

$\vec{f_s}$ is known to be produced from various mechanisms: chemical reactions in battery, piezo-electric crystals subjected to mechanical pressure, motion of a conductor in a $\vec{B}$ or a changing $\vec{B}$....

Take the line-integral of eq (1),

$$\oint \vec{f}\cdot\vec{dl} = \oint \vec{f_s}\cdot\vec{dl} + \underbrace{\oint \vec{E}\cdot\vec{dl}}_{\to 0}$$ (electrostatic field is conservative)

$$\Rightarrow \oint \vec{f_s}\cdot\vec{dl} = \oint \vec{f}\cdot\vec{dl} = \mathcal{E}$$

[$\mathcal{E}$ is known as e.m.f.]

$$\boxed{\mathcal{E} = \oint \vec{f_s}\cdot\vec{dl}}$$

For an ideal emf source (without any resistance) supplying current to a perfectly conducting circuit (i.e. σ = conductivity $\to \infty$),

net force $\vec{f}$ to drive the charge $= 0$.

$\therefore$ from eq (1),

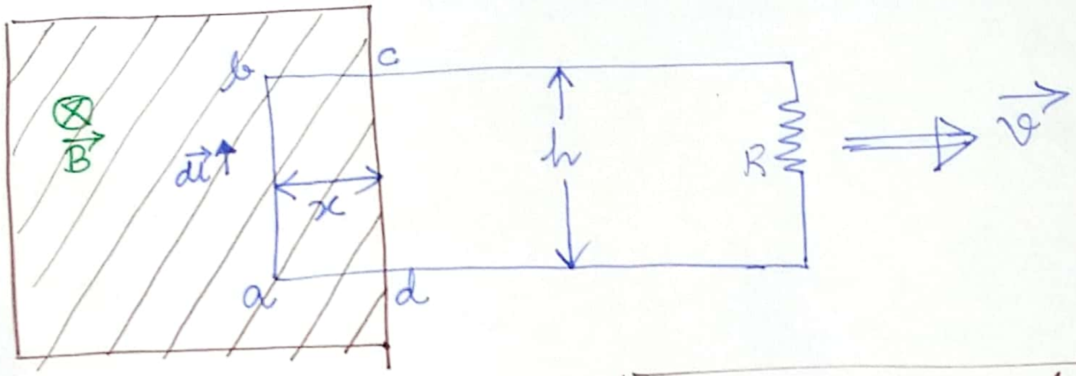
$$\vec{f_s} + \vec{E} = 0$$

$$\Rightarrow \vec{f_s} = -\vec{E}$$

$$\Rightarrow \int_A^B \vec{f_s}\cdot\vec{dl} = -\int_A^B \vec{E}\cdot\vec{dl} = (V_B - V_A)$$

$$\Rightarrow \boxed{\mathcal{E} = V_{BA}}$$

Motional e.m.f:



$$\mathcal{E} = \oint \vec{f}_{mag} \cdot \vec{dl}$$

$\boxed{\vec{dl} \text{ points along } ab}$ (for the non-zero part of the line-integral)

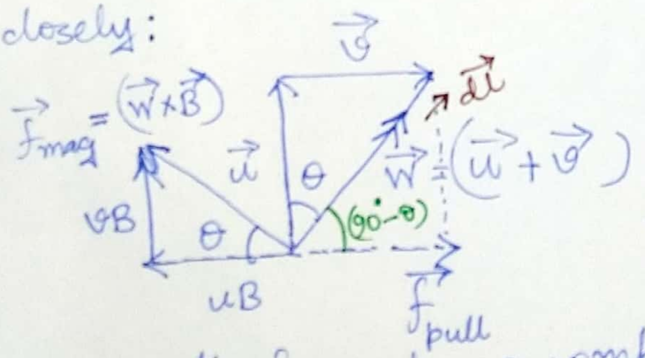
$$= (vBh)$$

Does something do not add up here? You have learnt earlier that Lorentz force (just the magnetic part) wouldn't do any work! $(\vec{v} \times \vec{B})$ is perpendicular to $\vec{v}$. So, it can move the charge but cannot do any work.

Who is then supplying the energy needed to heat up the resistor?

— The person who pulls the circuit!

Look closely:



Resultant velocity of a charge in the segment ab is $\vec{W}$.

$$\vec{f}_{mag} = (\vec{W} \times \vec{B})$$

The magnetic force has a component uB directed towards left. To counteract this, we need to pull the circuit towards right with a force $\vec{f}_{pull}$ $(|\vec{f}_{pull}| = uB)$.

$$\therefore \text{Work done} = \int \vec{f}_{pull} \cdot \vec{dl} = uB\left(\frac{h}{\cos\theta}\right) \cdot \sin\theta = vBh = \mathcal{E}$$

Rate at which flux decreases:

$$\frac{d\phi}{dt} = -Bh.\frac{dx}{dt} = -Blv$$

$$\Rightarrow \mathcal{E} = -\left(\frac{d\phi}{dt}\right)$$

$$\Rightarrow \oint \vec{E}.\vec{dl} = -\frac{d\phi}{dt}$$ [$\vec{E}$ = induced electric field due to changing flux]

$$= -\int \frac{\partial(\vec{B}.\vec{dS})}{\partial t}$$

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{E}).\vec{dS} = -\int_S \frac{\partial \vec{B}}{\partial t}.\vec{dS}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ — Faraday's law differential form.

Inconsistency in Ampere's law:

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} = 0 \quad \text{——— (1)}$$

In contrary, equation of continuity says:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad \text{——— (2)}$$

Thus (1) and (2) are consistent only for D.C. i.e. ρ = constant.

But continuity equation comes from law of conservation of charge and must always be true. So, we need to modify Ampere's law for electrodynamic (i.e. time varying) cases. This is exactly what Maxwell did.

From eqn. (2), we have:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial(\vec{\nabla}\cdot\vec{D})}{\partial t} \qquad \left[\text{Recall Gauss's law: } \vec{\nabla}\cdot\vec{D} = \rho\right]$$

$$\Rightarrow \vec{\nabla}\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) = 0$$

The inconsistency goes away when Ampere's law is modified as;

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)}$$

Maxwell-Ampere law.

Physically, this means, $\vec{B}$ can be produced by two methods:

① A conduction current density ($\vec{J}$)

② A displacement current density ($\frac{\partial \vec{D}}{\partial t}$) or a time-varying $\vec{E}$-field. [We don't need a current-carrying conductor to produce $\vec{B}$]

This gives rise to a nice symmetry between $\vec{E}$ and $\vec{B}$-fields! According to Faraday's law, a time-varying $\vec{B}$ produces an $\vec{E}$-field. On the other hand, Maxwell-Ampere's law says that a time-varying $\vec{E}$ produces $\vec{B}$!

So far, we have derived all 4 Maxwell's equations that governs electromagnetism

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$$ —— Gauss's law for $\vec{E}$

$$\vec{\nabla}\cdot\vec{B} = 0$$ —— Gauss's law for $\vec{B}$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ —— Faraday's law of electromagnetic induction.

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$ —— Maxwell-Ampere's law.

Potential in Electrodynamics:

In Faraday's law, we have seen:

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d\phi_B}{dt} \neq 0 \text{ in case of electrodynamics.}$$

So, for time-varrying case, $\vec{E}$ is not conservative and $\vec{E} \neq -\vec{\nabla}\phi$.

However, we can still write the fields in terms of potentials:

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \quad [\vec{A} \text{ is magnetic vector potential}] \qquad \text{——— (3)}$$

Also, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$= -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$= -\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}} \qquad \text{——— (4)}$$

Thus, $\vec{E}$ now depends on both a scalar potential ϕ and vector potential $\vec{A}$.

$$\vec{E} = -\vec{\nabla}\phi \quad \text{only when } \frac{\partial}{\partial t} \equiv 0$$

i.e. the static case.