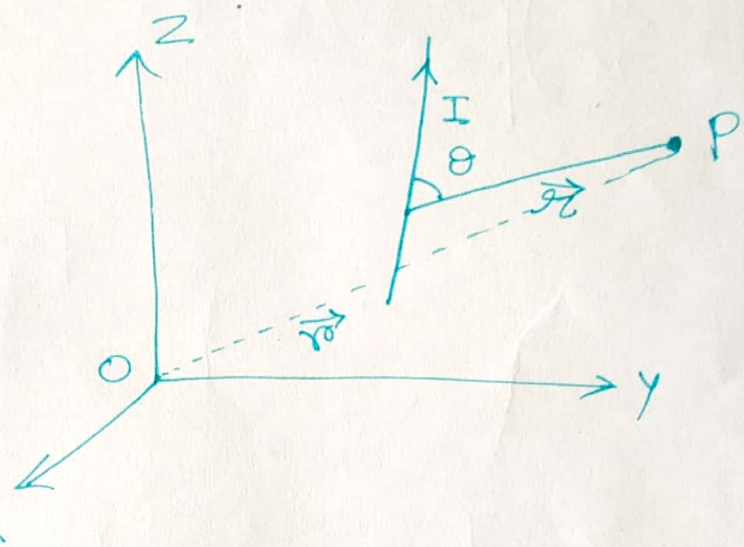
Steady current:

$$\vec{\nabla}\cdot\vec{J} = -\left(\frac{\partial \rho}{\partial t}\right) = 0$$

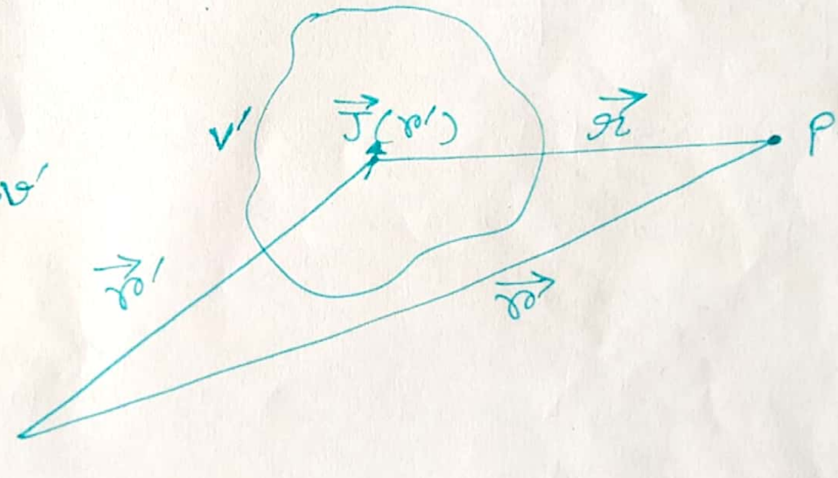
Imp: a point charge can never give rise to a steady current.

Biot-Savart law:

$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$



$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dv' \qquad (5)$$



$\boxed{\vec{\nabla}\cdot\vec{B}}$

Recall, $\vec{B}$ is dependent on (x, y, z)

$\vec{J}$ " " " (x', y', z')

$\vec{\mathscr{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$

$dv' = dx'\,dy'\,dz'$

And the vector identity: $\vec{\nabla}\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\vec{\nabla}\times\vec{P}) - \vec{P}\cdot(\vec{\nabla}\times\vec{Q})$

From eq (5),

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int_{V'} \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{r}}{r^2}\right) dv'$$

$$= \frac{\mu_0}{4\pi}\int\left[\left(\frac{\hat{r}}{r^2}\right)\cdot\underbrace{\cancel{(\vec{\nabla}\times\vec{J})}}_{0} - \vec{J}\cdot\underbrace{\cancel{\left(\vec{\nabla}\times\frac{\hat{r}}{r^2}\right)}}_{0}\right] dv'$$

0 (because, $\vec{J}$ depends on x', y', z' and not on x, y, z)

[well, you are very familiar with this function: $\frac{\hat{r}}{r^2} = \vec{\nabla}\left(\frac{1}{r}\right)$ and $\vec{\nabla}\times\left(\vec{\nabla}(\text{of some scalar})\right) = 0$]

$\therefore \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ — Gauss's law for $\vec{B}$ (differential form)

Always! In magnetostatics.... electrodynamics....

Physically, we already got the same result based on the fact that magnetic monopole doesn't exist!

So, now we know that magnetic field is solenoidal

$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

So, it is ~~also~~ always possible to write: $\vec{B} = (\vec{\nabla}\times\vec{A})$

$\vec{A}$ is known as magnetic vector potential.

What happens if I add another gradient of a scalar to $\vec{A}$? i.e. $\vec{A}_{new} = (\vec{A} + \vec{\nabla}\lambda)$.

$$\vec{\nabla}\times\vec{A}_{new} = \vec{\nabla}\times(\vec{A}+\vec{\nabla}\lambda) = \vec{\nabla}\times\vec{A} + \underbrace{\cancel{\vec{\nabla}\times(\vec{\nabla}\lambda)}}_{0}$$

$$= \vec{B} \text{ !!}$$

So, we have freedom towards choosing any $\vec{A}_{new}$ in place of $\vec{A}$, just by adding $\vec{\nabla}\lambda$ and this doesn't change $\vec{B}$.

To make life simple, choose $\vec{A}_{new}$ in such a way that

$$\vec{\nabla}\cdot\vec{A}_{new} = 0 \Rightarrow \vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\lambda) = 0$$

This would tell you what should be our λ.

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = -\nabla^2\lambda$$

$$\boxed{\nabla^2\lambda = -(\vec{\nabla}\cdot\vec{A})}$$

Solve this "poisson's eq.", get λ, add $\vec{\nabla}\lambda$ to $\vec{A}$.
This should make $\vec{\nabla}\cdot\vec{A}_{new} = 0$.

Evaluation of magnetic vector potential:

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{v'} \vec{J}(r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, dv'$$

$$= \frac{\mu_0}{4\pi}\int \left\{-\vec{J}(r')\times\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right\} dv' \qquad \text{—— (6)}$$

Use the vector identity: $\vec{\nabla}\times(f\vec{p}) = f(\vec{\nabla}\times\vec{p}) - \vec{p}\times(\vec{\nabla}f)$

$$\Rightarrow -\vec{p}\times(\vec{\nabla}f) = \vec{\nabla}\times(f\vec{p}) - f(\vec{\nabla}\times\vec{p})$$

$$\therefore -\vec{J}(r')\times\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}\times\left(\frac{\vec{J}(r')}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\left(\cancel{\vec{\nabla}\times\vec{J}(r')}\right)$$

0

[$\vec{J}$ is a function of x', y', z' and independent of x, y, z]

$\therefore$ From eq. (6),

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{v'}\left(\vec{\nabla}\times\frac{\vec{J}(r')}{\mathscr{r}}\right)dv'$$

$$= \vec{\nabla}\times\left[\frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{J}(r')\,dv'}{\mathscr{r}}\right]$$

$$\therefore \boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{J}(r')}{\mathscr{r}}\,dv'}$$

$\vec{\nabla} \times \vec{B}$:

① $\vec{\nabla} \times (\vec{P} \times \vec{Q}) = (\vec{Q} \cdot \vec{\nabla})P - (\vec{P} \cdot \vec{\nabla})Q + \vec{P}(\vec{\nabla} \cdot \vec{Q}) - Q(\vec{\nabla} \cdot \vec{P})$

② $J(x', y', z')$ is independent of (x, y, z).

$$\therefore \vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\vec{J}(r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) dv' \quad \text{——— (7)}$$

$$\vec{\nabla} \times \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \vec{J} \left(\vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (\vec{J} \cdot \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

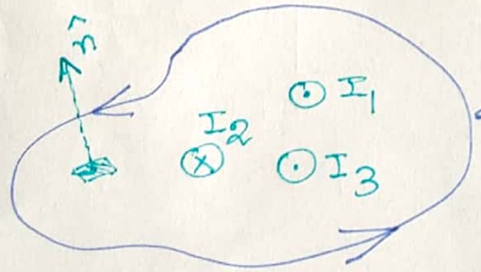
$$= \vec{J}\, 4\pi \delta^3(\mathscr{r}) - (\vec{J} \cdot \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{——— (8)}$$

contribution of the second term into the integral of eq(7) can be shown to be zero (we'll do that in a bit).

~~$\vec{\nabla} \times \vec{B}(r)$~~ $\therefore \vec{\nabla} \times \vec{B}(r) = \dfrac{\mu_0}{4\pi} \displaystyle\int \vec{J}(r')\, 4\pi \delta^3(\vec{r} - \vec{r}')\, dv'$

$$= \mu_0 \vec{J}(r)$$

$$\boxed{(\vec{\nabla} \times \vec{B}) = \mu_0 \vec{J}}$$ ——— Ampere's law in differential form



← Amperian loop.

(⊙ current to be treated as positive
⊗ current to be treated as negative)

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \int \vec{J} \cdot d\vec{S}$$

$$\Rightarrow \boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$ —— Ampere's law in integral form.

Let's show that contribution of second term of eq.(8) into eq.(7) is zero indeed.

$$-(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\cdot\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

consider just the x-component:

$$(\vec{J}\cdot\vec{\nabla}')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \vec{\nabla}'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right] - \cancel{\left(\frac{x-x'}{\mathscr{r}^3}\right)(\vec{\nabla}'\cdot\vec{J})}$$

0 ($\vec{\nabla}'\cdot\vec{J}=0$ steady current)

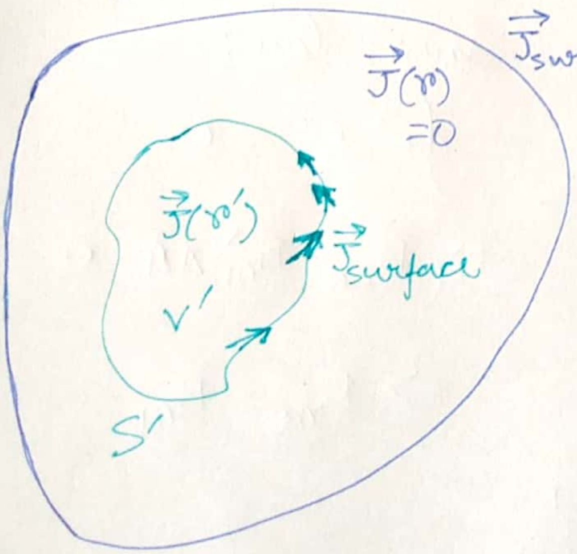
[Using the vector identity:

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

$$\Rightarrow \vec{A}\cdot(\vec{\nabla}f) = \vec{\nabla}\cdot(f\vec{A}) - f(\vec{\nabla}\cdot\vec{A})\ ]$$

So, contribution to eq.(7) from the x-component of second term of eq(8) can be written as:

$$\int_{V'}\vec{\nabla}'\cdot\left(\frac{x-x}{\mathscr{r}^3}\vec{J}\right)dv' = \oint_{S'}\left(\frac{x-x}{\mathscr{r}^3}\right)\vec{J}\cdot d\vec{s}'$$



$$\Rightarrow \int_{\text{Any volume} > V'}\vec{\nabla}'\cdot\left(\frac{x-x}{\mathscr{r}^3}\vec{J}\right)dv'$$

$$= \oint\left(\frac{x-x}{\mathscr{r}^3}\right)\vec{J}_{surface}\cdot d\vec{s}'$$

$$= 0$$