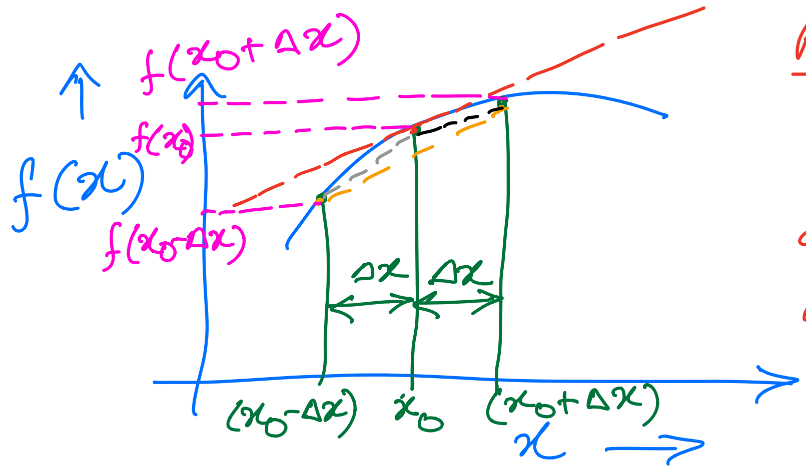
Lecture 17 (ECE545)

Finite difference approximation

Difference form of first order derivative:



Assumption

The values of the function are known at discrete points $x = x_0, x_0 \pm \Delta x, x_0 \pm 2\Delta x, \ldots$

Fig. 1

Consider the function $f(x)$, shown in fig. 1. From calculus, you know that the derivative of the function at $x = x_0$ is given by:

$$f'(x_0) = \left.\frac{df}{\partial x}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \qquad \text{——— (1)}$$

This is called forward difference.

Alternatively, we can define backward difference as:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x} \qquad \text{——— (2)}$$

and, central difference as:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} \qquad (3)$$

In fig. 1, the forward difference corresponds to the slope of the dotted black line, the backward difference corresponds to the slope of the dotted grey line and the central difference corresponds to the slope of the dotted orange line.

Now a quick look at the slopes of these lines show that the dotted orange line mimics $f'(x_0)$, the slope of the dotted red line more closely than the others. Can this be put on a solid ground by mathem-atical proof? Let's try it out.

$$f(x_0 \pm \Delta x) = f(x_0) \pm \Delta x\, f'(x_0) + \frac{(\Delta x)^2}{2} f''(x_0) \pm \frac{(\Delta x)^3}{3!} f'''(x_0) + \cdots \qquad (4)$$

$$\therefore \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = f'(x_0) + \left(\frac{\Delta x}{2}\right) f''(x_0) + \frac{(\Delta x)^2}{3!} f'''(x_0) + \cdots \qquad (5)$$

Also,

$$\frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}$$

$$= f'(x_0) - \left(\frac{\Delta x}{2}\right) f''(x_0) + \frac{(\Delta x)^2}{3!} f'''(x_0) - \cdots \quad \text{——— (6)}$$

Again from eqn (4), we can write:

$$f(x_0 + \Delta x) - f(x_0 - \Delta x)$$

$$= 2\Delta x\, f'(x_0) + \frac{(\Delta x)^3}{3!} f''(x_0) + \cdots$$

$$\Rightarrow \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$$

$$= f'(x_0) + \frac{(\Delta x)^2}{3!} f'''(x_0) + \cdots \quad \text{——— (7)}$$

Note that if we truncate the serieses on the R.H.S. of eqn. (5), (6) and (7) to approximate $f'(x_0)$ by forward, backwar-d and central differences, the errors in (5) and (6) are $O(\Delta x)$. In contrast, the error in (7) is $O(\Delta x^2)$.

Thus, central difference approximates the derivate of a function more accurately.

Difference form of second order derivative:

In order to obtain the difference form of second order derivative, we use eqn (4):

$$f(x_0+\Delta x)+f(x_0-\Delta x)$$

$$=2f(x_0)+(\Delta x)^2 f''(x_0)+\cdots$$

$$\Rightarrow f''(x_0)\simeq\frac{f(x_0+\Delta x)-2f(x_0)+f(x_0-\Delta x)}{(\Delta x)^2} \qquad (8)$$

Finite difference time domain (FDTD) method:

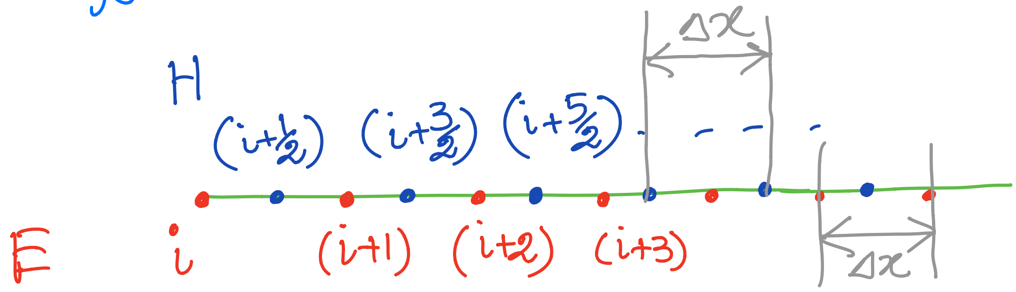
Consider TEM wave propagation along x direction. The $\vec{E}$ and $\vec{H}$ are given by $E_y\hat{y}$ and $H_z\hat{z}$, respectively.

The Maxwell's eqns. are:

$$\varepsilon_0\frac{\partial E_y}{\partial t}=-\frac{\partial H_z}{\partial x}\Rightarrow\frac{\partial E_y}{\partial t}=-\frac{1}{\varepsilon_0}\frac{\partial H_z}{\partial x} \qquad (9)$$

$$\mu_0\frac{\partial H_z}{\partial t}=-\frac{\partial E_y}{\partial x}\Rightarrow\frac{\partial H_z}{\partial t}=-\frac{1}{\mu_0}\frac{\partial E_y}{\partial x} \qquad (10)$$

We'll express these derivatives using finite differences.



$\boxed{\text{Fig: 2}}$ Spatial grids in FDTD analysis.

E-fields are evaluated at $i\Delta x, (i+1)\Delta x, (i+2)\Delta x$ ..... etc. H-fields are evaluated at $(i+\frac{1}{2})\Delta x, (i+\frac{3}{2})\Delta x, \ldots$ etc.

This is known as staggered grid.

Similarly, the discrete time instants are denoted by an index 'n'. The $\vec{E}$-field is evaluated at $n = 1, 2, 3, \ldots$ etc. and $\vec{H}$-field is evaluated at $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ etc.

With these notations, difference forms of eqn (9) and (10) becomes [Note: we'll apply central difference at $x_i = i\Delta x$ and $t_n = (n+\frac{1}{2})\Delta t$]:

$$\frac{E_y^{n+1}(i) - E_y^{n}(i)}{(\Delta t)} = -\frac{1}{\varepsilon_0}\,\frac{H_z^{(n+\frac{1}{2})}(i+\frac{1}{2}) - H_z^{(n+\frac{1}{2})}(i-\frac{1}{2})}{(\Delta x)}$$

———— (11)

and,

$$\frac{H_z^{(n+\frac{1}{2})}(i+\tfrac{1}{2}) - H_z^{(n-\frac{1}{2})}(i+\tfrac{1}{2})}{(\Delta t)} = -\frac{1}{\mu_0}\,\frac{E_y^{(n+1)}(i) - E_y^{n}(i)}{(\Delta x)} \qquad (12)$$

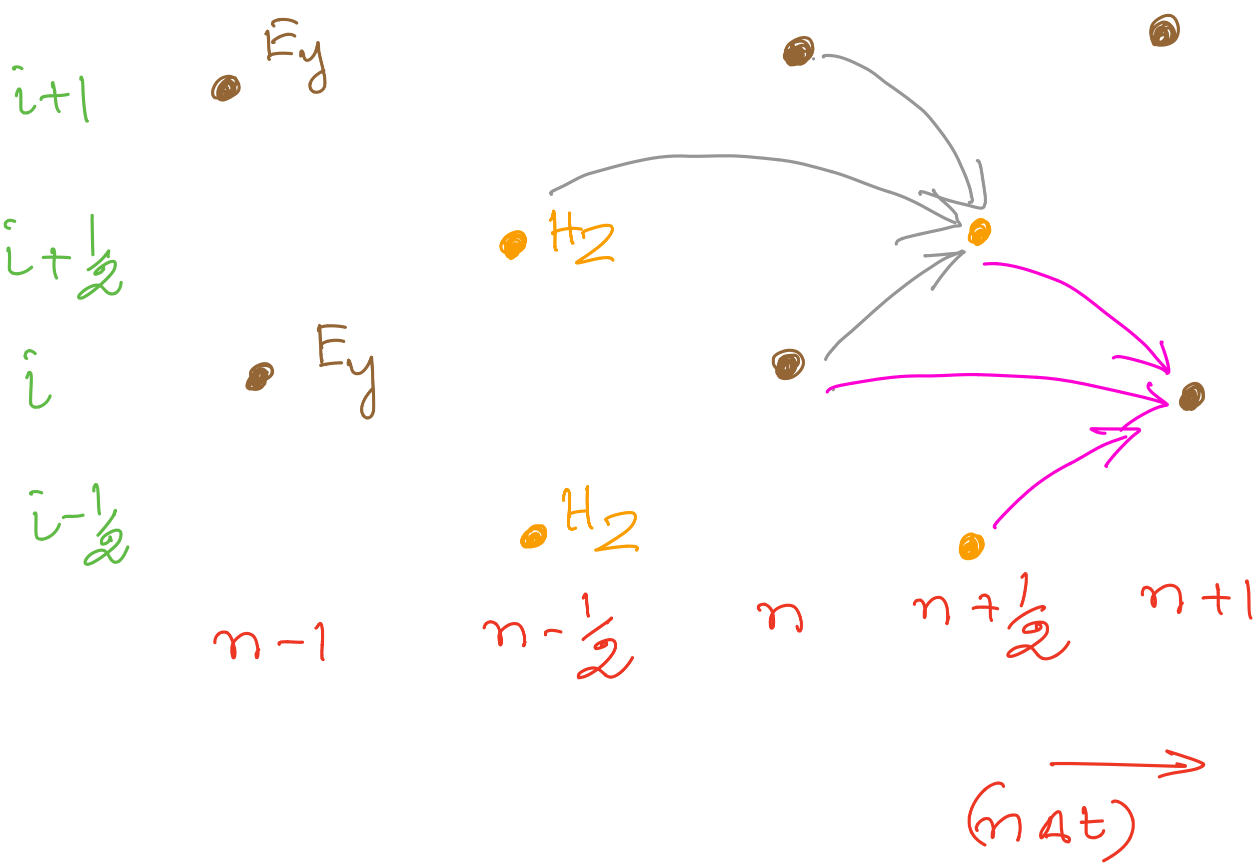
We readily obtain an algorithm to evaluate $\vec{H}$ and $\vec{E}$ by writing eqn. (12) and (11) as:

$$H_z^{(n+\frac{1}{2})}(i+\tfrac{1}{2}) = H_z^{(n-\frac{1}{2})}(i+\tfrac{1}{2}) - \frac{\Delta t}{\mu_0 \Delta x}\left[E_y^{n}(i+1) - E_y^{n}(i)\right] \qquad (13)$$

$$E_y^{n+1}(i) = E_y^{n}(i) - \frac{\Delta t}{\varepsilon_0 \Delta x}\left[H_z^{(n+\frac{1}{2})}(i+\tfrac{1}{2}) - H_z^{(n+\frac{1}{2})}(i-\tfrac{1}{2})\right] \qquad (14)$$

Note that $\vec{E}$ and $\vec{H}$ needs to be updated at alternate time instants (i.e. n & $n+\frac{1}{2}$). This cuts down the processor time by $\frac{1}{2}$.

A pictorial representation of spatial and temporal dependence of $\vec{E}$ and $\vec{H}$ can be obtained as:



Leap-frog model for 1D FDTD

Numerical dispersion:

In order to solve differential equations such as wave equation, we assume x and t to be continuous variables and obtain dispersion relation (i.e. the $\omega - k$ relation).

For example, consider the 1D wave propagation in free space:

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \quad \text{———} \quad (15)$$

Substitute the plane wave solution $E_y(x,t) = e^{j(\omega t - kx)}$. This would yield a linear dispersion:

$$\omega = \pm ck \quad \text{———} \quad (16)$$

We ask a simple question at this point: If we take the difference form of eqn.(15) and consider a discrete representation of $E_y(x,t)$ i.e. $E_y(i,n) = e^{j(n\omega\Delta t - ki\Delta x)}$, shall we get back the correct linear dispersion of free space?

The answer is: NO!

Let's prove it. The approach is going to be similar to the continuous variable case: substitute $E_y(i,n)$ into difference form of eqn.(15):

$$\frac{E_y^{(n+1)}(i) - 2E_y^n(i) + E_y^{(n-1)}(i)}{(\Delta t)^2}$$

$$= c^2 \frac{E_y^n(i+1) - 2E_y^n(i) + E_y^n(i-1)}{(\Delta x)^2} \qquad (17)$$

$$e^{j[\omega(n+1)\Delta t - k i \Delta x]}$$

$$= \left(\frac{c\Delta t}{\Delta x}\right)^2 \left[e^{j[\omega n \Delta t - k(i+1)\Delta x]} - 2e^{j[\omega n \Delta t - k i \Delta x]} + e^{j[\omega n \Delta t - k(i-1)\Delta x]} \right]$$

$$+\left[2e^{j(\omega n\Delta t - k i\Delta x)} - e^{j[\omega(n-1)\Delta t - k i \Delta x]}\right]$$

$$\Rightarrow e^{j\omega\Delta t} = \left(\frac{c\Delta t}{\Delta x}\right)^2\left[e^{-jk\Delta x} - 2 + e^{jk\Delta x}\right] + \left(2 - e^{-j\omega\Delta t}\right)$$

$$\Rightarrow \cos(\omega\Delta t) = \left(\frac{c\Delta t}{\Delta x}\right)^2\left[\cos(k\Delta x) - 1\right] + 1$$

$$\Rightarrow \sin\left(\frac{\omega\Delta t}{2}\right) = \pm\left(\frac{c\Delta t}{\Delta x}\right)\sin\left(\frac{k\Delta x}{2}\right) \qquad \text{———— (18)}$$

$\boxed{\text{Numerical dispersion}}$

↓

- Non-linear (in contrast to actual linear dispersion of free space: $\omega = \pm ck$)
- The departure from the correct dispersion is entirely due to discretization of space and time.

▨ $\left.\sin\left(\frac{\omega \Delta t}{2}\right)\right|_{max} = 1$

This is possible only if $\left(\frac{c\Delta t}{\Delta x}\right) \leq 1$

[Because $\left.\sin\left(\frac{k\Delta x}{2}\right)\right|_{max} = 1$]

This factor $\left(\frac{c\Delta t}{\Delta x}\right) = S$ is called the stability factor / Courant number and for a physically possible dispersion (i.e. a stable solution) $S \leq 1$ (In 1D)

▨ It can be shown that for 2D,

$$S = \frac{c\Delta t}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq \frac{1}{\sqrt{2}}$$

and for 3D,

$$S = \frac{c\Delta t}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} \leq \frac{1}{\sqrt{3}}$$

Each of these stability criteria for 1D, 2D and 3D is known as Courant - Friedrich-Levy (CFL) stability condition.