Lecture 7 (ECE 545)

Surface plasmon polariton (SPP)

Unlike previous lectures, we will denote the relative permittivity of materials by ε.

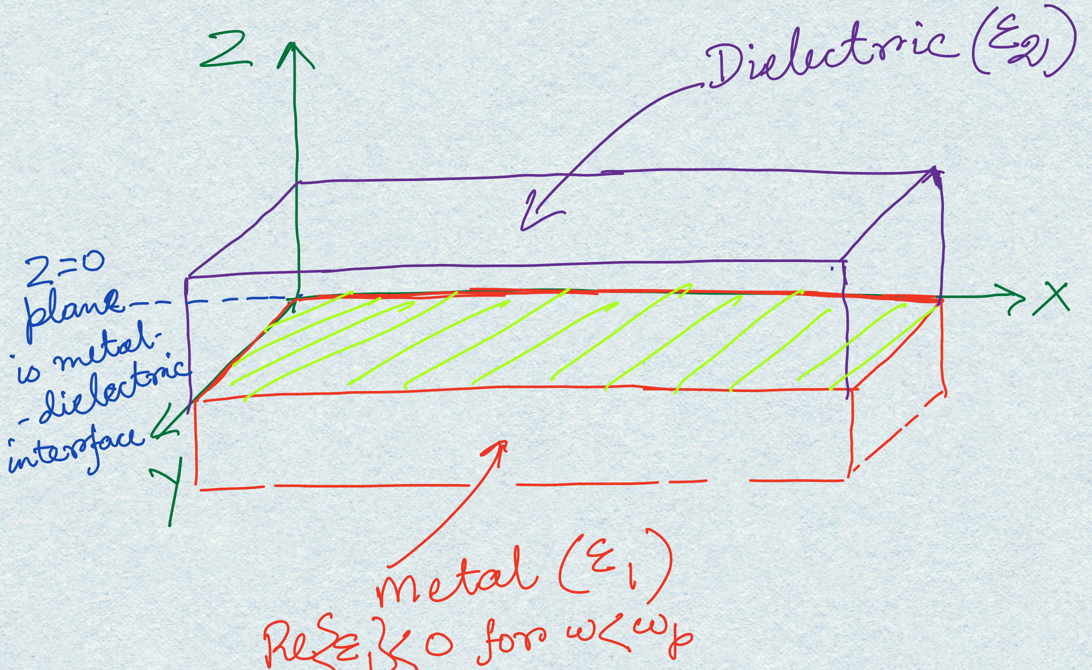
In this new notation, Helmholtz equation can be written as:

$$\nabla^2 \vec{E} + \omega^2 \mu_0 (\varepsilon_0 \varepsilon) \vec{E} = 0$$

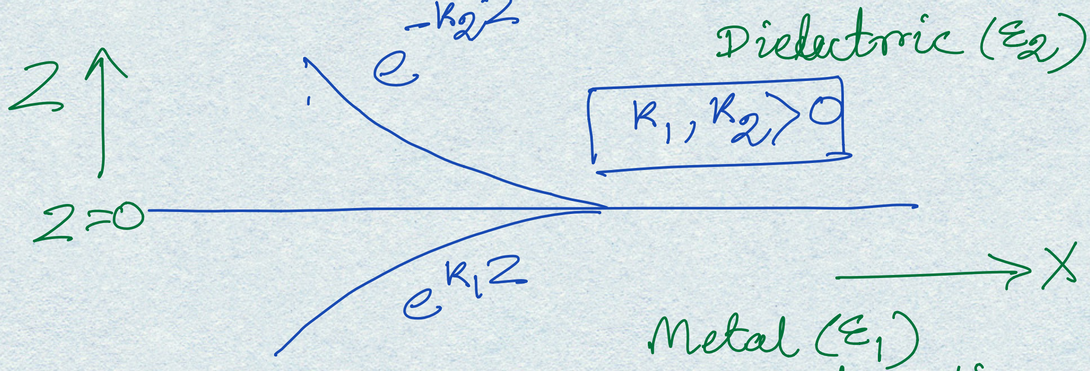
$$\Rightarrow \nabla^2 \vec{E} + k_0^2 \varepsilon \vec{E} = 0 \qquad (1)$$

$$\left[\text{where, } k_0^2 = \omega^2 \mu_0 \varepsilon_0 = \frac{\omega^2}{c^2}\right]$$

Our aim is to solve the above equation for the following geometry:



We are interested in a solution of the wave eqn (1), where the fields will decay exponentially (along Z direction) on either sides of the interface but the wave propagates along X-direction (i.e. evanescent along Z but propagating along x)



[frequency of operation is $\omega < \omega_p$]

Such waves are known as surface plasmon polaritons (SPP).

A huge advantage of such wave is: the energy is mostly concentrated around the metal-dielectric interface. Typically, this confinitement length is 10-100nm at optical frequencies i.e. much smaller than the optical wavelength (~300-650nm). Due to this deep sub-wavelength scale energy confinement, SPP is a very useful technology for nanoscale applications.

In other words, we are interested in solutions of the form: $\vec{E}(x,y,z) = \vec{E}(z)\, e^{-i\beta x}$ ——— (2)

[there's no y-dependence as the structure is assumed to be infinite along y-direction]

Putting the field of eqn(2) into eqn(1),

$$\frac{\partial^2\{\vec{E}(z)\, e^{-i\beta x}\}}{\partial x^2} + \frac{\partial^2\{\vec{E}(z)\, e^{-i\beta x}\}}{\partial z^2}$$

$$+ k_0^2 \varepsilon \vec{E}(z)\, e^{-i\beta x} = 0 \qquad \left[\text{Again, } \frac{\partial}{\partial y} \equiv 0\right]$$

$$\Rightarrow -\beta^2 \vec{E}(z)\, e^{-i\beta x} + \frac{\partial^2 \vec{E}(z)}{\partial z^2} \cdot e^{-i\beta x}$$

$$+ k_0^2 \varepsilon \vec{E}(z)\, e^{-i\beta x} = 0$$

$$\Rightarrow \frac{\partial^2 \vec{E}(z)}{\partial z^2} + (k_0^2 \varepsilon - \beta^2)\vec{E}(z) = 0$$

——— (3)

Now, let's use the curl equations of EM fields to make life simpler.

$$\vec{\nabla} \times \vec{E} = -i\omega\mu_0 \vec{H}$$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -i\omega\mu_0 \vec{H}$$

$$\Rightarrow \hat{x}\left(-\frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x}\right) = -i\omega\mu_0 \vec{H}$$

$$\left.\begin{aligned} \therefore\ i\omega\mu_0 H_x &= \frac{\partial E_y}{\partial z} \\ i\omega\mu_0 H_y &= \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) \\ i\omega\mu_0 H_z &= -\left(\frac{\partial E_y}{\partial x}\right) \end{aligned}\right\} \quad (4)$$

$\boxed{\text{Note}}$ We have assumed propagation along x-direction. So, there's a $e^{-i\beta x}$ type x-dependence.

$$\therefore \frac{\partial(\ )}{\partial x} \equiv -i\beta(\ )$$

Using this, the above set of eqns. become:

$$i\omega\mu_0 H_x = \left(\frac{\partial E_y}{\partial z}\right) \text{———} (5)$$

$$i\omega\mu_0 H_y = -i\beta E_z - \left(\frac{\partial E_x}{\partial z}\right) \text{———} (6)$$

$$i\omega\mu_0 H_z = i\beta E_y \text{———} (7)$$

$$\vec{\nabla}\times\vec{H} = i\omega\varepsilon_0\varepsilon\vec{E}$$

$$\Rightarrow i\omega\varepsilon_0\varepsilon\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \hat{x}\left(-\frac{\partial H_y}{\partial z}\right) - \hat{y}\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x}\right)$$

$$= -\hat{x}\frac{\partial H_y}{\partial z} + \hat{y}\left(i\beta H_z + \frac{\partial H_x}{\partial z}\right) + \hat{z}\left(-i\beta H_y\right)$$

$$\therefore\ i\omega\varepsilon_0\varepsilon E_x = -\left(\frac{\partial H_y}{\partial z}\right) \quad \text{———} \quad (8)$$

$$i\omega\varepsilon_0\varepsilon E_y = i\beta H_z + \frac{\partial H_x}{\partial z} \quad \text{———} \quad (9)$$

$$i\omega\varepsilon_0\varepsilon E_z = -i\beta H_y \quad \text{———} \quad (10)$$

$$\boxed{\text{TM mode: } H_x = 0}$$

$$\xrightarrow{\text{Eqn (5)}} \boxed{E_y = 0} \xrightarrow{\text{Eqn (7)}} \boxed{H_z = 0}$$

For TM mode the non-zero field components are:

E_x, E_z and H_y

$$\boxed{\text{TE mode: } E_x = 0}$$

$$\xrightarrow{\text{Eqn (8)}} \boxed{H_y = 0} \xrightarrow{\text{Eqn (10)}} \boxed{E_z = 0}$$

For TE mode the non-zero field components are:

$$E_y, H_x, H_z$$

Using the above conditions, we can simplify the equations for TE and TM modes:

comes from eqn (8)

TM
$$E_x = \frac{i}{\omega \varepsilon_0 \varepsilon}\left(\frac{\partial H_y}{\partial z}\right) \quad \text{—— (11)}$$

$$E_z = -\frac{\beta}{\omega \varepsilon_0 \varepsilon} H_y \quad \text{—— (12)}$$

comes from eqn (10)

Wave equation for TM mode:

$$\frac{\partial^2 H_y}{\partial z^2} + (k_0^2 \varepsilon - \beta^2) H_y = 0 \quad \text{—— (13)}$$

TE
$$H_x = -\frac{i}{\omega \mu_0}\left(\frac{\partial E_y}{\partial z}\right) \quad \text{—— (14)}$$

$$H_z = \frac{\beta}{\omega \mu_0}(E_y) \quad \text{—— (15)}$$

Wave equation for TE mode:

$$\frac{\partial^2 E_y}{\partial z^2} + (k_0^2 \varepsilon - \beta^2) E_y = 0 \quad \text{——— (16)}$$

Case I (TM mode)

$$z>0 \begin{cases} H_y(z) = A_2 e^{-i\beta x} \cdot e^{-k_2 z} & \text{——— (17)} \\ E_x(z) = -\dfrac{i A_2}{\omega \varepsilon_0 \varepsilon_2} k_2 e^{-i\beta x} e^{-k_2 z} & \text{——— (18)} \\ E_z(z) = -\dfrac{\beta A_2}{\omega \varepsilon_0 \varepsilon_2} e^{-i\beta x} e^{-k_2 z} & \text{——— (19)} \end{cases}$$

$$z<0 \begin{cases} H_y(z) = A_1 e^{-i\beta x} e^{k_1 z} & \text{——— (20)} \\ E_x(z) = \dfrac{i A_1}{\omega \varepsilon_0 \varepsilon_1} k_1 e^{-i\beta x} e^{k_1 z} & \text{——— (21)} \\ E_z(z) = -\dfrac{\beta}{\omega \varepsilon_0 \varepsilon_1} A_1 e^{-i\beta x} e^{k_1 z} & \text{——— (22)} \end{cases}$$

Apply boundary conditions at $z=0$:

i) H_y should be continuous across $z=0$

ii) E_x " " " " $z=0$

From eqns. (17) and (20),

$$\boxed{A_1 = A_2}$$

From eqns (18) and (21),

$$-\frac{k_2}{\varepsilon_2} = \frac{k_1}{\varepsilon_1}$$

$$\Rightarrow \boxed{\frac{k_1}{k_2} = -\frac{\varepsilon_1}{\varepsilon_2}} \qquad (23)$$

Now, for evanescent decay to exist, we must have $k_1 > 0$ and $k_2 > 0$.

$\therefore$ From (23), ε_1 and ε_2 must have opposite signs for SPP to exist!!

[This is why metal on one side of the interface and dielectric on the other side is a must for SPP]

Further, putting eqns (17) and (20) into eqn (13), we have:

$$k_2^2 = \beta^2 - k_0^2 \varepsilon_2 \qquad (24)$$

$$k_1^2 = \beta^2 - k_0^2 \varepsilon_1 \qquad (25)$$

Dividing eqn. (25) by (24),

$$k_1^2 / k_2^2 = \frac{\beta^2 - k_0^2 \varepsilon_1}{\beta^2 - k_0^2 \varepsilon_2}$$

$$\Rightarrow \varepsilon_1^2 / \varepsilon_2^2 = \frac{\beta^2 - k_0^2 \varepsilon_1}{\beta^2 - k_0^2 \varepsilon_2} \quad \text{[From eqn. (23)]}$$

$$\Rightarrow \beta^2 \varepsilon_2^2 - k_0^2 \varepsilon_1 \varepsilon_2^2 = \beta^2 \varepsilon_1^2 - k_0^2 \varepsilon_1^2 \varepsilon_2$$

$$\Rightarrow \beta^2 (\varepsilon_1^2 - \varepsilon_2^2) = k_0^2 \varepsilon_1 \varepsilon_2 (\varepsilon_1 - \varepsilon_2)$$

$$\Rightarrow \beta^2 = k_0^2 \left(\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right)$$

$$\Rightarrow \boxed{\beta = k_0 \sqrt{\frac{\varepsilon_1 \varepsilon_2}{(\varepsilon_1 + \varepsilon_2)}}} \qquad \text{——— (26)}$$

Dispersion relation of SPP

Note that, in the above eqn, $\varepsilon_1 \varepsilon_2 < 0$.
In order for β to be real, we must have

$$\boxed{(\varepsilon_1 + \varepsilon_2) < 0}$$

For metal, $\varepsilon_1 = \varepsilon_1(\omega)$

$$= \left(1 - \omega_p^2/\omega^2\right)$$

Thus, the requirement for β to be real boils down to:

$$1 - \omega_p^2/\omega^2 + \varepsilon_2 < 0$$

$$\Rightarrow \omega_p^2/\omega^2 > (1+\varepsilon_2)$$

$$\Rightarrow \boxed{\omega < \frac{\omega_p}{\sqrt{(1+\varepsilon_2)}}} \quad \text{——— (27)}$$

Frequency below which SPP can exist.

For example, at an air-metal interface,

$$\omega < \frac{\omega_p}{\sqrt{1+1}} \quad \text{i.e. } \omega < \frac{\omega_p}{\sqrt{2}} \quad \text{for SPP}$$

to exist.