Lecture 6 (ECE545)

Recall: $\mathcal{F}\left\{\dfrac{dx}{dt}\right\} = i\omega X(\omega)$

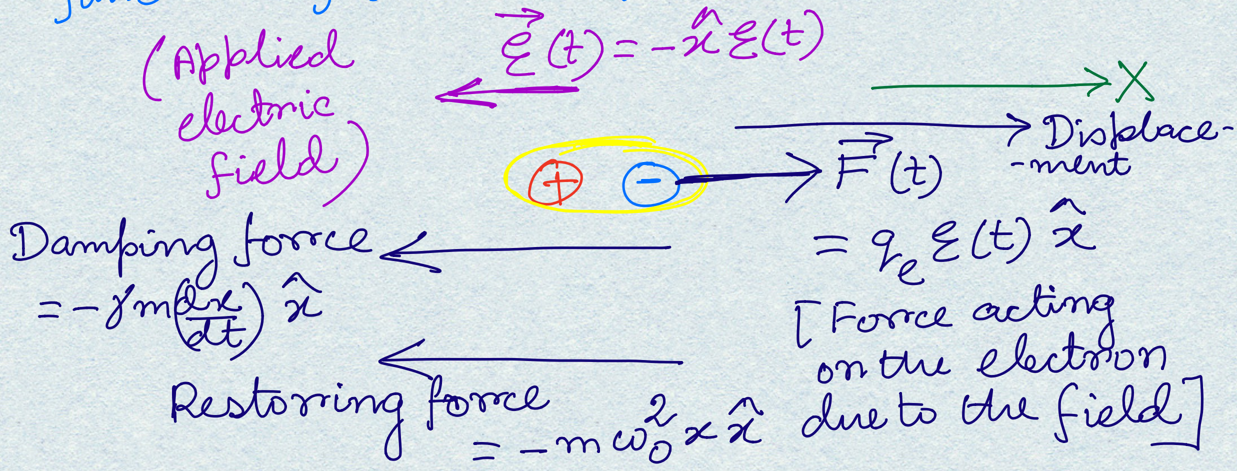
Dielectric constant of dispersive materials:

Materials with frequency dependent dielectric constants are known as dispersive materials. In this section we obtain the frequency dependence of dielectric constant.

Some useful quantities/relations:

- Polarization, $\vec{P}(\omega)$: dipole moment per unit volume.
- $\vec{P}(\omega) = \varepsilon_0 \chi_e(\omega) \vec{E}(\omega)$
  where, $\chi_e(\omega)$ is known as electrical susceptibility.
- Relative permittivity $= \varepsilon_r(\omega) = 1 + \chi_e(\omega)$

Now, we aim to show that $\vec{P}$ is indeed a function of frequency!



$$q_e \mathcal{E}(t) - \gamma m \frac{dx}{dt} - m\omega_0^2 x = m\left(\frac{d^2x}{dt^2}\right)$$

Taking Fourier transform,

$$q_e E(\omega) - i\omega\gamma m X(\omega) - m\omega_0^2 X(\omega)$$
$$= -\omega^2 X(\omega)$$

$$\Rightarrow \frac{q_e E(\omega)}{m} = X(\omega)\left[\omega_0^2 - \omega^2 + i\omega\gamma\right]$$

$$\Rightarrow X(\omega) = \frac{q_e E(\omega)}{m} \frac{1}{\left[\omega_0^2 - \omega^2 + i\omega\gamma\right]} \qquad \text{——— (1)}$$

$\therefore$ Dipole moment of a molecule with single electron $(p) = q_e X(\omega)$

$$= \frac{q_e^2 E(\omega)}{m} \frac{1}{\left[\omega_0^2 - \omega^2 + i\omega\gamma\right]} \qquad \text{——— (2)}$$

Suppose, a molecule has Z electrons. Out of these Z, f_j electrons have natural frequency ω_j and damping constant γ_j.

Also, $\sum_j f_j = Z$

Dipole moment of such a molecule would be a slight modification of eqn (2):

$$p(\omega) = \frac{q_e^2 E(\omega)}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

Now, suppose there are N molecules per unit volume.

So, the polarization P i.e. dipole moment per unit volume will be:

$$P(\omega) = N p(\omega) = \frac{N q_e^2 E(\omega)}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)} \quad \text{——— (3)}$$

Comparing the above with $P(\omega) = \varepsilon_0 \chi_e(\omega) E(\omega)$,

$$\varepsilon_0 \chi_e(\omega) = \frac{N q_e^2 E(\omega)}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

$$\Rightarrow \chi_e(\omega) = \frac{N q_e^2 E(\omega)}{m \varepsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

So, relative permittivity can be written as:

$$\varepsilon_r(\omega) = 1 + \frac{N q_e^2 E(\omega)}{m \varepsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 + i\omega\gamma_j)}$$

Going back to eqn. (2), imagine a sea of free electrons in a metal. If the metal is assumed to be lossless, there's no damping i.e. $\gamma = 0$.

Let N = electron density inside the metal.

Also, for free electrons, $\omega_0 \to 0$. (or, $\omega \gg \omega_0$)

$$\therefore P(\omega) = \frac{N q_e^2 E(\omega)}{m} \left(-\frac{1}{\omega^2}\right)$$

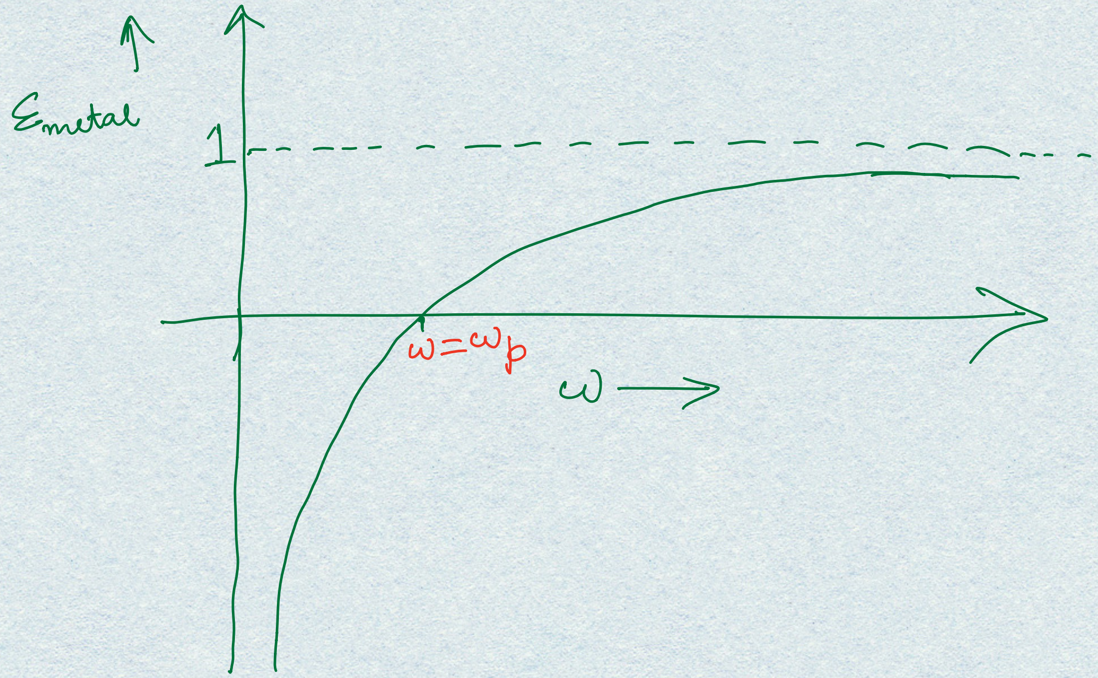
$$\therefore \chi_e(\omega) = \frac{N q_e^2}{m \varepsilon_0} \left(-\frac{1}{\omega^2}\right)$$

$$= -\left(\frac{\omega_p^2}{\omega^2}\right)$$

$\omega_p = \sqrt{\dfrac{N q_e^2}{m \varepsilon_0}}$ is known as plasma frequency of the metal

(typically falls in the UV range).

$$\therefore \varepsilon_{metal}(\omega) = 1 + \chi_e(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$


ε_{metal}
1
$\omega=\omega_p$
ω