Tutorial 2 Solutions

Q1. $y(t) = x(t) * h(t)$

$y(t) = x(t) * (\delta(t+2) + 2\delta(t+1))$

$y(t) = x(t+2) + 2x(t+1)$

$$x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Finding $x(t+2)$,

$x(t)$

$t \to t+2$ shifting

$x(t+2)$

$$x(t+2) = \begin{cases} (t+2)+1 & 0 \le (t+2) \le 1 \\ 2-(t+2) & 1 < (t+2) \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore x(t+2) = \begin{cases} t+3 & -2 \le t \le -1 \\ -t & -1 < t \le 0 \\ 0 & \text{elsewhere} \end{cases}$$

Finding $2x(t+1)$,

$x(t)$

$t \to t+1$

$x(t+1)$

$$\therefore x(t+1) = \begin{cases} t+2 & -1 \le t \le 0 \\ 1-t & 0 < t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

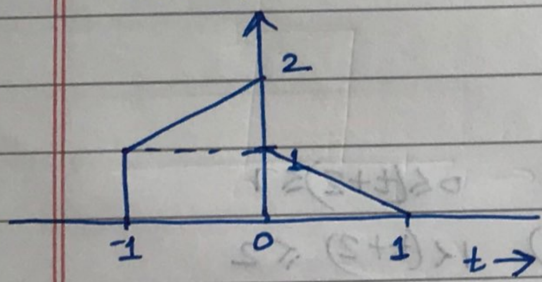
$$2x(t+1) = \begin{cases} 2(t+2) & -1 \le t \le 0 \\ 2(1-t) & 0 < t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Finding $x(t+2) + 2x(t+1) = y(t)$

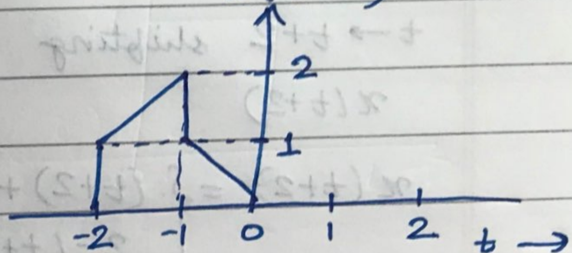
$$y(t) = \begin{cases} t+3 & -2 \leq t \leq -1 \\ t+4 & -1 \leq t \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Graphically,

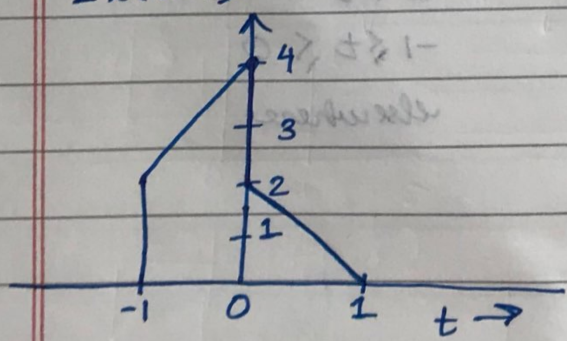
$x(t+1)$



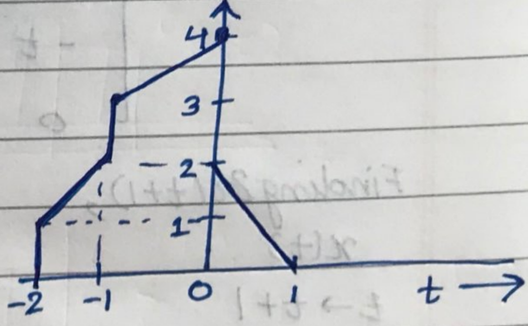
$x(t+2)$



$2x(t+1)$



$y(t) = x(t+2) + 2x(t+1)$



Q2. Given $h(t) = e^{2t}u(-t+4) + e^{-2t}u(t-5)$

we can write as,

putting $t = \tau$ & arranging in intervals,

$$h(\tau) = \begin{cases} e^{2\tau} & \tau < 4 \\ e^{-2\tau} & \tau > 5 \\ 0 & 4 < \tau < 5 \end{cases}$$

$\tau \rightarrow \tau + t$

$$h(\tau+t) = \begin{cases} e^{2(\tau+t)} & \tau+t < 4 \\ e^{-2(\tau+t)} & \tau+t > 5 \\ 0 & 4 < \tau+t < 5 \end{cases}$$

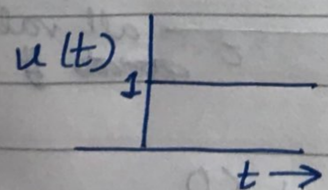
$\tau \rightarrow -\tau$

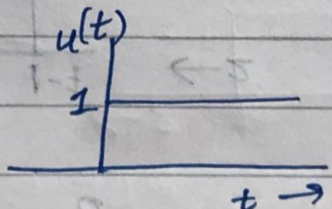
$$h(-\tau+t) = \begin{cases} e^{2(-\tau+t)} & -\tau+t < 4 \\ e^{-2(-\tau+t)} & -\tau+t > 5 \\ 0 & 4 < -\tau+t < 5 \end{cases}$$

$$= \begin{cases} e^{2(t-\tau)} & \tau > t-4 \\ e^{-2(t-\tau)} & \tau < t-5 \\ 0 & t-5 < \tau < t-4 \end{cases}$$

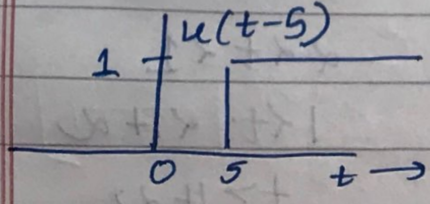
comparing we get

$A = t-5, \; B = t-4$

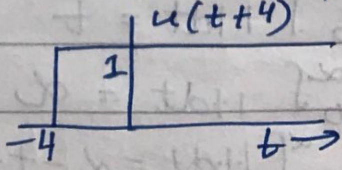




$t \rightarrow t-5$

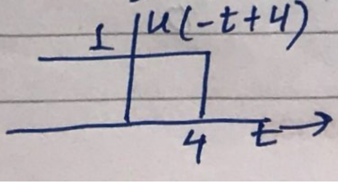


$t \rightarrow t+4$



$e^{-2t} \cdot 1$ for $t > 5$

$t \rightarrow -t$



$e^{2t} \cdot 1$ for $t < 4$

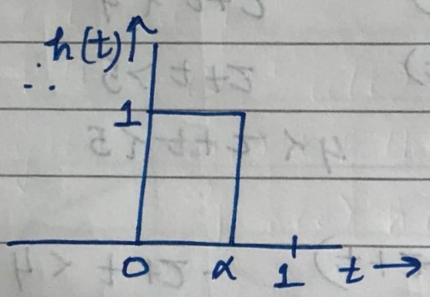
Q3. $x(t) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

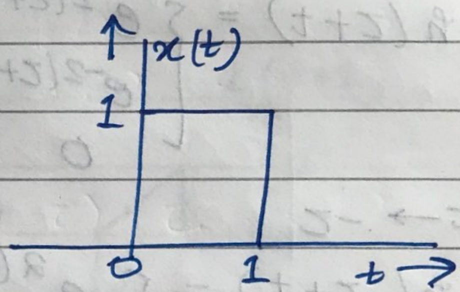
$h(t) = x\left(\frac{t}{\alpha}\right)$, $0 < \alpha < 1$

$t \to t/\alpha$

$h(t) = \begin{cases} 1 & 0 \le t/\alpha \le 1 \\ 0 & \text{elsewhere} \end{cases}$

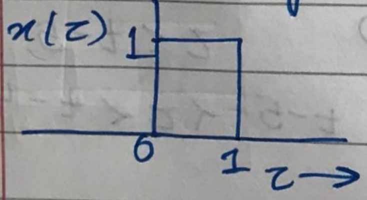
$= \begin{cases} 1 & 0 < t < \alpha \\ 0 & \text{elsewhere} \end{cases}$ & $0 < \alpha < 1$

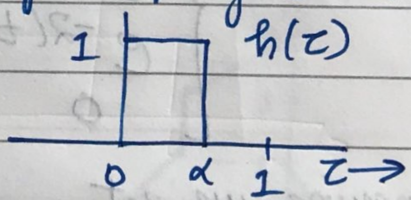


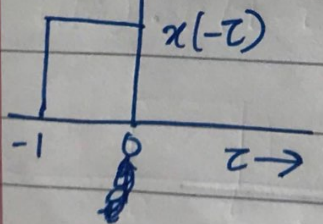


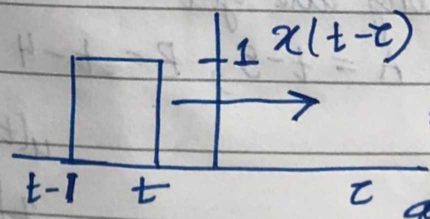
a) $y(t) = x(t) * h(t)$

Graphically solving. Plotting on τ axis.





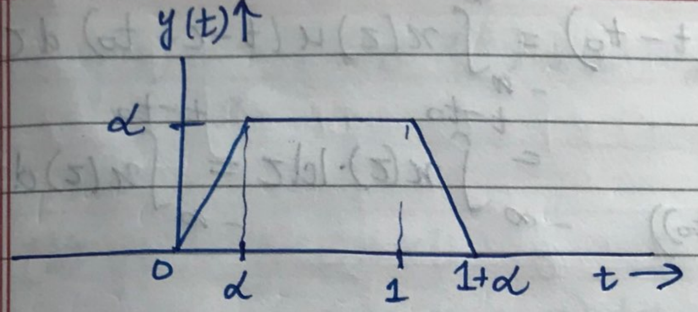




moving this plot over all values of τ

$$y(t) = \begin{cases} 0 & t < 0 \\ \int_0^t 1 \cdot 1\, dt = t & 0 < t < \alpha \\ \int_0^\alpha 1 \cdot 1\, dt = \alpha & \alpha < t < 1 \\ \int_{t-1}^\alpha 1 \cdot 1\, dt = \alpha - t + 1 & 1 < t < 1 + \alpha \\ 0 & t > 1 + \alpha \end{cases}$$

b)



discontinuities in $y(t) \longrightarrow 0, \alpha, 1, 1+\alpha$.

If $\frac{dy(t)}{dt}$ to have 3 discontinuities, $\alpha = 1$.

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t \leq 1 \\ 2-t & 1 < t < 2 \end{cases}$$

Q4. a) $x(t) * \delta(t) = x(t)$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t) \int \delta(t-\tau)\,d\tau$$

$$= x(t)\cdot 1 = x(t)$$

b) $x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau-t_0)\,d\tau = x(\tau)\Big|_{\tau = t-t_0} = x(t-t_0)$

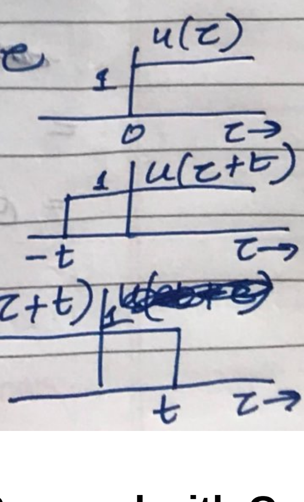
or

$$\delta(t-t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\tau-t_0)\,x(t-\tau)\,d\tau$$

$$= x(t-\tau)\Big|_{\tau=t_0} \int_{-\infty}^{\infty} \delta(\tau-t_0)\,d\tau = x(t-t_0)$$

c) $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$

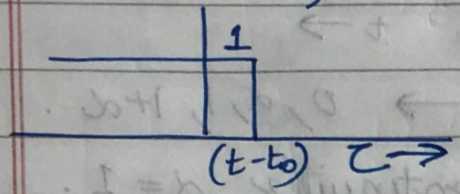
$$= \int_{-\infty}^{t} x(\tau)\,d\tau$$



d) $x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau-t_0)\, d\tau$

$= \int_{-\infty}^{t-t_0} x(\tau)\cdot 1\, d\tau = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$

$u(-\tau+(t-t_0))$



Q5. $x(t) = u(t)$

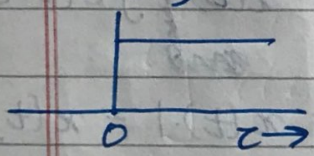
$h(t) = e^{-\alpha t} u(t)\,,\ \alpha > 0$

$y(t) = x(t) * h(t)$

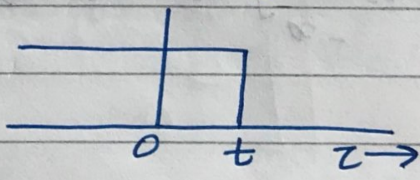
$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} u(\tau)\, e^{-\alpha(t-\tau)}\, u(t-\tau)\, d\tau$

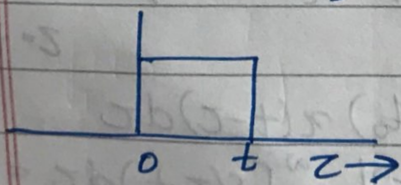
$u(\tau)$



$u(t-\tau)$



$u(\tau)\cdot u(t-\tau)$



$= \int_{0}^{t} e^{-\alpha(t-\tau)} \cdot 1 \cdot d\tau$

$= e^{-\alpha t} \int_{0}^{t} e^{\alpha\tau}\, d\tau$

$= e^{-\alpha t} \left. \dfrac{e^{\alpha\tau}}{\alpha} \right|_{0}^{t} = \dfrac{e^{-\alpha t}}{\alpha}\left[e^{\alpha t} - 1\right]$

$= \dfrac{1 - e^{-\alpha t}}{\alpha}$

$$\therefore\ y(t) = \frac{1}{\alpha}\left(1-e^{-\alpha t}\right)u(t) \quad \text{as for } t<0 \ \ y(t)=0.$$

(b) $y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau \qquad , \ \alpha > 0$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau}\,u(\tau)\,u(t-\tau)\,d\tau$$

$$= \int_{0}^{t} e^{-\alpha\tau}\,d\tau$$

$$= \left.\frac{e^{-\alpha\tau}}{-\alpha}\right|_{0}^{t}$$

$$= \frac{-1}{\alpha}\left[e^{-\alpha t} - 1\right]$$

$$= \frac{1-e^{-\alpha t}}{\alpha}$$

$$\therefore\ y(t) = \left(\frac{1-e^{-\alpha t}}{\alpha}\right)u(t)$$

Q6. $\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow u(t-\tau) - u(t-2\tau)$

$$y(t) = x(t) * h(t)$$

$$u(t-\tau) - u(t-2\tau) = \delta(t-\tau) * h(t)$$

if $h(t) = u(t) - u(t-\tau)$ then

$y(t)$ will be $u(t-\tau) - u(t-2\tau)$

$$\therefore\ h(t) = u(t) - u(t-\tau)$$

a) $y(t) = x(t) * h(t)$

$= x(t) * [u(t) - u(t-\tau)]$

$\cancel{= \int_{\tau'=-\infty}^{\infty} x(\tau') \cdot u(\tau')}$

$= \int_{\tau'=-\infty}^{\infty} x(\tau')[u(t-\tau') - u(t-\tau'-\tau)]\,d\tau'$

$= \underbrace{\int_{\tau'=-\infty}^{\tau'=\infty} x(\tau')\,u(t-\tau')\,d\tau'}_{\text{I}} - \underbrace{\int_{-\infty}^{\infty} x(\tau')\,u(t-\tau'-\tau)\,d\tau'}_{\text{II}}$

Solving I,

$\int_{-\infty}^{\infty} x(\tau')\,u(t-\tau')\,d\tau'$

From Q4 part (c),

$\text{I} = \int_{-\infty}^{t} x(\tau')\,d\tau'$

For time invariance,

$y_{\text{I}}(t) = \int_{-\infty}^{t} x(\tau')\,d\tau'$

$y(t) \longrightarrow y(t-t_0)$

$y_{\text{I}}(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau')\,d\tau'$ — ①

Delaying input,

$y'(t) = \int_{\tau'=-\infty}^{\tau'=t} x(\tau'-t_0)\,d\tau'$

Let $p = \tau' - t_0$

$dp = d\tau'$

$$y'(t) = \int_{p=-\infty}^{t-t_0} x(p)\,dp \quad \text{— ②}$$

$\therefore$ as ① = ② $\therefore$ I is time invariant.

solving (II),

$$\text{II} = \int_{-\infty}^{\infty} x(\tau')\,u(t-\tau'-\tau)\,d\tau'$$

For Q 4 part (d),

$$= \int_{-\infty}^{t-\tau} x(\tau')\,d\tau'$$

For time invariance,

$$y_{\text{II}}(t) = \int_{\tau'=-\infty}^{\tau'=t-\tau} x(\tau')\,d\tau'$$

$$y_{\text{II}}(t-t_0) = \int_{-\infty}^{t-t_0-\tau} x(\tau')\,d\tau' \quad \text{— ③}$$

Delaying input,

$$\int_{\tau'=-\infty}^{\tau'=t-\tau} x(\tau'-t_0)\,d\tau'$$

Let $p = \tau' - t_0 \qquad dp = d\tau'$

$$\int_{p=-\infty}^{p=t-\tau-t_0} x(p)\,dp \quad \text{— ④}$$

as ③ = ④ $\therefore$ II is time invariant.

$\therefore$ system is time invariant.

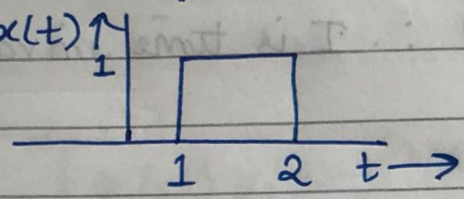
b). $h(t) = u(t) - u(t-\tau)$

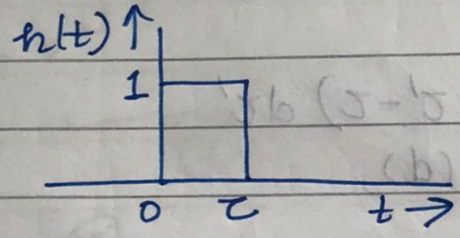
causality will depend on τ

$\tau > 0$ causal

$\tau < 0$ not causal

c) $x(t) = u(t-1) - u(t-2)$



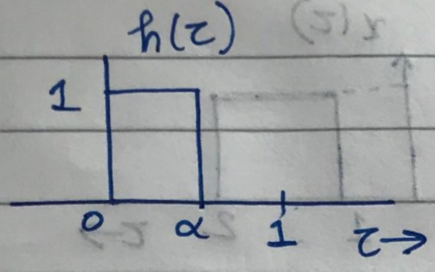


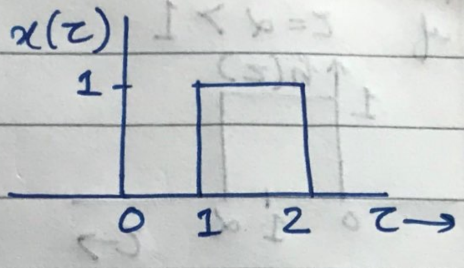
There can be 2 cases $0 < \tau < 1$ or $\tau > 1$, considering $\tau > 0$.

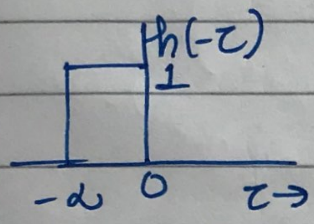
similar to problem no 3 (a).

Let $\tau = \alpha$ to reduce confusion

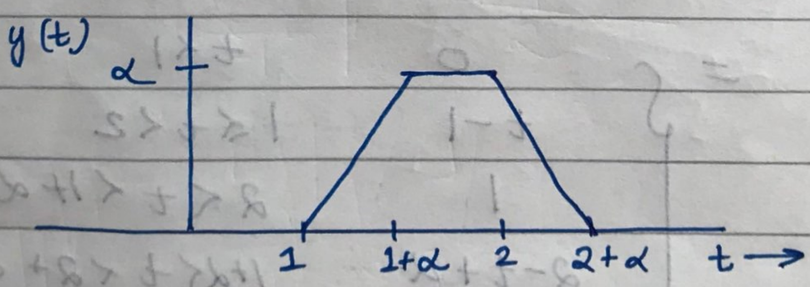
if $0 < \tau = \alpha < 1$



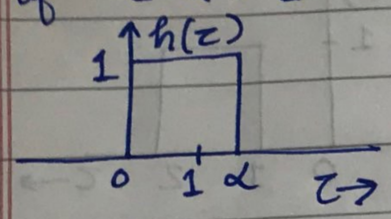


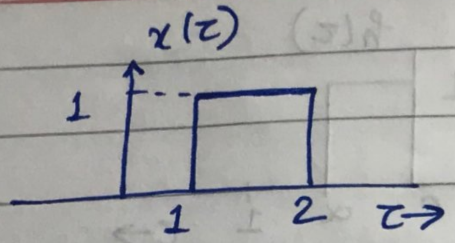


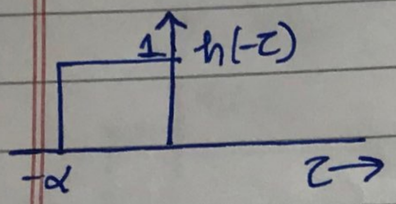
$$y(t) = \begin{cases} 0 & t < 1 \\ \int_1^t 1 \cdot dt = t-1 & 1 < t < 1+\alpha \\ \int_{t-\alpha}^{t} 1 \cdot dt = t - t + \alpha = \alpha & 1+\alpha < t < 2 \\ \int_{t-\alpha}^{2} 1 \cdot dt = 2 - t + \alpha & 2 < t < 2+\alpha \\ 0 & t > 2+\alpha \end{cases}$$



if $\tau = \alpha > 1$

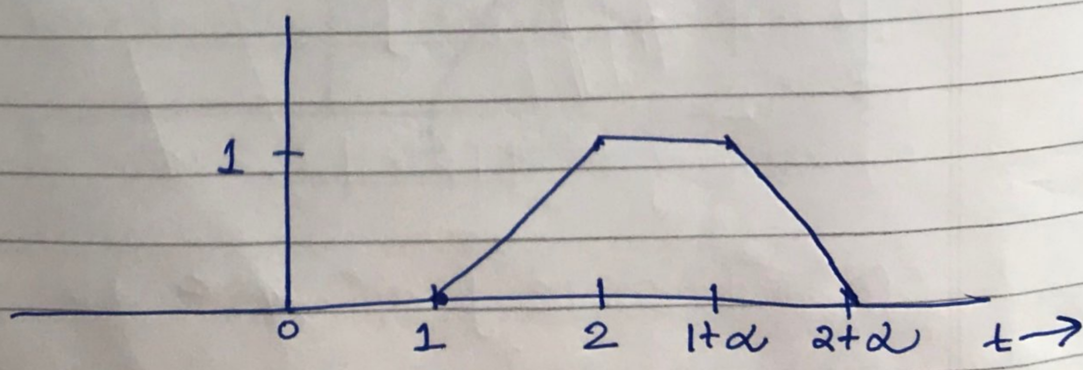






$$y(t) = \begin{cases} 0 & t < 1 \\ \int_1^t 1\cdot dt = t-1 & 1 \leq t < 2 \\ \int_1^2 1\cdot dt = 1 & 2 \leq t < 1+\alpha \\ \int_{t-\alpha}^{2} 1\cdot dt = 2-t+\alpha & 1+\alpha < t < 2+\alpha \\ 0 & t > 2+\alpha \end{cases}$$

$$= \begin{cases} 0 & t < 1 \\ t-1 & 1 \leq t < 2 \\ 1 & 2 < t < 1+\alpha \\ 2-t+\alpha & 1+\alpha < t < 2+\alpha \\ 0 & t > 2+\alpha \end{cases}$$



substitute back $\alpha = \tau$.

Q7. $w(t) = x(t) * h_1(t)$

$y(t) = w(t) * h_2(t)$

$y(t) = (x(t) * h_1(t)) * h_2(t)$

$= x(t) * (h_1(t) * h_2(t))$

$= x(t) * h(t)$

a) $\therefore h(t) = h_1(t) * h_2(t)$

$$= \int_{-\infty}^{\infty} h_1(\tau)\, h_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, 2e^{-(t-\tau)} u(t-\tau)\, d\tau$$

$$= 2e^{-t} \int_0^t e^{-2\tau} e^{\tau} d\tau$$

$$= 2e^{-t} \int_0^t e^{-\tau}\, d\tau$$

$$= 2e^{-t} \left. \frac{e^{-\tau}}{-1} \right|_0^t$$

$$= \frac{2e^{-t}}{-1} [e^{-t} - 1]$$

$$= 2e^{-t}[1 - e^{-t}]$$

$$= 2[e^{-t} - e^{-2t}]$$

$\therefore h(t) = 2[e^{-t} - e^{-2t}]$ as $h(t) = 0$ for $t < 0$.

b) For stability $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 2\int_0^{\infty} e^{-\tau} - e^{-2\tau}\, d\tau$$

$$= \left. \frac{2e^{-\tau}}{-1} \right|_0^{\infty} - \left. \frac{2e^{-2\tau}}{-2} \right|_0^{\infty} = -2[0-1] + [0-1]$$

$$= 2 - 1 = 1 < \infty$$

$\therefore$ BIBO stable.