Conditional P.M.F, Expectations and Joint Distribution

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Conditional P.M.F

- Random variable X has P.M.F $p_X(x)$
- A is an event
- Conditional P.M.F - $p_{X|A}(x) = \mathbf{P}\left(X = x \mid A\right)$
- All properties of P.M.Fs hold the same way.

Example: Conditional P.M.F

- Let $A = \{X \geq 2\}$
- $p_{X|A}(x) =$
- $\mathbf{E}[X \mid A]$

Total Expectation Theorem

- Partition of sample space into disjoint events $A_1, A_2 \ldots A_N$.
- $\mathbf{P}(B) = \mathbf{P}(B \mid A_1)\mathbf{P}(A_1) + \cdots + \mathbf{P}(B \mid A_N)\mathbf{P}(A_N)$
- $p_X(x) = p_{X \mid A_1}(x)\mathbf{P}(A_1) + \cdots + p_{X \mid A_N}(x)\mathbf{P}(A_N)$
- $\mathbf{E}[X] = \mathbf{E}_{X \mid A_1}[X]\mathbf{P}(A_1) + \cdots + \mathbf{E}_{X \mid A_N}[X]\mathbf{P}(A_N)$
- Mean of a Geometric r.v can be found using the above theorem

$$A_1 = \{X = 1\}, A_2 = \{X > 1\}$$

$$\mathbf{E}[X] = \mathbf{E}_{X=1}[X]\mathbf{P}(X = 1) + \mathbf{E}_{X>1}[X]\mathbf{P}(X > 1)$$

- Solve for $\mathbf{E}[X] = 1/p$

Joint P.M.F

- $p_{X,Y}(x,y) = \mathbf{P}\left(X = x \text{ and } Y = y\right)$
- $\sum_x \sum_y p_{X,Y}(x,y)$
- Marginal distribution: $p_Y(y) = \sum_x p_{X,Y}(x,y)$
- $p_{X \mid Y}(x \mid y) = \mathbf{P}\left(X = x \mid Y = y\right) = p_{X,Y}(x,y)/p_Y(y)$
- $\sum_x p_{X,Y}(x \mid y)$

Independent Random Variables

$$p_{X,Y,Z}(x,y,z) = p_X(x)\, p_{Y\,|\,X}(y\,|\,x)\, p_{Z\,|\,X,Y}(z\,|\,x,y)$$

- Random Variables X, Y, Z are independent if:

$$p_{X,Y,Z}(x,y,z) = p_X(x)p_Y(y)p_Z(z) \quad \forall x, y, z$$

- Conditional independence works the same as “normal” independence.

Expectations

$\mathbf{E}[X] = \sum_x x p_X(x)$
$\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$

- In general, $\mathbf{E}[g(X, Y)] \neq g(\mathbf{E}[X], \mathbf{E}[Y])$
- $\mathbf{E}[\alpha X + \beta] = \alpha \mathbf{E}[X] + \beta$
- $\mathbf{E}[X + Y + Z] = \mathbf{E}[X] + \mathbf{E}[\mathrm{Y}] + \mathbf{E}[Z]$
- If X, Y are independent:
  - $\mathbf{E}[XY] = \mathbf{E}[Y]\mathbf{E}[X]$
  - $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[h(Y)]\mathbf{E}[g(X)]$

Variances

- $var(\alpha X) = \alpha^2\, var(X)$
- $var(X + \alpha) = var(X)$
- When X and Y are independent and Z$= X + Y$

$$var(X + Y) = var(X) + var(Y) \tag{1}$$

- Examples:
  - If $X = Y$, $var(X + Y) =$
  - When X and Y are independent, $Z = X - 3Y$

$$var(Z) = var(X) + ... \tag{2}$$

  - Binomial R.V.
  - Hat in the ring problem.