Lecture 4: Random Variables, Discrete PMFs, Expectation and Variance

IIIT Delhi

praveshb@iiitd.ac.in

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Random Variable

- R.V's is way of assigning numerical values to outcomes
- Function from sample space Ω to Real line **R**.
- Can have multiple random variable coming from the same sample space.
- Note: Its not a variable, its a function, $f_X(x)$
- X denotes the function, x a specific value in the range.

Example: R.V.s

- Perform an experiment: Four coin tosses
- Possible Variables: Number of heads, number of tales, number of consecutive heads....
- Sample the height of this class.
- Interest is to find the likelihood of each value of this variable.

Probability Mass Function (PMF)

- Tells us the probability distribution of X
- Notation:

$$\begin{aligned} p_X(x) &= \mathbf{P}(X = x) \\ &= \mathbf{P}(\{w \in \Omega, s.t, X(w) = x\}) \end{aligned} \quad (1)$$

- "Probability of those Sample Space points whose random variable function value is x"
- $p_X(x) \geq 0$, $\sum_x p_X(x) = 1$

Examples of PMFs

- Bernoulli PMF or Bernoulli Distribution:
  - A random variable is called Bernoulli r.v if it can take only two possible values.
  - $p_X(1) = p$, $p_X(0) = 1 - p$
  - Example: head or tail on one coin toss
- Binomial PMF or Binomial Distribution:
  - R.V: The number of successes (or 1's) in n Bernoulli trials.
  - $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ (How ?)
  - Number of heads in "n" coin tosses
  - Can be extended to "multinomial" distribution

Examples of PMFs

- Poisson Distribution
  - Bernoulli Trials: n large, p small, $\lambda = np$
  - Poisson distribution approximates Binomial
  - $p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$
- Geometric:
  - k is the first time a head occurs during a repeated coin toss.
  - $p_X(k) = (1-p)^{k-1} p$

More Examples

- *Birthdays*: Probability that in a company of 500, exactly k people have birthdays on a particular day?
- *Misprints*: If there is a constant prob. of any letter being misprint, then we have as many Bernoulli trials as there are letters. Total number of misprints in a page is approximately Poisson.
- -*Defective items*: Small probability of any item in a box being defective. What is the probability that no defective items in the box of 100 items.

Expectation of a Random Variable

- Whats a typical value a R.V takes? One good answer: its expectation.
- Definition:

$$\mathbf{E}[X] = \sum_x x \, p_X(x)$$

- Examples:
  - Expectation of a Bernoulli r.v $= p$
  - Expectation of a Binomial r.v $= np$
    Interpretation: np successes in n trials
  - Expectation of a Poisson r.v $= \lambda$

Properties of Expectations

- Let X be a r.v and let Y=g(X)
- $\mathbf{E}[Y] = \sum_x g(x)p_X(x)$
- $\mathbf{E}[\alpha] = \alpha$
- $\mathbf{E}[\alpha X] = \alpha \mathbf{E}[X]$

Variance of a R.V

- Indicates the spread of an r.v.

$$\begin{aligned} var(X) &= \mathbf{E}[(x - \mathbf{E}[X])^2] \\ &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \end{aligned}$$

- Standard deviation $\sigma_X = \sqrt{var(X)}$
- Examples:
  - Variance of Bernoulli, Poisson and Binomial r.vs are?
  - Notion of mean square error of an estimate !
- Properties:
  - $var\{\alpha X\} = \alpha^2 var\{X\}$