Engineering Optimization: Lecture 2

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Notations

- Scalars : a, b, c
- Vectors: $\mathbf{a}, \mathbf{b}, \mathbf{c}$
- Matrices: $\mathbf{A}, \mathbf{B}, \mathbf{C}$
- Matrices: A_{ij} denote an element.

Vector Arithmetic

- Addition: Element wise

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}+\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}=\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}, \quad \mathbf{a}\in R^m, \mathbf{b}\in R^m$$

- Scalar Multiplication: Element wise.
- Vector Multiplication: Inner Product

$$\mathbf{a},\mathbf{b}\in R^m, \mathbf{a}^\mathbf{T}\mathbf{b}=\langle \mathbf{a},\mathbf{b}\rangle=\sum_i a_i b_i$$

- Inner product for complex vectors :

$$\mathbf{a},\mathbf{b}\in C^m,\ \mathbf{a}^\mathbf{H}\,\mathbf{b}=\langle \mathbf{a},\mathbf{b}\rangle=\sum_i a_i^* b_i$$

- Orthogonal vectors: If for any two vectors $\mathbf{x}^\mathbf{H}\mathbf{y}=\mathbf{0}$, then they are orthogonal.

Matrix Operations and Some Identities

- Transpose :

$$(\mathbf{A}^{\mathbf{T}})^{\mathbf{T}} = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}}$$

$$(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}.\mathbf{A}^{\mathbf{T}}$$

- Conjugate transpose ($\mathbf{A}^H$) for complex matrices.
- Inverse for $\mathbf{A}, \mathbf{B} \in R^{n \times n}$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

System of Linear Equations

- General form of a linear equations:

$$a_1 x_1 + a_2 x_2 \cdots + a_n x_n = b \quad \text{or} \quad \mathbf{a}^T \mathbf{x} = b$$

- Simultaneous linear equations:

$$a_{11} x_1 + a_{12} x_2 \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 \cdots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{n1} x_1 + a_{n2} x_2 \cdots + a_{nn} x_n = b_n$$

Or,

$$\mathbf{Ax} = \mathbf{b}$$

Linear Independence

- Linear Combination

$$y = a_1 x_1 + a_2 x_2 \cdots + a_n x_n$$

- Linear independence

$$\sum_i a_i \mathbf{x}_i = 0 \text{ iff } a_i = 0, \quad \forall i$$

- How many linear independent vectors can exist in 1-d, 2-d.. spaces?
- Span: A collection of all possible linear combination of vectors in a given set.
- Connection between linear independence and span ?

Subspaces

- Any subset X of R^n is a subspace if for all $\mathbf{x}, \mathbf{y} \in X$,

$$\alpha \mathbf{x} + \beta \mathbf{y} \in X \quad \forall \alpha, \beta \in R$$

,

- Basis: Set of vectors which are linearly independent and whose span is a subspace.
- Examples: ?
- Is the basis of a subspace unique?
- Prove that any basis set of a subspace has same number of elements. Also called the *dim* of a subspace.
- Every subspace of non-zero dimension has a basis that is orthogonal.

Matrices and Subspaces

- If X is a subset of R^n and $\mathbf{A}$ is an $m \times n$ matrix, then the image of X under $\mathbf{A}$ denoted by $\mathbf{A}X$.

$$AX = \{\mathbf{Ax} \,|\, \mathbf{x} \in X\}$$

- If X is a subspace, so is AX. Proof?
- *Range Space of A, denoted by* $R(A)$ *is the set of all* $\mathbf{y} \in R^m$ *such that* $\mathbf{y} = \mathbf{Ax}$ *for some* $\mathbf{x} \in R^n$.
- *The Null Space of A, denoted by* $N(A)$ *is the set of all* $\mathbf{x} \in R^n$ *such that* $\mathbf{Ax} = \mathbf{0}$.

Matrix Products

- Matrix Multiplication:

$$\mathbf{A} \in R^{p \times m},\ \mathbf{B} \in R^{m \times q},\ \mathbf{a}_i, \mathbf{b}_i \in R^m \quad \forall i$$

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2 \ldots \mathbf{a}_p]^T, \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_q]$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_1 & \ldots & \mathbf{a}_1^T\mathbf{b}_q \\ \ldots & \ldots & \ldots \\ \mathbf{a}_p^T\mathbf{b}_1 & \ldots & \mathbf{a}_p^T\mathbf{b}_q \end{bmatrix}$$

- Linear Combination of rows and columns

Rank of a Matrix

- Number of linearly independent columns/rows of a matrix is its rank or (dimension of $R(A)$).
- Examples: ?
- Rank $\leq$ number of rows/columns. Proof? H.W
- Row rank is equal to column rank. Proof? H.W

Square Matrices and Eigen Values

- A square matrix is singular if its determinant is zero.
- If a matrix A is non-singular, for every non-zero $\mathbf{x} \in R^n$, we have $\mathbf{Ax} \neq \mathbf{0}$.
- A characteristic polynomial of a square matrix A is defined by $\phi(\lambda) = det(\lambda\mathbf{I} - \mathbf{A})$.
- Roots of $\phi(\lambda)$ gives eigen values. A vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ is called eigen vector.
- Geometric interpretation of E.Vs?
- Eigen values of a singular matrix?
- Spectral norm of a matrix $\mathbf{A}$, denoted by $\rho(\mathbf{A})$ is the maximum absolute eigen value of $\mathbf{A}$.

Symmetric Matrices

- For real symmetric matrices we have the following properties:
- All eigen values of a real symmetric matrix are real.
- Eigen vectors corresponding to distinct eigen values are orthogonal.
- All eigen vectors are linearly independent.
- Any symmetric real matrix can be written as

$$\mathbf{A} = \sum_i \lambda_i \mathbf{x}_i \mathbf{x}_i^T$$

Positive Definite Matrices

- A symmetric $n \times n$ matrix $\mathbf{A}$ is called positive definite if $\mathbf{x}^\mathbf{T}\mathbf{A}\mathbf{x} > \mathbf{0}$ for all $\mathbf{x} \in R^n$. Its called positive semi-definite if $\mathbf{x}^\mathbf{T}\mathbf{A}\mathbf{x} \geq \mathbf{0}$ for all $\mathbf{x} \in R^n$.
- PSD-ness implies symmetric behaviour. Converse not true.

Singular Value Decomposition

Norms

$f : \mathbf{R}^n \rightarrow \mathbf{R}$ is a norm if for all $x, y \in \mathbf{R}^n$, $t \in \mathbf{R}$ s

1. $f(x) \geq 0$
2. $f(tx) = |t|f(x)$
3. $f(x + y) \leq f(x) + f(y)$

$f(x)$ usually denoted as $|| \; ||_{mark}$

Example: $||l||_2$, called l_2 norm or the euclidean norm.

In general l_p, norms are $||l||_p, \;\; p \geq 1$, where,

$$||x||_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$