Lecture – 8

Date: 30.01.2017

- Matched, Lossless, and Reciprocal 3-port Network
- Scattering Parameters and Circuit Symmetry

Matched, Lossless, Reciprocal Devices

Matched Device

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will come out from a port if a signal is incident on that port (but **only** that port!).

- **In other words:** $V_m^- = S_{mm} V_m^+ = 0$ For all m → When all the ports 'm' are matched

- It is apparent that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

$$\mathbf{S} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

Lossless Device

- For a lossless device, all of the power that is delivered to each device port must eventually find its way **out**!
- In other words, power is not **absorbed** by the network—no power to be **converted to heat**!
- The **power incident** on some port m is related to the amplitude of the **incident wave** (V_m^+) as:

$$P_m^+ = \frac{|V_m^+|^2}{2Z_0}$$

- The power of the **wave exiting** the port is:

$$P_m^- = \frac{|V_m^-|^2}{2Z_0}$$

Matched, Lossless, Reciprocal Devices (contd.)

- power absorbed by that port is the **difference** of the incident power and reflected power:

$$\Delta P_m = P_m^+ - P_m^- = \frac{|V_m^+|^2}{2Z_0} - \frac{|V_m^-|^2}{2Z_0}$$

- For an N-port device, the total incident power is:

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{1}{2Z_0}\sum_{m=1}^{N}|V_m^+|^2$$

$$|V_m^+|^2 = (\mathbf{V}^+)^H \mathbf{V}^+$$

$(\mathbf{V}^+)^H$ is the conjugate transpose of the row vector $\mathbf{V}^+$

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{(\mathbf{V}^+)^H \mathbf{V}^+}{2Z_0}$$

Similarly, the total reflected power

$$P^- = \sum_{m=1}^{N} P_m^- = \frac{(\mathbf{V}^-)^H \mathbf{V}^-}{2Z_0}$$

- Recall that the incident and reflected wave amplitudes are **related** by the **scattering matrix** of the device as:

$$\mathbf{V}^- = \mathbf{S}\mathbf{V}^+$$

- Therefore:

$$P^- = \frac{(\mathbf{V}^-)^H \mathbf{V}^-}{2Z_0} = \frac{(\mathbf{V}^+)^H \mathbf{S}^H\mathbf{S}\mathbf{V}^+}{2Z_0}$$

- Therefore the **total power delivered** to the N-port device is:

$$\Delta P = P^+ - P^- = \frac{(\mathbf{V}^+)^H \mathbf{V}^+}{2Z_0} - \frac{(\mathbf{V}^+)^H \mathbf{S}^H\mathbf{S}\mathbf{V}^+}{2Z_0}$$

$$\Rightarrow \Delta P = \frac{(\mathbf{V}^+)^H}{2Z_0}\left(I - \mathbf{S}^H\mathbf{S}\right)\mathbf{V}^+$$

- For a lossless device: **ΔP=0**

$$\Rightarrow \frac{(\mathbf{V}^+)^H}{2Z_0}\left(I - \mathbf{S}^H\mathbf{S}\right)\mathbf{V}^+ = 0$$

For all $\mathbf{V}^+$

Matched, Lossless, Reciprocal Devices (contd.)

- Therefore: $I - S^H S = 0$ $\Rightarrow S^H S = I$

a special kind of matrix known as a **unitary matrix**

If a network is **lossless**, then its scattering matrix **S** is **unitary**

- **How to recognize a unitary matrix?**

The **columns** of a unitary matrix form an **orthonormal set**!

Example:

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

each **column** of the scattering matrix will have a **magnitude equal to one**

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1$$ **For all n**

inner product (i.e., dot product) of **dissimilar columns** must be **zero**

$$\sum_{m=1}^{N} S_{mi} S_{mj}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + ... + S_{Ni} S_{Nj}^* = 0$$ **For all i≠j**

dissimilar columns are orthogonal

- eg, for a lossless **3-port** device: say signal is incident on port 1, and **all** other ports are **terminated**. The power **incident** on port 1 is therefore: $P_1^+ = \frac{|V_1^+|^2}{2Z_0}$

- the power **exiting** the device at each port is: $P_m^- = \frac{|V_m^-|^2}{2Z_0} = \frac{|S_{m1} V_1^+|^2}{2Z_0} = |S_{m1}|^2 P_1^+$

Matched, Lossless, Reciprocal Devices (contd.)

- The **total** power exiting the device is therefore:

$$P^- = P_1^- + P_2^- + P_3^- = |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+$$

$$\Rightarrow P^- = \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2\right) P_1^+$$

- As the device is **lossless**, then the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^- = P_1^+$). This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1$$

- Of course, this will be true if the incident wave is placed on **any** of the **other** ports of this lossless device:

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1$$

- We can state in general then that: $\sum_{m=1}^{N} |S_{mn}|^2 = 1$ For all n

- In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

- An **example** of a (unitary) scattering matrix for a 4-port **lossless** device is:

$$\mathbf{S} = \begin{bmatrix} 0 & 1/2 & j\sqrt{3}/2 & 0 \\ 1/2 & 0 & 0 & j\sqrt{3}/2 \\ j\sqrt{3}/2 & 0 & 0 & 1/2 \\ 0 & j\sqrt{3}/2 & 1/2 & 0 \end{bmatrix}$$

Matched, Lossless, Reciprocal Devices (contd.)

Reciprocal Device

- Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.
- For reciprocal network, the elements of the s-matrix are **related** as: $S_{mn} = S_{nm}$
- For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as: $S^T = S$ **where T indicates transpose.**
- An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\mathbf{S} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10-j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Example – 1

- A **lossless**, **reciprocal** 3-port device has S-parameters of $S_{11} = {}^1/_2$, $S_{31} = {}^1/_{\sqrt{2}}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.

→ Find the remaining **6** scattering parameters.

Example – 1 (contd.)

Q: This problem is clearly **impossible**—you have not provided us with sufficient **information**!

A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

- Start with what we **currently** know: $\mathbf{S}=\begin{bmatrix} 1/2 & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$

- As the device is **reciprocal**, we then also know:

$$\boldsymbol{S_{12} = S_{21}}$$
$$\boldsymbol{S_{13} = S_{31} = 1/\sqrt{2}}$$
$$\boldsymbol{S_{32} = S_{23}}$$

- And therefore: $\mathbf{S}=\begin{bmatrix} 1/2 & S_{21} & 1/\sqrt{2} \\ S_{21} & S_{22} & S_{32} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$

- Now, since the device is **lossless**, we know that:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1 \quad \Rightarrow \quad (1/2)^2 + |S_{21}|^2 + (1/\sqrt{2})^2 = 1$$

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1 \quad \Rightarrow \quad |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1 \quad \Rightarrow \quad (1/2)^2 + |S_{32}|^2 + (1/\sqrt{2})^2 = 1$$

Columns have unit magnitude

Example – 1 (contd.)

$$0 = S_{11}S_{12}^{*} + S_{21}S_{22}^{*} + S_{31}S_{32}^{*} = \frac{1}{2}S_{12}^{*} + S_{21}S_{22}^{*} + \frac{1}{\sqrt{2}}S_{32}^{*}$$

$$0 = S_{11}S_{13}^{*} + S_{21}S_{23}^{*} + S_{31}S_{33}^{*} = \frac{1}{2}\frac{1}{\sqrt{2}} + S_{21}S_{32}^{*} + \frac{1}{\sqrt{2}}(0)$$

$$0 = S_{12}S_{13}^{*} + S_{22}S_{23}^{*} + S_{32}S_{33}^{*} = S_{21}\left(\frac{1}{\sqrt{2}}\right) + S_{22}S_{32}^{*} + S_{32}(0)$$

Dissimilar columns are **orthogonal**

We can simplify these expressions and can further simplify them by using the fact that the elements are all **real**, and therefore $S_{21} = S_{21}{}^{*}$ (etc.).

Q: I count the simplified expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is **no** solution!

A: Actually, we have **six** real equations and **six** real unknowns, since scattering element has a magnitude and phase. In this case we know the values are **real**, and thus the phase is either $0°$ or $180°$(i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

Example – 1 (contd.)

- the scattering matrix for the given **lossless**, **reciprocal** device is:

$$S=\begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

A Matched, Lossless, Reciprocal 3-Port Network

- Consider a 3-port device. Such a device would have a scattering matrix :

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

- Assuming the device is passive and made of simple (isotropic) materials, the device will be **reciprocal**, so that:

$$S_{21} = S_{12} \quad S_{31} = S_{13} \quad S_{23} = S_{32}$$

- Similarly, if it is **matched**, we know that:

$$S_{11} = S_{22} = S_{33} = 0$$

- As a result, a **matched**, **reciprocal** device would have a scattering matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{21} & 0 & S_{32} \\ S_{31} & S_{32} & 0 \end{bmatrix}$$

- if we wish for this network to be **lossless**, the scattering matrix must be **unitary**, and therefore:

$$|S_{21}|^2 + |S_{31}|^2 = 1 \qquad S_{31}^* S_{32} = 0$$
$$|S_{12}|^2 + |S_{32}|^2 = 1 \qquad S_{21}^* S_{32} = 0$$
$$|S_{13}|^2 + |S_{23}|^2 = 1 \qquad S_{31}^* S_{31} = 0$$

- Since each complex value S is represented by **two real numbers** (i.e., real and imaginary parts), the unitary equations result in **9** real equations. The problem is, the 3 complex values S_{21}, S_{31} and S_{32} are represented by only **6** real unknowns.

A Matched, Lossless, Reciprocal 3-Port Network (contd.)

We have **over constrained** our problem ! There are **no unique solutions** to these equations !

As unlikely as it might seem, this means that a matched, lossless, reciprocal **3-port** device of **any** kind is a **physical impossibility**!

You **can** make a lossless reciprocal 3-port device, **or** a matched reciprocal 3-port device, **or even** a matched, lossless (but non-reciprocal) 3-port network.

But try as you might, you **cannot** make a lossless, matched, **and** reciprocal three port component!

Guess what! I have determined that—unlike a **3-port** device—a matched, lossless, reciprocal **4-port** device **is** physically possible! In fact, I've found **two** general solutions!

Matched, Lossless, Reciprocal 4-Port Network

- The first solution is referred to as the **symmetric** solution:

$$\mathbf{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

- Note for the symmetric solution, every row and every column of the scattering matrix has the **same** four values (i.e., α, jβ, and two zeros)!
- The second solution is referred to as the **anti-symmetric** solution.

$$\mathbf{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Note that for anti-symmetric solution, **two** rows and **two** columns have the same four values (i.e., α, β, and two zeros), while the **other** two row and columns have (slightly) **different** values (α, -β, and two zeros)

- It is **quite** evident that each of these solutions are **matched** and **reciprocal**. However, to ensure that the solutions are indeed **lossless**, we must place an **additional** constraint on the values of α, β. Recall that a **necessary** condition for a lossless device is:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \qquad \textbf{For all } n$$

- For **symmetric** case:

$$|\alpha|^2 + |\beta|^2 = 1$$

- Similarly for the **anti-symmetric** case:

$$|\alpha|^2 + |\beta|^2 = 1$$

Matched, Lossless, Reciprocal 4-Port Network (contd.)

- It is evident that if the scattering matrix is **unitary** (i.e., lossless), the values α and β **cannot** be independent, but must be **related** as:

$$|\alpha|^2 + |\beta|^2 = 1$$

- **Generally** speaking, we can find that α ≥ β. Given the constraint on these two values, we can thus conclude that:

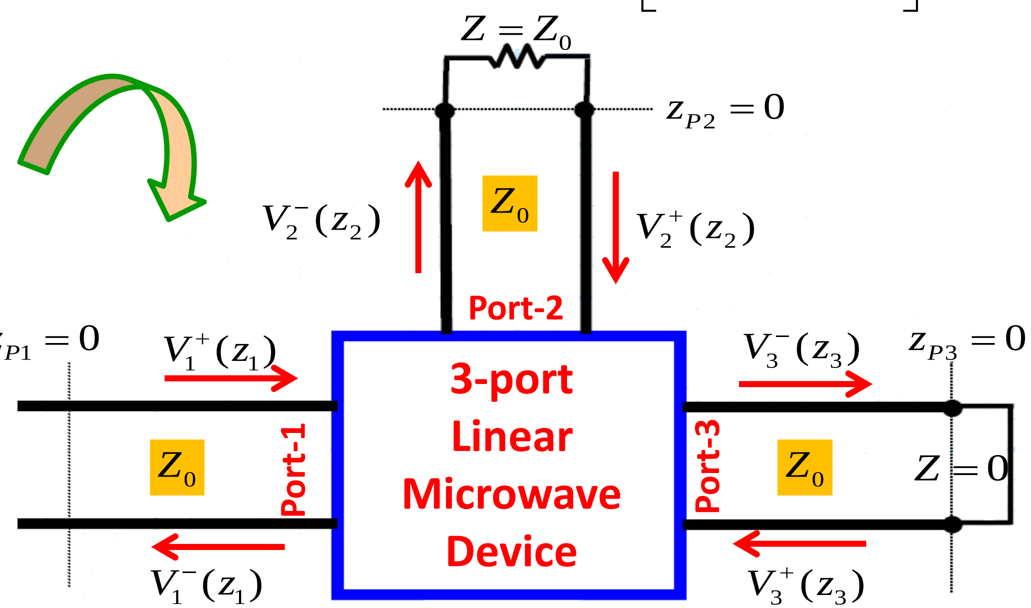
$$0 \le |\beta| \le \frac{1}{\sqrt{2}} \qquad \frac{1}{\sqrt{2}} \le |\beta| \le 1$$

Example – 2

- Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$S = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0 & 0.2 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

- A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 = \frac{V_1^-(z_{P1})}{V_1^+(z_{P1})}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} = \frac{V_2^-(z_{P2})}{V_1^+(z_{P1})}$$

Example – 2 (contd.)

Solution:

I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:

$$\Gamma_1 = \frac{V_1^-}{V_1^+} = S_{11} = 0.0 \quad \textbf{and} \quad T_{21} = \frac{V_2^-}{V_1^+} = S_{21} = 0.5$$

NO!!! The above solution is not correct!

Remember, $V_1^-/V_1^+ = S_{11}$ only if ports 2 and 3 are terminated in matched loads! In this problem port 3 is terminated with a short circuit.

Therefore: $\Gamma_1 = \frac{V_1^-}{V_1^+} \neq S_{11}$ **and similarly:** $T_{21} = \frac{V_2^-}{V_1^+} \neq S_{21}$

- To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_1^- = 0.2V_2^+ + 0.5V_3^+ \qquad V_2^- = 0.5V_1^+ + 0.2V_3^+ \qquad V_3^- = 0.5V_1^+ + 0.5V_2^+$$

- **and** the **two** equations provided by the **attached loads**: $V_2^+ = 0$ $\quad V_3^+ = -V_3^-$

- Solve those five expressions to find: $\Gamma_1 = \frac{V_1^-}{V_1^+} = -0.25 \qquad T_{21} = \frac{V_2^-}{V_1^+} = 0.4$

Example – 3

- Consider a **two-port device** with Z_0 =50Ω and scattering matrix (at some specific frequency ω_0):

$$S(\omega=\omega_0)=\begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

- Say that the transmission line connected to **port 2** of this device is terminated in a **matched** load, and that the wave **incident** on **port 1** is:

$$V_1^+(z_1)=-j2e^{-j\beta z_1}$$ **where** $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$

2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$

3. the net power flowing into port 1

Solution: **1.** Given the **incident** wave on port 1 is: $V_1^+(z_1)=-j2e^{-j\beta z_1}$

- we can conclude (since $z_{1P} = 0$): $V_1^+(z_1=z_{1P})=-j2e^{-j\beta z_{1P}}=-j2e^{-j\beta(0)}=-j2$
- since port 2 is **matched** (and **only** because its matched!): $V_1^-(z_1=z_{1P})=S_{11}V_1^+(z_1=z_{1P})=0.1(-j2)=-j0.2$
- The voltage at port 1 is thus:

$$V_1(z_1=z_{1P})=V_1^+(z_1=z_{1P})+V_1^-(z_1=z_{1P})=-j2+(-j0.2)=-j2.2=2.2e^{j(-\pi/2)}$$

Example – 3 (contd.)

- Similarly, since port 2 is **matched**: $V_2^+(z_2 = z_{2P}) = 0$

- Therefore: $V_2^-(z_2 = z_{2P}) = S_{21}V_1^+(z_1 = z_{1P}) = j0.7(-j2) = 1.4$

  - The voltage at port 2 is thus:

$$V_2(z_2 = z_{2P}) = V_2^+(z_2 = z_{2P}) + V_2^-(z_2 = z_{2P}) = 0 + 1.4 = 1.4 = 1.4e^{-j0}$$

2. **The port currents** can be determined from the results of the previous section

$$I_1(z_1 = z_{1P}) = I_1^+(z_1 = z_{1P}) - I_1^-(z_1 = z_{1P}) = \frac{V_1^+(z_1 = z_{1P})}{Z_0} - \frac{V_1^-(z_1 = z_{1P})}{Z_0}$$

$$\Rightarrow I_1(z_1 = z_{1P}) = -j\frac{2.0}{50} + j\frac{0.2}{50} = -j\frac{1.8}{50} = -j0.036 = 0.036e^{-j\pi/2}$$

$$I_2(z_2 = z_{2P}) = I_2^+(z_2 = z_{2P}) - I_2^-(z_2 = z_{2P}) = \frac{V_2^+(z_2 = z_{2P})}{Z_0} - \frac{V_2^-(z_2 = z_{2P})}{Z_0}$$

$$\Rightarrow I_2(z_2 = z_{2P}) = \frac{0}{50} - \frac{1.4}{50} = -\frac{1.4}{50} = -0.028 = 0.028e^{+j\pi}$$

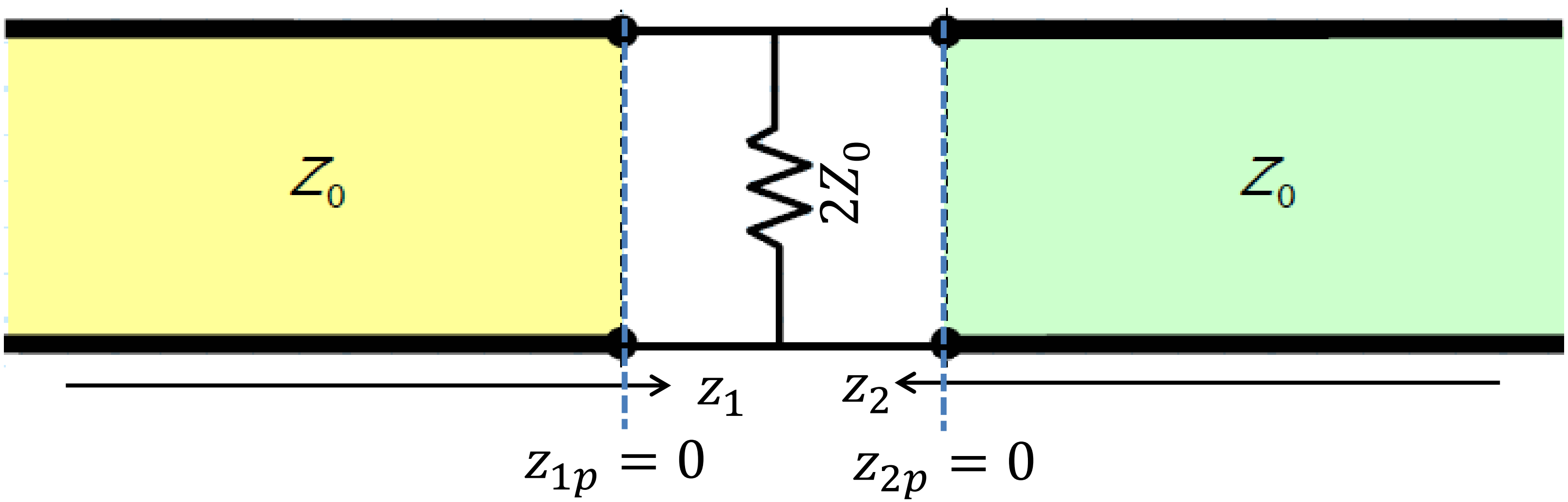
Example – 3 (contd.)

3. The **net power** flowing into port 1 is: $\Delta P_1 = P_1^+ - P_1^-$

$$\Rightarrow \Delta P_1 = \frac{\left|V_1^+\right|^2}{2Z_0} - \frac{\left|V_1^-\right|^2}{2Z_0} \qquad \Rightarrow \Delta P_1 = \frac{(2)^2 - (0.2)^2}{2(50)} = 0.0396 \text{ Watts}$$

Example – 4

- determine the **scattering matrix** of this two-port device:



Q: OK, but how can we **determine** the scattering matrix of a device?
A: We must carefully apply our **transmission line theory**!

Q: Determination of the Scattering Matrix of a multi-port device would seem to be particularly laborious. Is there any way to simplify the process?
A: Many (if not most) of the useful devices made by us humans exhibit a high degree of **symmetry**. This can greatly **simplify** circuit analysis—if we **know how** to exploit it!

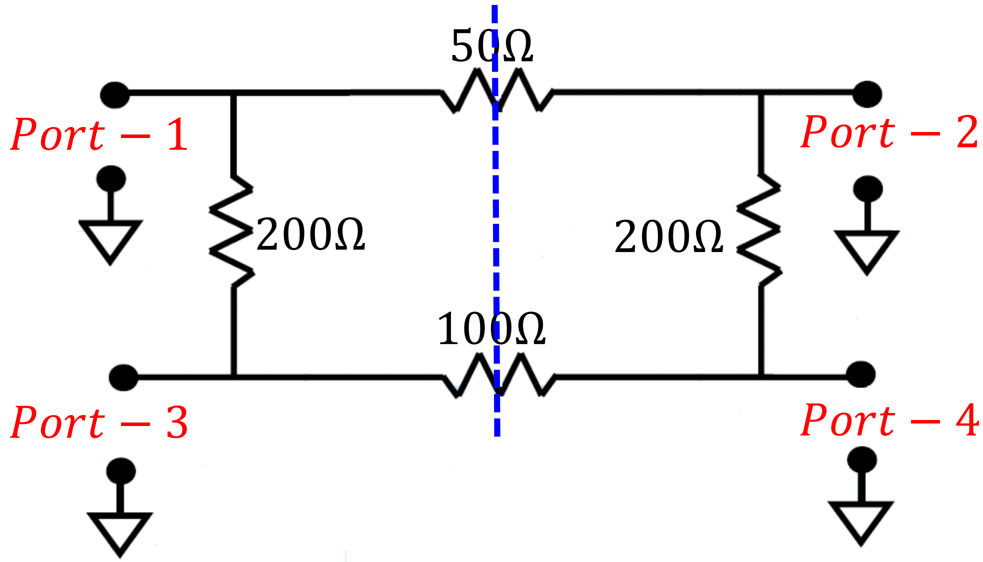
Q: Is there any **other** way to use circuit symmetry to our advantage?
A: Absolutely! One of the most **powerful** tools in circuit analysis is **Odd-Even Mode** analysis.

Circuit Symmetry

- For example consider these symmetric multi-port circuits:

$1 \rightarrow 2$ $3 \rightarrow 4$

$2 \rightarrow 1$ $4 \rightarrow 3$

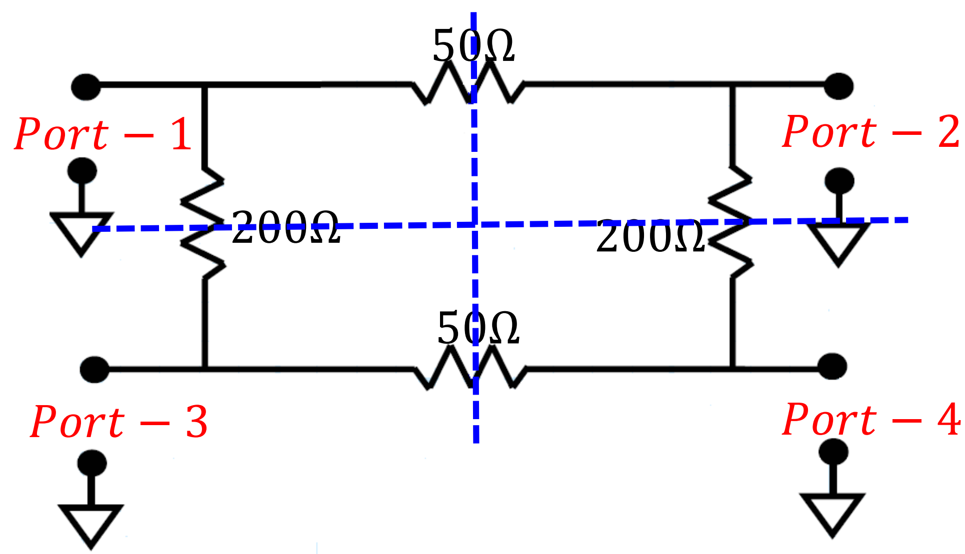


- Or this circuit: which is **congruent** under these permutations:

$1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 2$

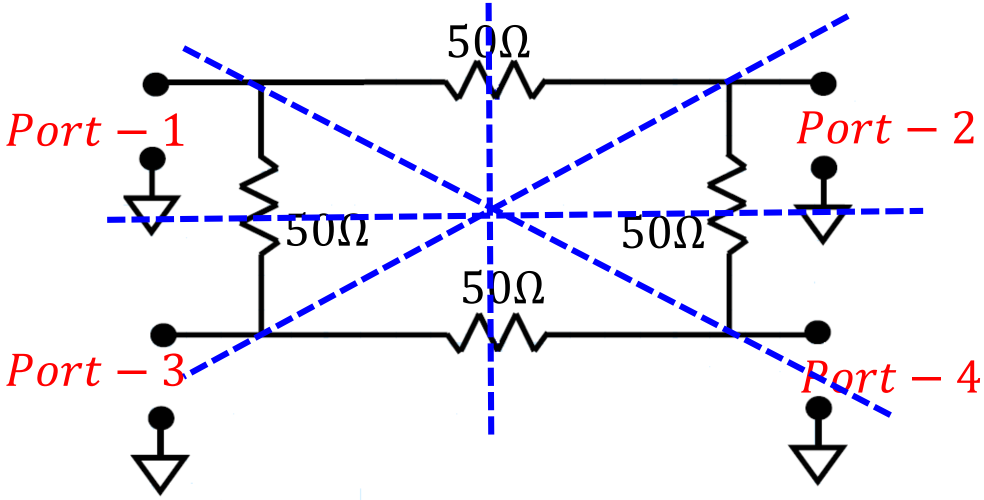
$1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 3$

$1 \rightarrow 4, 2 \rightarrow 3, 3 \rightarrow 2, 4 \rightarrow 1$



Circuit Symmetry (contd.)

- Or this circuit with: which is **congruent** under these permutations:



$$1 \to 3, 2 \to 4, 3 \to 1, 4 \to 2$$

$$1 \to 2, 2 \to 1, 3 \to 4, 4 \to 3$$

$$1 \to 4, 2 \to 3, 3 \to 2, 4 \to 1$$

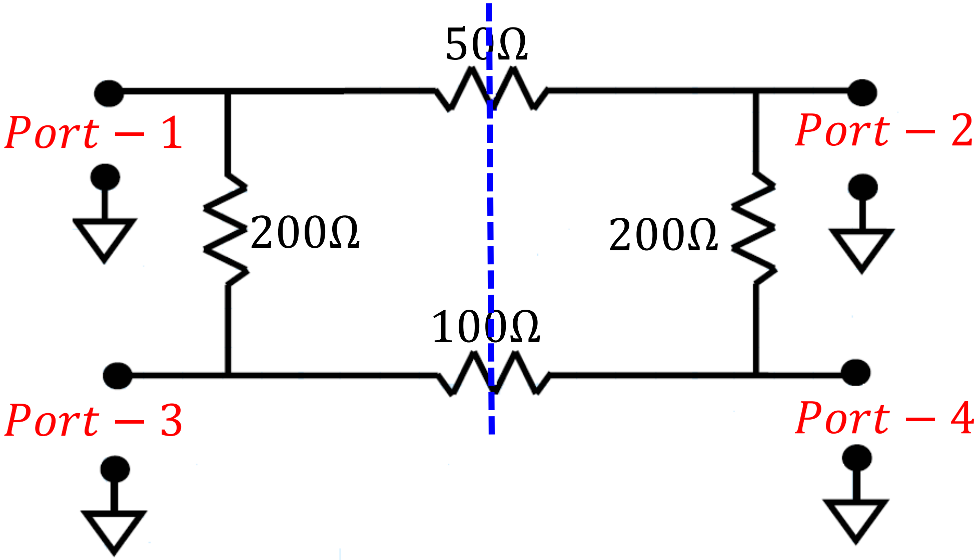
$$1 \to 4, 2 \to 2, 3 \to 3, 4 \to 1$$

$$1 \to 1, 2 \to 3, 3 \to 2, 4 \to 4$$

The **importance** of this can be seen when considering the scattering matrix, impedance matrix, or admittance matrix of these networks.

Circuit Symmetry (contd.)

- For **example**, consider again this **symmetric circuit**:



- This four-port network has a single plane of **reflection symmetry**, and thus is congruent under the permutation:

$$1 \to 2, 2 \to 1, 3 \to 4, 4 \to 3$$

- So, since (for example) 1→2, we find that for this circuit:

$$\boldsymbol{S_{11} = S_{22}} \qquad \boldsymbol{Z_{11} = Z_{22}} \qquad \boldsymbol{Y_{11} = Y_{22}}$$

must be true!

- Or, since 1→2 and 3→4 we find:

$$\boldsymbol{S_{13} = S_{24}} \qquad \boldsymbol{Z_{13} = Z_{24}} \qquad \boldsymbol{Y_{13} = Y_{24}}$$

$$\boldsymbol{S_{31} = S_{42}} \qquad \boldsymbol{Z_{31} = Z_{42}} \qquad \boldsymbol{Y_{31} = Y_{42}}$$

- Continuing for **all** elements of the permutation, for this symmetric circuit, the s-matrix **must** have **this** form:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{13} & S_{14} \\ S_{21} & S_{11} & S_{14} & S_{13} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

impedance and **admittance** matrices would likewise have this same form.

Circuit Symmetry (contd.)

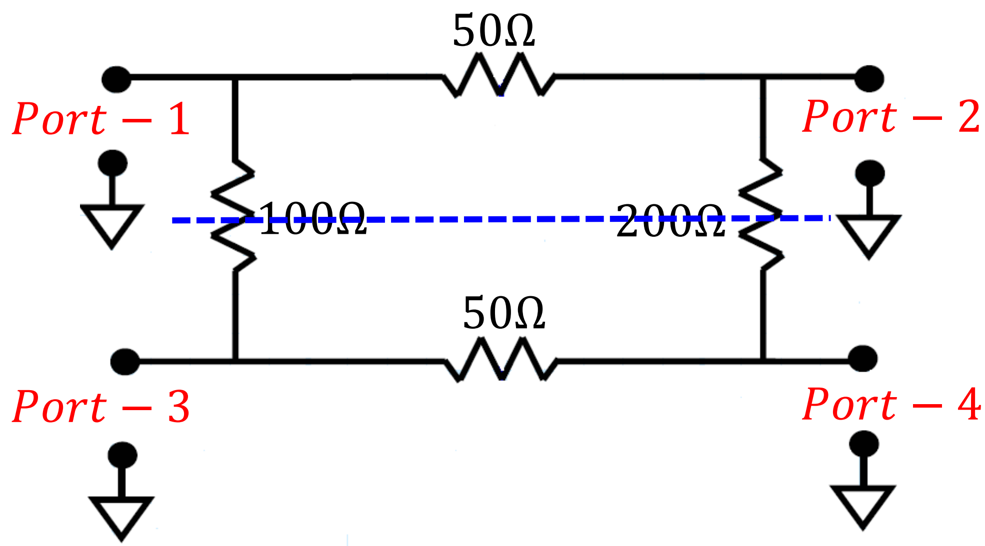
- Note there are just **8** independent elements in this matrix. If we also consider **reciprocity** (a constraint independent of symmetry) we find that $S_{31} = S_{13}$ and $S_{41} = S_{14}$, and the matrix reduces further to one with just **6** independent elements:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

- Or, for circuits with **this** symmetry:

$$1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 2$$

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{22} \end{bmatrix}$$

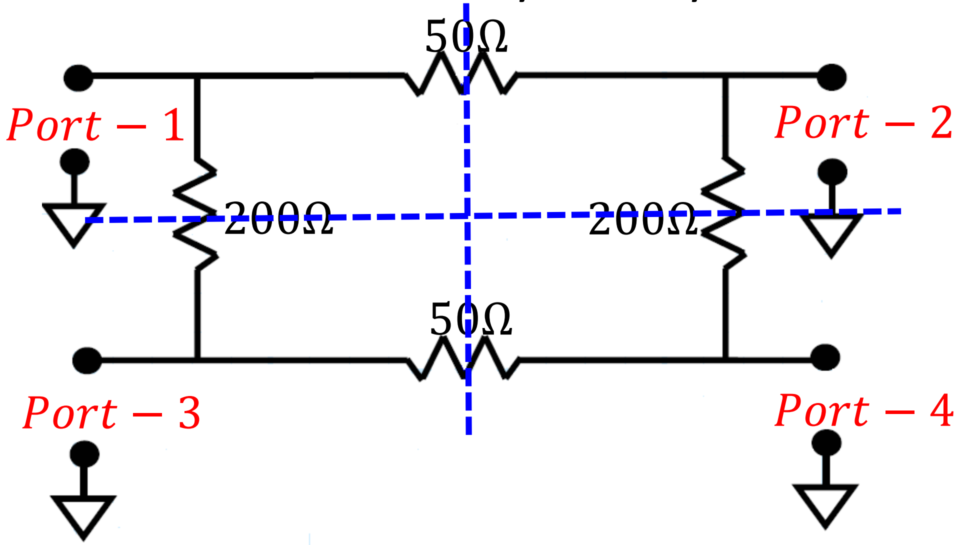


Q: Interesting. But **why do we care**?

A: This will greatly **simplify** the analysis of this symmetric circuit, as we need to determine **only** six matrix elements!

Circuit Symmetry (contd.)

- For a circuit with symmetry:

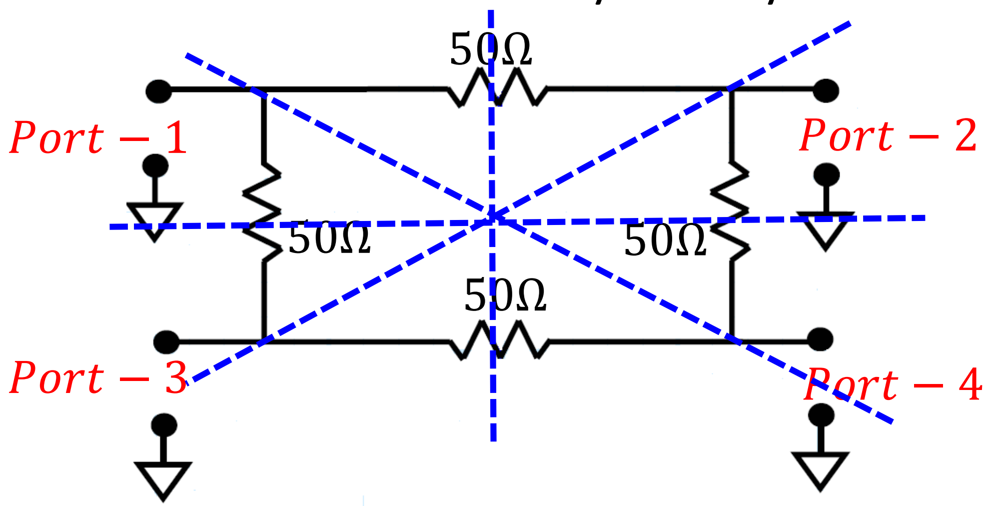


- the impedance (or scattering, or admittance) matrix has the form:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{11} \end{bmatrix}$$

Note: there are just **four** independent values!

- For a circuit with **such** symmetry:



- the admittance (or scattering, or impedance) matrix has the form:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{21} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{21} \\ S_{21} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{21} & S_{21} & S_{11} \end{bmatrix}$$

Note: there are just **three** independent values!

Circuit Symmetry (contd.)

- One more interesting thing (yet **another** one!); recall that we earlier found that a matched, lossless, reciprocal **4-port** device must have a scattering matrix with one of **two forms**:

$$S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Symmetric

$$S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Anti-symmetric

- The "symmetric solution" has the **same form** as the scattering matrix of a circuit with symmetry, reciprocity, and matched ports!

$$S = \begin{bmatrix} 0 & S_{21} & S_{31} & 0 \\ S_{21} & 0 & 0 & S_{31} \\ S_{31} & 0 & 0 & S_{21} \\ 0 & S_{31} & S_{21} & 0 \end{bmatrix}$$

Q: Does this mean that a matched, lossless, reciprocal four-port device with the "symmetric" scattering matrix **must** exhibit **certain type** of symmetry?

A: That's **exactly** what it means!

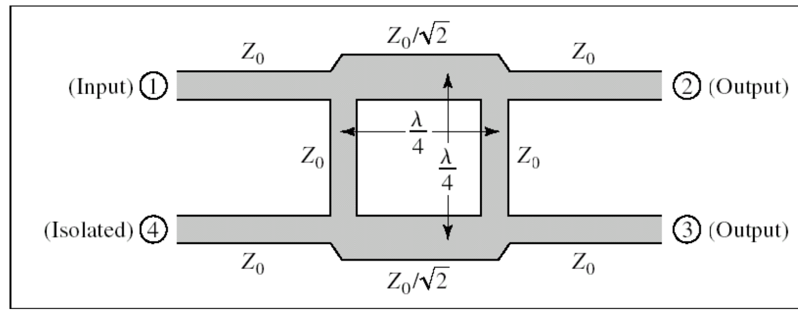
- Not only can we determine from the **form** of the scattering matrix **whether** a particular design is possible (e.g., a matched, lossless, reciprocal 3-port device is impossible), we can also determine the **general structure** of a possible solutions.

Circuit Symmetry (contd.)

- Likewise, the "anti-symmetric" matched, lossless, reciprocal four-port network **must** exhibit symmetry!

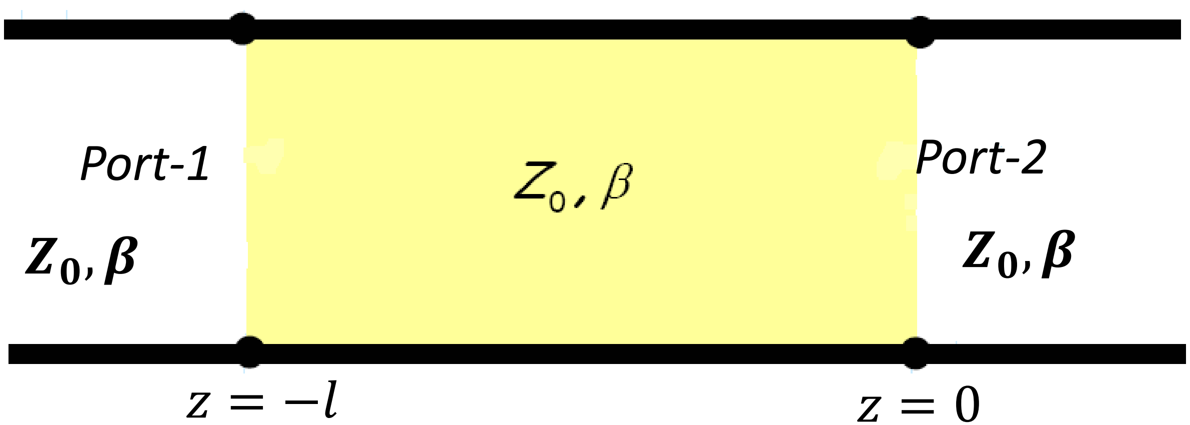
$$S = \begin{bmatrix} 0 & S_{21} & S_{31} & 0 \\ S_{21} & 0 & 0 & -S_{31} \\ S_{31} & 0 & 0 & S_{21} \\ 0 & -S_{31} & S_{21} & 0 \end{bmatrix}$$

We'll see just what these symmetric, matched, lossless, reciprocal four-port circuits actually are later in the course!



Example – 5

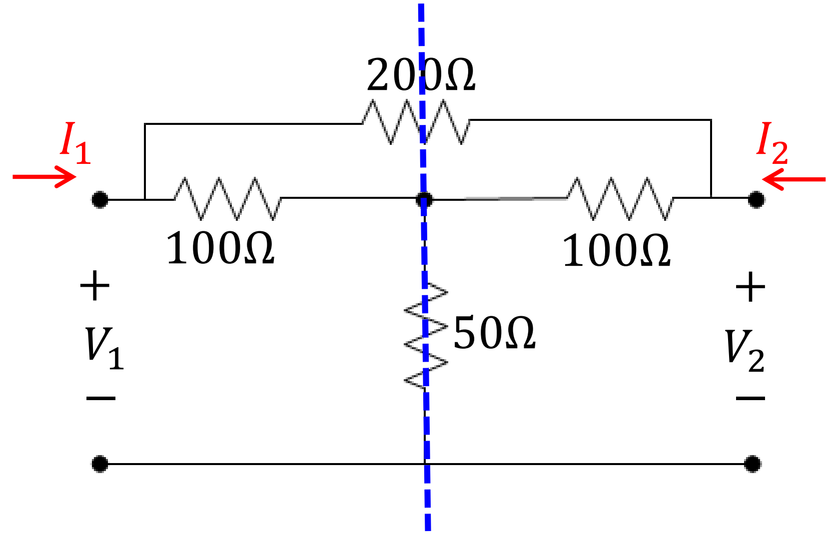
- determine the scattering matrix of the simple two-port device shown below:



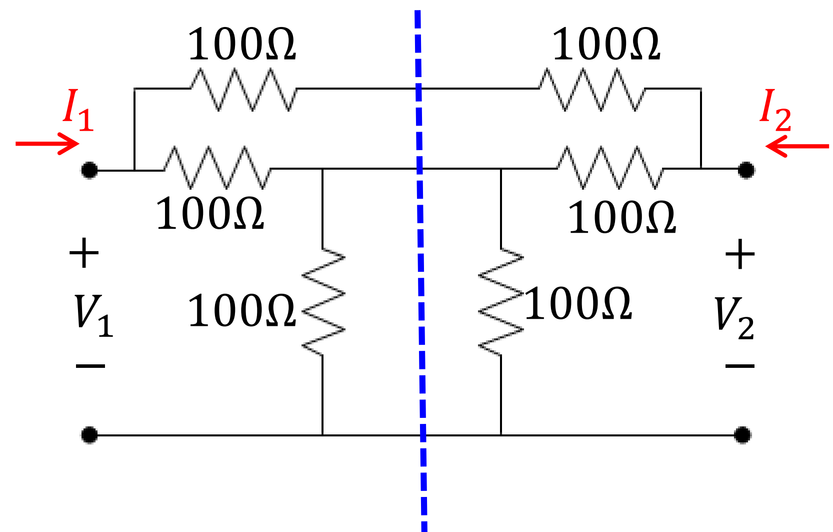
$$S = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix}$$

Symmetric Circuit Analysis

- Consider this symmetric **two-port** device:



Q: Yikes! The plane of reflection symmetry slices through two resistors. What can we do about that?



A: Resistors are easily split into two equal pieces: the 200Ω resistor into two 100Ω resistors in **series**, and the 50Ω resistor as two 100 Ω resistors in **parallel**.

- Recall that the **symmetry** of this 2-port device leads to **simplified** network matrices:

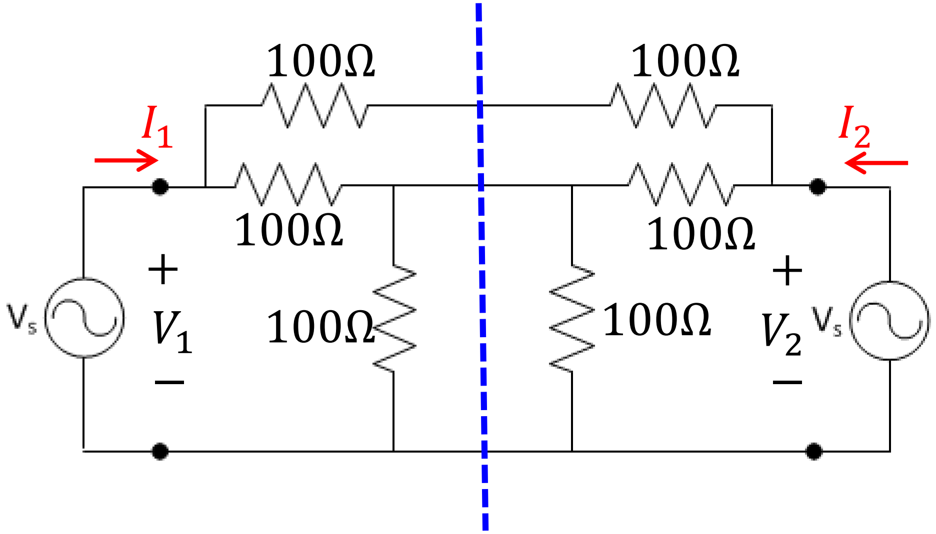
$$S = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{11} \end{bmatrix}$$

Symmetric Circuit Analysis (contd.)

Q: can circuit symmetry likewise simplify the procedure of **determining** these elements? In other words, can symmetry be used to **simplify circuit analysis**?

A: You bet!

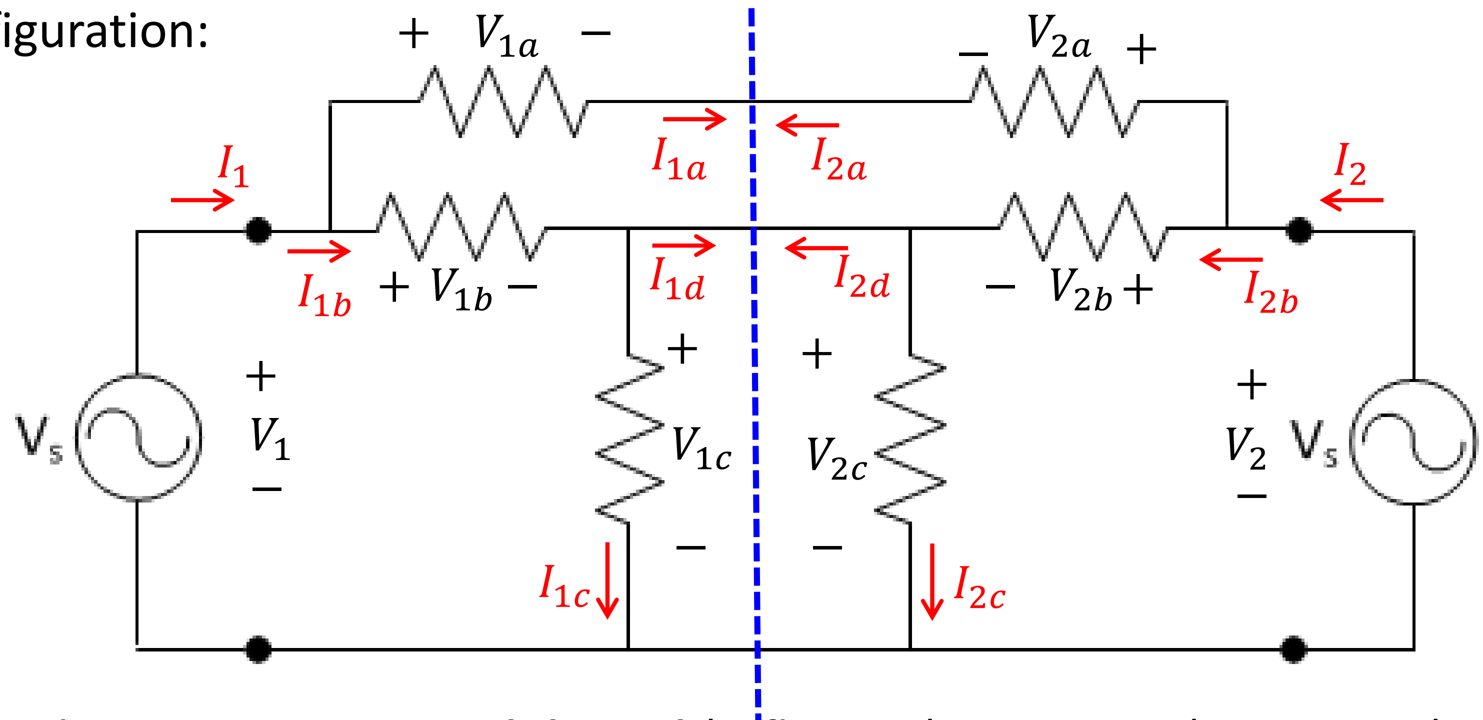
- First, consider the case where we **attach sources** to circuit in a way that **preserves** the circuit **symmetry**:



But remember! In order for **symmetry to be preserved**, the source values on both sides (i.e, V_s) must be **identical**!

Symmetric Circuit Analysis (contd.)

- Consider the **voltages** and **currents** within this circuit under this symmetric configuration:



- Since this circuit possesses **bilateral** (reflection) symmetry (1→2, 2→1), symmetric currents and voltages must be equal:

$$V_1 = V_2 \qquad V_{1a} = V_{2a} \qquad V_{1b} = V_{2b} \qquad V_{1c} = V_{2c}$$

$$I_1 = I_2 \qquad I_{1a} = I_{2a} \qquad I_{1b} = I_{2b} \qquad I_{1c} = I_{2c} \qquad I_{1d} = I_{2d}$$

Q: Wait! This **can't** possibly be correct! Look at currents I_{1a} and I_{2a}, as well as currents I_{1d} and I_{2d}. From KCL, **this** must be true:

$$I_{1a} = -I_{2a}$$
$$I_{1d} = -I_{2d}$$

Symmetric Circuit Analysis (contd.)

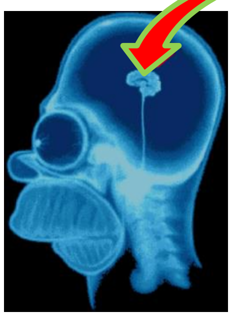
- Yet **you** say that **this** must be true: $I_{1a} = I_{2a}$ $I_{1d} = I_{2d}$

There is an obvious contradiction here! There is no way that both sets of equations can simultaneously be correct, is there?

A: Actually there **is**! There is **one** solution that will satisfy **both** sets of equations:

$$I_{1a} = I_{2a} = 0 \qquad I_{1d} = I_{2d} = 0$$

The currents are **zero**!



If you **think** about it, this makes **perfect sense**! The result says that **no current** will flow from one side of the symmetric circuit into the other.

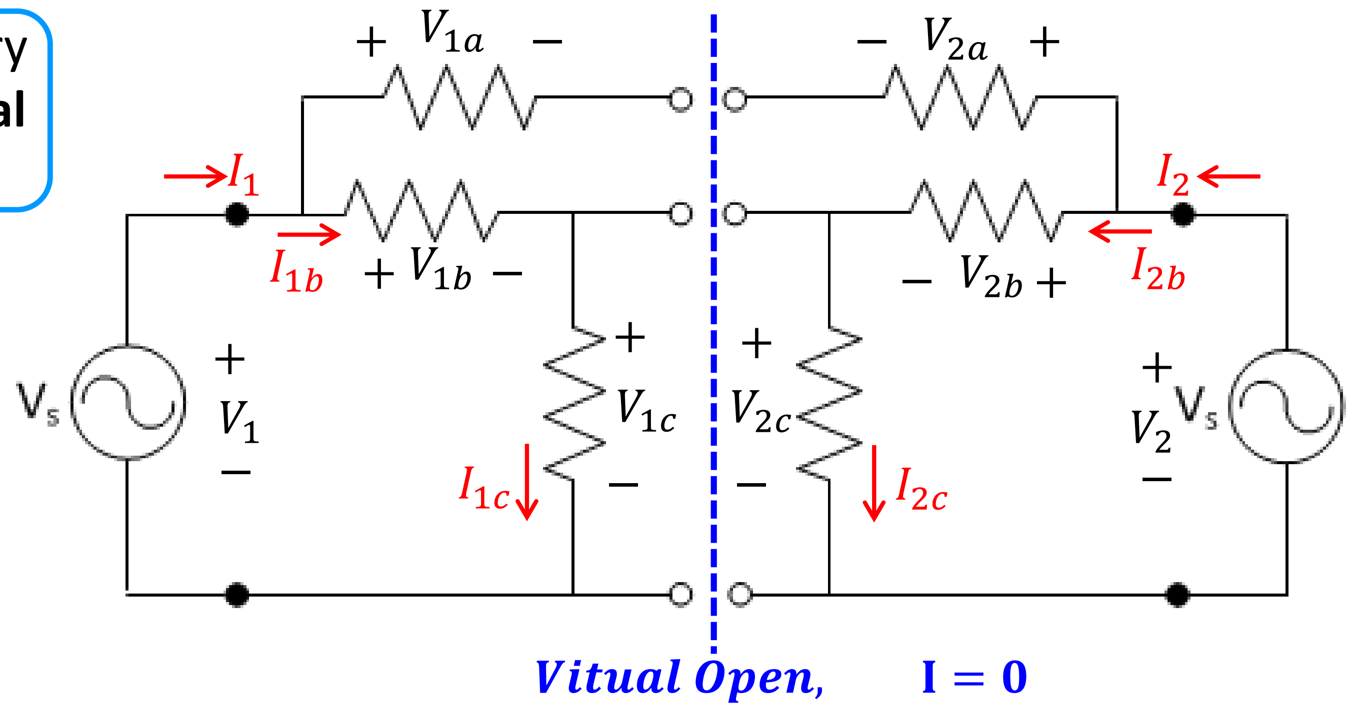
- If current **did** flow across the symmetry plane, then the circuit symmetry would be **destroyed**—one side would effectively become the "**source side**", and the other the "**load side**" (i.e., the source side delivers current to the load side).
- Thus, **no current** will flow **across** the reflection symmetry plane of a **symmetric circuit**—the symmetry plane thus acts as a **open circuit**!

Symmetric Circuit Analysis (contd.)

The plane of symmetry thus becomes a **virtual open**!

Q: So what?

A: So what! This means that our circuit can be split apart into two separate but identical circuits. Solve one half-circuit, and you have solved the other!



$$V_1 = V_2 = V_s$$

$$V_{1b} = V_{2b} = V_s/2$$

$$V_{1a} = V_{2a} = 0$$

$$V_{1c} = V_{2c} = V_s/2$$

$$I_1 = I_2 = V_s/200$$

$$I_{1c} = I_{2c} = V_s/200$$

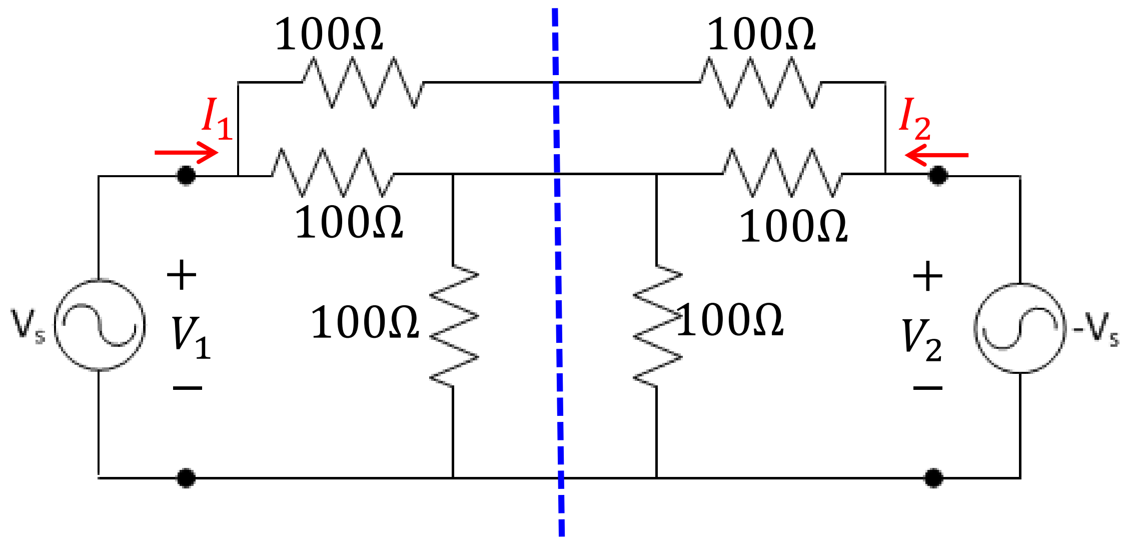
$$I_{1b} = I_{2b} = V_s/200$$

$$I_{1a} = I_{2a} = 0$$

$$I_{1d} = I_{2d} = 0$$

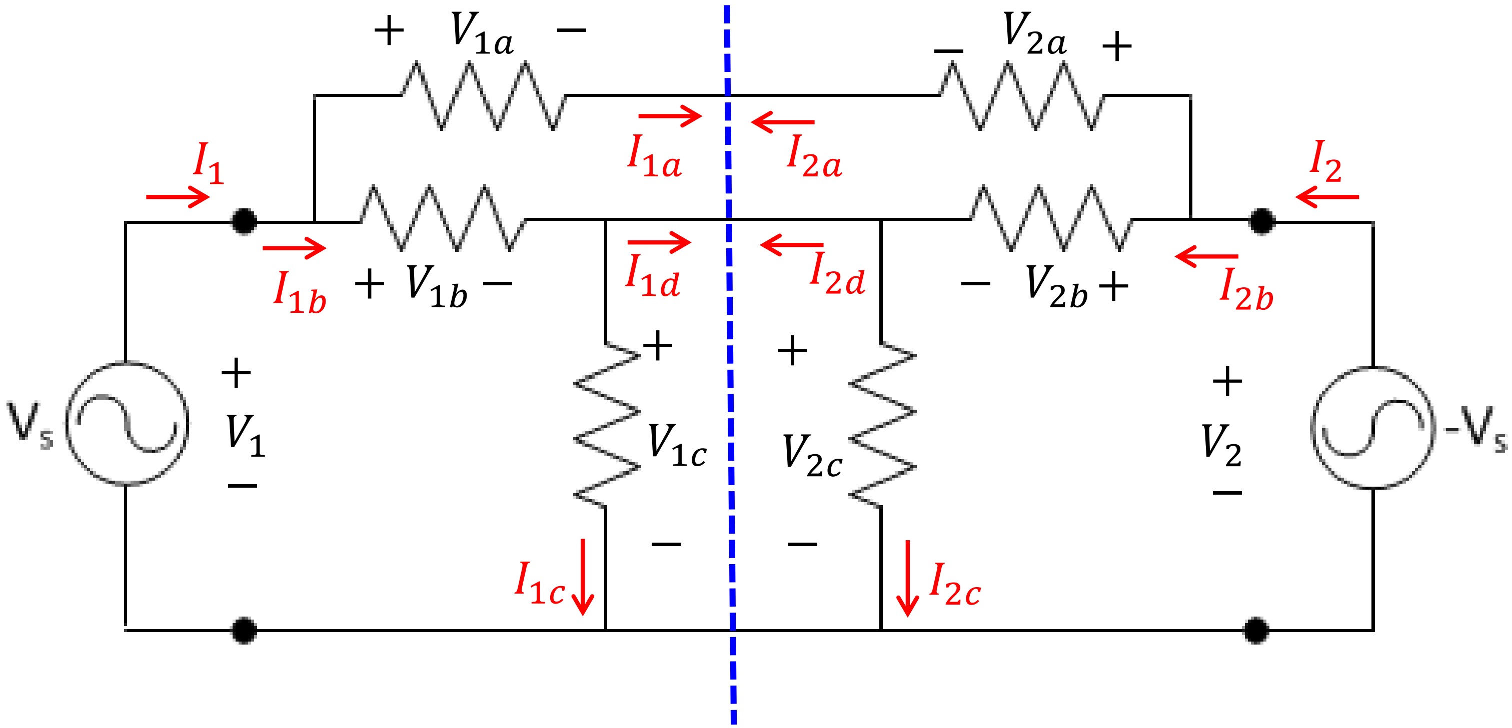
Asymmetric Circuit Analysis

- Now, consider **another** type of symmetry, where the sources are **equal** but **opposite** (i.e., **180 degrees** out of phase).



This situation still preserves the **symmetry** of the circuit— **somewhat.** The **voltages** and **currents** in the circuit will now posses **odd symmetry**—they will be **equal but opposite** (180 degrees out of phase) at symmetric points across the symmetry plane.

Asymmetric Circuit Analysis (contd.)

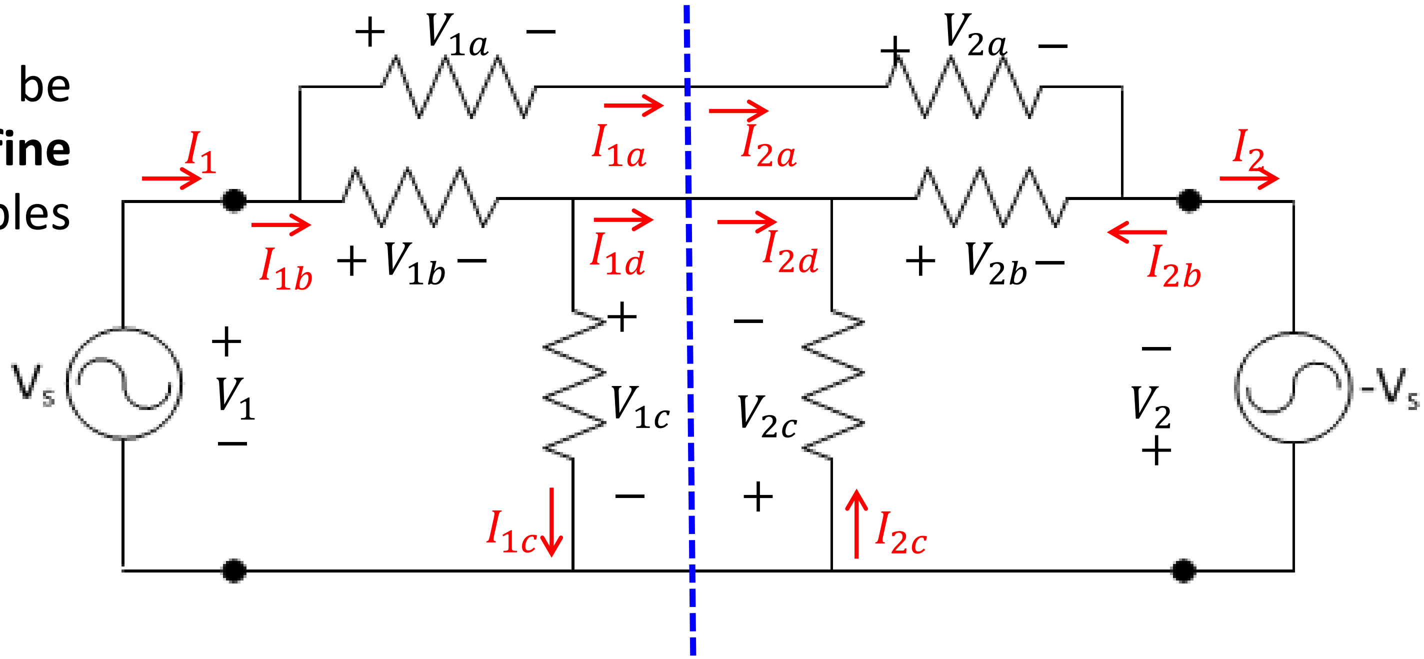


$$V_1 = -V_2 \qquad V_{1a} = -V_{2a} \qquad V_{1b} = -V_{2b} \qquad V_{1c} = -V_{2c}$$

$$I_1 = -I_2 \qquad I_{1a} = -I_{2a} \qquad I_{1b} = -I_{2b} \qquad I_{1c} = -I_{2c} \qquad I_{1d} = -I_{2d}$$

Asymmetric Circuit Analysis (contd.)

- Perhaps it would be easier to **redefine** the circuit variables as:



$V_1 = V_2 \qquad V_{1a} = V_{2a} \qquad V_{1b} = V_{2b} \qquad V_{1c} = V_{2c}$

$I_1 = I_2 \qquad I_{1a} = I_{2a} \qquad I_{1b} = I_{2b} \qquad I_{1c} = I_{2c} \qquad I_{1d} = I_{2d}$

Q: But wait! **Again** I see a problem. By **KVL** it is evident that: $V_{1c} = -V_{2c}$

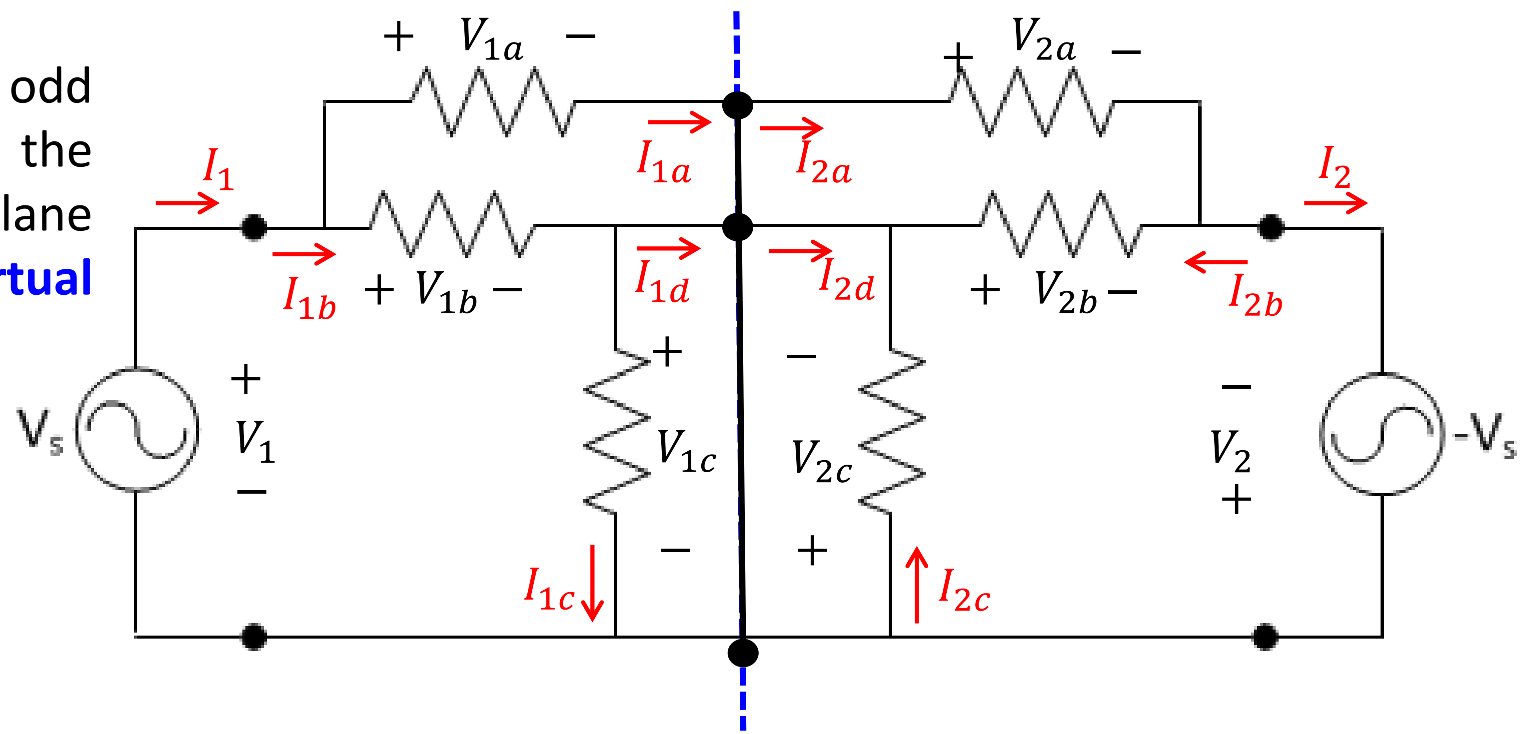
Yet **you** say that $V_{1c} = V_{2c}$ must be true!

Asymmetric Circuit Analysis (contd.)

A: Again, the solution to **both** equations is **zero**! $V_{1c} = V_{2c} = 0$

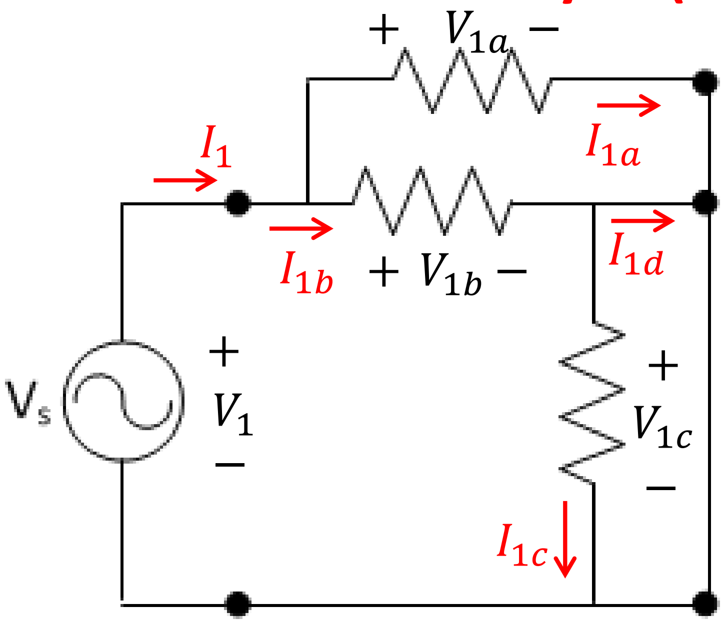
For the case of **odd symmetry**, the symmetric plane must be a plane of **constant potential** (i.e., constant voltage)—just like a **short circuit**!

- Thus, for odd symmetry, the symmetric plane forms a **virtual short**.



Vitual Short, $\mathbf{V = 0}$

Asymmetric Circuit Analysis (contd.)



$V_1 = V_s$

$V_{1b} = V_s$

$V_{1a} = V_s$

$V_{1c} = 0$

$I_1 = {}^{V_s}/_{50}$

$I_{1a} = {}^{V_s}/_{100}$

$I_{1b} = {}^{V_s}/_{100}$

$I_{1c} = 0$

$I_{1d} = {}^{V_s}/_{100}$