Lecture-16

• Continuity Equation and Relaxation Time
• Electrostatic Boundary Conditions
Continuity Equation

- The charge conservation principle says: the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume.

- Therefore, current $I_{\text{out}}$ coming out of the closed surface is:

\[
I_{\text{out}} = \oint \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_{\text{in}}}{dt}
\]

Q_{\text{in}} is the charge enclosed by the closed surface.

From Divergence Theorem

\[
\oint \mathbf{J} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{J} \, dv
\]

- We know that:

\[
\frac{-dQ_{\text{in}}}{dt} = -\frac{d}{dt} \int_{v} \rho_{v} \, dv
\]

If we agree to keep the volume constant:

\[
\frac{-dQ_{\text{in}}}{dt} = -\int_{v} \frac{\partial \rho_{v}}{\partial t} \, dv
\]
Continuity Equation (contd.)

- Therefore:

\[ \nabla \cdot J = \frac{\partial \rho_v}{\partial t} \]

Continuity Equation or Continuity of Current Equation

Continuity equation is derived from the principle of conservation of charge → It states that there can be no accumulation of charge at any point

For steady currents, \( \frac{\partial \rho_v}{\partial t} = 0 \), and therefore, \( \nabla \cdot \vec{J} = 0 \)

Total charge leaving the volume is the same as the charge entering the volume ← precursor to Kirchoff’s Current Law
Electrostatic Boundary Conditions

- A vector field is said to be spatially continuous if it doesn’t exhibit abrupt changes in either magnitude or direction as a function of position.
- Even though the electric field may be continuous in adjoining dissimilar media, it may well be discontinuous at the boundary between them.
- Boundary conditions specify how the components of fields tangential and normal to an interface between two media relate across the interface.
- To determine boundary conditions, we need to use Maxwell’s equations:

\[ \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} = 0 \]

\[ \nabla \times \mathbf{E} = 0 \]

\[ \oint_{S} \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} \]

\[ \nabla \cdot \mathbf{D} = \rho_v \]

Needless to say, these boundary conditions are equally valid for Electrodynamics
Dielectric – Dielectric Boundary Conditions

- Consider the interface between two dissimilar dielectric regions:

\[
\begin{array}{ccc}
\varepsilon_1 & \vec{E}_1(\vec{r}) & \vec{D}_1(\vec{r}) \\
\varepsilon_2 & \vec{E}_2(\vec{r}) & \vec{D}_2(\vec{r}) \\
\end{array}
\]

- Say that an electric field is present in both regions, thus producing also an electric flux density \( \vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r}) \)

**Q:** How are the fields in dielectric region 1 related to the fields in region 2?

**A:** They must satisfy the dielectric boundary conditions!
Dielectric – Dielectric Boundary Conditions (contd.)

- First, let’s write the fields at the dielectric interface in terms of their normal \( \vec{E}_n(\vec{r}) \) and tangential \( \vec{E}_t(\vec{r}) \) vector components:

\[
\begin{align*}
\vec{E}_1(\vec{r}) &= \vec{E}_{1n}(\vec{r}) + \vec{E}_{1t}(\vec{r}) \\
\vec{E}_2(\vec{r}) &= \vec{E}_{2n}(\vec{r}) + \vec{E}_{2t}(\vec{r})
\end{align*}
\]

- Our first boundary condition states that the tangential component of the electric field is continuous across a boundary.

- In other words:

\[ \vec{E}_{1t}(\vec{r}_b) = \vec{E}_{2t}(\vec{r}_b) \]

where \( \vec{r}_b \) denotes any point on the boundary (e.g., dielectric interface).
The **tangential** component of the electric field at one side of the dielectric boundary is **equal** to the tangential component at the other side!

- We can likewise consider the **electric flux densities** on the dielectric interface in terms of their **normal** and **tangential** components:

\[
\begin{align*}
\vec{D}_{1n}(\vec{r}) &= \varepsilon_1 \vec{E}_1(\vec{r}) \\
\vec{D}_{1t}(\vec{r}) &= \varepsilon_1 \vec{E}_1(\vec{r}) \\
\vec{D}_{2n}(\vec{r}) &= \varepsilon_2 \vec{E}_2(\vec{r}) \\
\vec{D}_{2t}(\vec{r}) &= \varepsilon_2 \vec{E}_2(\vec{r})
\end{align*}
\]
Dielectric – Dielectric Boundary Conditions (contd.)

- The second dielectric boundary condition states that the **normal** vector component of the **electric flux density** is **continuous** across the dielectric boundary.
- In other words:

\[
\bar{D}_{1n}(\bar{r}_b) = \bar{D}_{2n}(\bar{r}_b)
\]

where \(\bar{r}_b\) denotes any point on the boundary (e.g., dielectric interface).

- Since \(\bar{D}(\bar{r}) = \varepsilon \bar{E}(\bar{r})\), these boundary conditions can **likewise** be expressed as:

\[
\frac{\bar{D}_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{\bar{D}_{2t}(\bar{r}_b)}{\varepsilon_2}
\]

\[
\varepsilon_1 \bar{E}_{1n}(\bar{r}_b) = \varepsilon_2 \bar{E}_{2n}(\bar{r}_b)
\]
MAKE SURE YOU UNDERSTAND THIS:

These boundary conditions describe the relationships of the vector fields at the dielectric interface only (i.e., at points $r = r_b$)!!!! They say nothing about the value of the fields at points above or below the interface.
Dielectric – Dielectric Boundary Conditions (contd.)

Proof

- To derive the boundary conditions for tangential components of $\vec{E}$ and $\vec{D}$, let us consider the closed rectangular loop $abcd$.
- The line integral along this closed loop is ZERO.
- If $\Delta h \to 0$, the contributions to the line integral by the segments $bc$ and $da$ vanish.
Dielectric – Dielectric Boundary Conditions (contd.)

• Therefore:

\[
\oint \mathbf{E} \cdot d\mathbf{l} = \int_{a}^{b} \mathbf{E}_1 \cdot \mathbf{\hat{a}}_{l1} \, dl + \int_{c}^{d} \mathbf{E}_2 \cdot \mathbf{\hat{a}}_{l2} \, dl = 0
\]

Where, \( \mathbf{\hat{a}}_{l1} \) and \( \mathbf{\hat{a}}_{l2} \) are the unit vectors along segments \( ab \) and \( cd \).

• Next, we decompose \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) into components normal and tangential to the boundary:

\[
\mathbf{E}_1 = \mathbf{E}_{1n} + \mathbf{E}_{1t} \quad \mathbf{E}_2 = \mathbf{E}_{2n} + \mathbf{E}_{2t}
\]

• We also know that:

\( \mathbf{\hat{a}}_{l1} = -\mathbf{\hat{a}}_{l2} \)

• Thus the contour integral can be simplified to:

\[
\left( \mathbf{E}_1 - \mathbf{E}_2 \right) \cdot \mathbf{\hat{a}}_{l1} = 0 \quad \Rightarrow \quad \mathbf{E}_{1t} = \mathbf{E}_{2t}
\]

Thus the tangential component of the electric field is continuous across the boundary between any two media.
Dielectric – Dielectric Boundary Conditions (contd.)

- Upon decomposing $\vec{D}_1$ and $\vec{D}_2$ into components normal and tangential to the boundary and noting that $\vec{D}_{1t} = \varepsilon_1 \vec{E}_{1t}$ and $\vec{D}_{2t} = \varepsilon_2 \vec{E}_{2t}$, the boundary condition on the tangential component of the electric flux density is:

\[
\frac{\vec{D}_{1t}}{\varepsilon_1} = \frac{\vec{D}_{2t}}{\varepsilon_2}
\]
Dielectric – Dielectric Boundary Conditions (contd.)

- Now, apply Gauss’s law to determine boundary conditions on the normal components of \( \vec{E} \) and \( \vec{D} \).

- The total outward flux through the three surfaces of the small cylinder must equal the total charge enclosed in the cylinder.
- By letting the cylinder’s height \( \Delta h \to 0 \), the contribution to the total flux through the side surface goes to ZERO.
Dielectric – Dielectric Boundary Conditions (contd.)

• Even if each of the two media happens to contain free charge densities, the only free charge remaining in the collapsed cylinder is that distributed on the boundary \( Q_{enc} = \Delta s \times \rho_s \).

\[
\int_S \overline{D}.dS = Q_{enc} = \Delta s \times \rho_s \quad \Rightarrow \quad \int_{\text{top}} \overline{D}_1 \hat{a}_{n1} dS + \int_{\text{bottom}} \overline{D}_2 \hat{a}_{n2} dS = \Delta s \times \rho_s
\]

• It is important to remember that the normal unit vector at the surface of any medium is always defined to be in the outward direction away from that medium.

• Since, \( \hat{a}_{n1} = -\hat{a}_{n2} \)

\[
(\overline{D}_1 - \overline{D}_2) \cdot \hat{a}_{n2} = \rho_s
\]

• If \( \overline{D}_{1n} \) and \( \overline{D}_{2n} \) denoted the normal components of \( \overline{D}_1 \) and \( \overline{D}_2 \) along \( \hat{a}_{n2} \)

\[
\overline{D}_{1n} - \overline{D}_{2n} = \rho_s
\]
Dielectric – Dielectric Boundary Conditions (contd.)

• If no free charge exist at the boundary (i.e., charges are not deliberately placed at the boundary) then:

\[ \overrightarrow{D}_{1n} - \overrightarrow{D}_{2n} = 0 \]

• Thus the normal component of \( \overrightarrow{D} \) is continuous across the interface, that is \( D_n \) undergoes no change at the boundary.

• Furthermore:

\[ \varepsilon_1 \overrightarrow{E}_{1n} = \varepsilon_2 \overrightarrow{E}_{2n} \]

• The boundary conditions are usually applied in finding the electric field on one side of the boundary given the field on the other side.

• Beside this, we can use the boundary conditions to determine the “refraction” of the electric field across the interface.
Conductor – Dielectric Boundary Conditions

• Consider the case where region 2 is a **perfect conductor**:

$$\varepsilon_1 \frac{\hat{E}_1(r)}{r} = \hat{E}_{1n}(r)$$

$$\sigma_2 = \infty \text{ (i.e., perfect conductor)}$$

• Recall $\vec{E}(r) = 0$ in a perfect conductor. This of course means that **both** the tangential and normal component of $\vec{E}_2(r)$ are also equal to zero:

$$\vec{E}_{2t}(r) = 0 = \vec{E}_{2n}(r)$$
And, since the **tangential** component of the electric field is **continuous** across the boundary, we find that **at the interface**:

\[
\overrightarrow{E}_{1t}(\vec{r}_b) = \overrightarrow{E}_{2t}(\vec{r}_b) = 0
\]

**Think about what this means!** The **tangential** vector component in the dielectric (at the dielectric/conductor boundary) is **zero**. Therefore, the electric field **at the boundary** only has a **normal** component:

\[
\overrightarrow{E}_1(\vec{r}_b) = \overrightarrow{E}_{1n}(\vec{r}_b)
\]

Therefore, we can say:

**The electric field on the surface of a conductor is orthogonal** (i.e., normal) to the conductor.
Conductor – Dielectric Boundary Conditions (contd.)

**Q1:** What about the **electric flux density** \( \vec{D}_1(\vec{r}) \)?

**A1:** The relation \( \vec{D}_1(\vec{r}) = \varepsilon_1 \vec{E}_1(\vec{r}) \) is still of course valid, so that the **electric flux density** at the surface of the conductor must also be orthogonal to the conductor.

For boundary with surface charge density \( (\rho_s) \), \( \vec{D}_{1n}(\vec{r}) = \varepsilon_1 \vec{E}_{1n}(\vec{r}) = \rho_s \)

**Q2:** But, we learnt that the **normal** component of the **electric flux density** is **continuous** across an interface. If \( \vec{D}_{2n}(\vec{r}) = 0 \), why isn’t \( \vec{D}_{1n}(\vec{r}_b) = 0 \)?

**A2:** Great question! The answer comes from a more **general** form of the boundary condition.
Conductor – Dielectric Boundary Conditions (contd.)

- Consider again the interface of two dissimilar dielectrics. This time, however, there is some surface charge distribution $\rho_S(\vec{r}_b)$ (i.e., free charge!) at the dielectric interface:

  \[ \hat{a}_n \left[ \vec{D}_{1n}(\vec{r}_b) - \vec{D}_{2n}(\vec{r}_b) \right] = \rho_S(\vec{r}_b) \]

- The boundary condition for this situation turns out to be:

  \[ \vec{D}_{1n}(\vec{r}_b) - \vec{D}_{2n}(\vec{r}_b) = \rho_S(\vec{r}_b) \]
Conductor – Dielectric Boundary Conditions (contd.)

- Note that if \( \rho_S(\vec{r}_b) = 0 \), this boundary condition returns (both physically and mathematically) to the case studied earlier—the normal component of the electric flux density is continuous across the interface.

- This more general boundary condition is useful for the dielectric/conductor interface. Since \( \vec{D}_2(\vec{r}) = 0 \) in the conductor, we find that:

\[
\hat{a}_n \cdot \vec{D}_{ln}(\vec{r}_b) = \rho_S(\vec{r}_b) \quad \Rightarrow \quad D_{ln}(\vec{r}_b) = \rho_S(\vec{r}_b)
\]

In other words, the normal component of the electric flux density at the conductor surface is equal to the charge density on the conductor surface.
Conductor – Dielectric Boundary Conditions (contd.)

• Note in a perfect conductor, there is **plenty** of **free** charge available to form this charge density! Therefore, we find in **general** that $\vec{D}_{1n}(\vec{r}) \neq 0$ at the surface of a conductor.

\[ \varepsilon_1 \hat{a}_n \]

\[ \vec{D}_1(\vec{r}_b) \]

\[ \vec{D}_2(\vec{r}) = 0 \]

$\sigma_2 = \infty$ (i.e., perfect conductor)
Conductor – Dielectric Boundary Conditions (contd.)

**Summary:**

\[ \vec{E}_{1t}(\vec{r}_b) = 0 \quad \text{and} \quad \vec{D}_{1t}(\vec{r}_b) = 0 \]

\[ \vec{D}_{1n}(\vec{r}_b) = \rho_s(\vec{r}_b) \]

\[ \vec{E}_{1n}(\vec{r}_b) = \frac{\rho_s(\vec{r}_b)}{\varepsilon_1} \]

Again, these boundary conditions describe the fields **at the conductor/dielectric interface**. They say **nothing** about the value of the fields at locations above this interface.
Conductor – Dielectric Boundary Conditions (contd.)

• Thus under static conditions, the following conclusions can be made about a perfect conductor:

1. No electric field may exist within a conductor, i.e.,
\[ \rho_v = 0, \quad \vec{E} = 0 \]

2. Since, \( \vec{E} = -\nabla V = 0 \), there can be no potential difference between any two points in the conductor; that is, a conductor is an equipotential body.

3. An electric field must be external to the conductor and must be normal to its surface. i.e.,
\[ \vec{D}_t = \varepsilon_0 \varepsilon_r \vec{E}_t = 0, \quad \vec{D}_n = \varepsilon_0 \varepsilon_r \vec{E}_n = \rho_s \]

An important use of this concept is in the design of Electrostatic Shielding
Conductor – Free Space Boundary Conditions

• It is a special case of conductor-dielectric boundary conditions.
• Replace by $\varepsilon_r = 1$ in the expressions to get:

\[
\vec{D}_t = \varepsilon_0 \vec{E}_t = 0, \quad \vec{D}_n = \varepsilon_0 \vec{E}_n = \rho_s
\]

It should be noted once again that the electric field must approach a conducting surface normally.
Example: Boundary Conditions

- Two slabs of dissimilar dielectric material share a common boundary, as shown below. The respective electric field is also shown.

\[ \varepsilon_1 = 6\varepsilon_0 \quad \vec{E}_1(\vec{r}) = E_{x1}\hat{x} + E_{y1}\hat{y} \]

\[ \varepsilon_2 = 3\varepsilon_0 \quad \vec{E}_2(\vec{r}) = 2\hat{x} + 6\hat{y} \]

*In each dielectric region, let’s determine (in terms of \( \varepsilon_0 \)):
  (1) the electric field,  (2) the electric flux density,  (3) the bound volume charge density (i.e., the equivalent polarization charge density) within the dielectric, and  (4) the bound surface charge density (i.e., the equivalent polarization charge density) at the dielectric interface.*
Example: Boundary Conditions (contd.)

- Since we already know the electric field in the second region, let’s evaluate region 2 first.
- We can easily determine the electric flux density within the region:

\[ \vec{D}_2(\vec{r}) = \varepsilon_2 \vec{E}_2(\vec{r}) \]

\[ \vec{D}_2(\vec{r}) = 3\varepsilon_0 \left(2\hat{a}_x + 6\hat{a}_y\right) \]

\[ \therefore \vec{D}_2(\vec{r}) = 6\varepsilon_0\hat{a}_x + 18\varepsilon_0\hat{a}_y \]

- Likewise, the polarization vector within the region is:

\[ \vec{P}_2(\vec{r}) = \varepsilon_0 \chi_{e2} \vec{E}_2(\vec{r}) \]

\[ \vec{P}_2(\vec{r}) = \varepsilon_0 \left(\varepsilon_{r2} - 1\right) \left(2\hat{a}_x + 6\hat{a}_y\right) \]

\[ \Rightarrow \vec{P}_2(\vec{r}) = \varepsilon_0 \left(3 - 1\right) \left(2\hat{a}_x + 6\hat{a}_y\right) \]

\[ \therefore \vec{P}_2(\vec{r}) = 4\varepsilon_0\hat{a}_x + 12\varepsilon_0\hat{a}_y \]
Example: Boundary Conditions (contd.)

Q: Why did we determine the polarization vector? It is **not** one of the quantities this problem asked for!

A: True! But the problem **did** ask for the equivalent **bound charge densities** (both volume and surface) within the dielectric. We need to know polarization vector $\vec{P}(\vec{r})$ to find this **bound** charge!

- Recall the bound **volume** charge density is: $\rho_{vp}(\vec{r}) = -\nabla \cdot \vec{P}(\vec{r})$
- and the bound **surface** charge density is: $\rho_{sp}(\vec{r}) = \vec{P}(\vec{r}) \cdot \hat{n}$

- Since the polarization vector $\vec{P}(\vec{r})$ is a **constant** (i.e., it has precisely the same magnitude and direction at every point within region 2), we find that the divergence of $\vec{P}(\vec{r})$ is **zero**, and thus the volume bound charge density is zero within the region:

  $\rho_{vp2}(\vec{r}) = -\nabla \cdot \vec{P}_2(\vec{r}) \quad \rho_{vp2}(\vec{r}) = -\nabla \cdot \left( 4\varepsilon_0 \hat{a}_x + 12\varepsilon_0 \hat{a}_y \right) \quad \therefore \rho_{vp2}(\vec{r}) = 0$
**Example: Boundary Conditions (contd.)**

- However, we find that the **surface** bound charge density is **not** zero!
- Note that the unit vector normal to the **surface** of the bottom dielectric slab is $\hat{a}_{n2} = \hat{a}_y$:

\[
\hat{a}_{n2} = \hat{a}_y
\]

- Since the polarization vector is constant, we know that its value at the **dielectric interface** is likewise equal to $4\varepsilon_0\hat{a}_x + 12\varepsilon_0\hat{a}_y$. Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the top of region 2 (at the dielectric interface) is:

\[
\rho_{sp2}(\vec{r}_b) = \vec{P}_2(\vec{r}_b).\hat{a}_{n2}
\]

\[
\rho_{sp2}(\vec{r}_b) = \left(4\varepsilon_0\hat{a}_x + 12\varepsilon_0\hat{a}_y\right).\hat{a}_y
\]

\[
\therefore \rho_{sp2}(\vec{r}_b) = 12\varepsilon_0
\]
Example: Boundary Conditions (contd.)

• Now, let’s determine these same quantities for region 1 (i.e., the top dielectric slab).

**Q1:** How the heck can we do this? We don’t know anything about the fields in region 1!

**A1:** True! We don’t know $\vec{E}_1(\vec{r})$ or $\vec{D}_1(\vec{r})$ or even $\vec{P}_1(\vec{r})$. However, we know the next best thing—we know $\vec{E}_2(\vec{r})$ and $\vec{D}_2(\vec{r})$ and even $\vec{P}_2(\vec{r})$!

**Q2:** Huh!?!?

**A2:** We can use boundary conditions to transfer our solutions from region 2 into region 1!
**Example: Boundary Conditions (contd.)**

- First, we note that at the dielectric interface, the vector components of the electric fields **tangential** to the interface are $\vec{E}_{1t}(\vec{r}_b) = E_{1x}\hat{a}_x$ and $\vec{E}_{2t}(\vec{r}_b) = 2\hat{a}_x$:

  $$\vec{E}_{1t}(\vec{r}_b) = E_{1x}\hat{a}_x$$

  $$\vec{E}_{2t}(\vec{r}_b) = 2\hat{a}_x$$

- Thus, applying the **boundary condition** $\vec{E}_{1t}(\vec{r}_b) = \vec{E}_{2t}(\vec{r}_b)$, we find:

  $$E_{1x}\hat{a}_x = 2\hat{a}_x$$

  $\therefore E_{1x} = 2$
Example: Boundary Conditions (contd.)

- Likewise, we note that at the dielectric interface, the vector components of the electric fields normal to the interface are $\vec{E}_{1n}(\vec{r}_b) = E_{1y}\hat{a}_y$ and $\vec{E}_{2n}(\vec{r}_b) = 6\hat{a}_y$:

- Here, we can apply a second boundary condition, $\varepsilon_1\vec{E}_{1n}(\vec{r}_b) = \varepsilon_2\vec{E}_{2n}(\vec{r}_b)$:

  $6\varepsilon_0 E_{y1}\hat{a}_y = 3\varepsilon_0 \times 6\hat{a}_y \quad \Rightarrow \quad E_{1y}\hat{a}_y = 3\hat{a}_y$ 

- Thus, the electric field in the top region is:

  $\vec{E}_1(\vec{r}) = E_{1x}\hat{a}_x + E_{1y}\hat{a}_y$ 

  $\Rightarrow \quad \vec{E}_1(\vec{r}) = 2\hat{a}_x + 3\hat{a}_y$
Example: Boundary Conditions (contd.)

- We can then find the electric flux density by multiplying by the permittivity of region 1 ($\varepsilon_1 = 6\varepsilon_0$).

$$\vec{D}_1(\vec{r}) = \varepsilon_1 \vec{E}_1(\vec{r}) \quad \Rightarrow \quad \vec{D}_1(\vec{r}) = 12\varepsilon_0 \hat{a}_x + 18\varepsilon_0 \hat{a}_y$$

- Note we could have solved this problem another way!
- Instead of applying boundary conditions to $\vec{E}_2(\vec{r})$, we could have applied them to electric flux density $\vec{D}_2(\vec{r})$:

$$\vec{D}_2(\vec{r}) = 6\varepsilon_0 \hat{a}_x + 18\varepsilon_0 \hat{a}_y$$

- We know that the electric flux density within region 1 must be constant, i.e.:

$$\vec{D}_1(\vec{r}) = D_{1x} \hat{a}_x + D_{1y} \hat{a}_y$$
Example: Boundary Conditions (contd.)

- The vector fields $\vec{D}_1(\vec{r})$ and $\vec{D}_2(\vec{r})$ at the interface are related by the boundary conditions:

\[
\frac{\vec{D}_{1t}(\vec{r}_b)}{\varepsilon_1} = \frac{\vec{D}_{2t}(\vec{r}_b)}{\varepsilon_2}, \quad \vec{D}_{1n}(\vec{r}_b) = \vec{D}_{2n}(\vec{r}_b)
\]

- After simplification, we find that the electric flux density in region 1 is:

\[
\vec{D}_1(\vec{r}) = 12\varepsilon_0 \hat{a}_x + 18\varepsilon_0 \hat{a}_y
\]

Precisely the same result as before!

- We can then find the electric field in region 1 by dividing the obtained electric flux density by the dielectric permittivity:

\[
\vec{E}_1(\vec{r}) = \frac{\vec{D}_1(\vec{r})}{\varepsilon_1} = 2\hat{a}_x + 3\hat{a}_y
\]

the same result as before!
Example: Boundary Conditions (contd.)

- Now, finishing this problem, we need to find the polarization vector $\vec{P}_1(\vec{r})$:

$$\vec{P}_1(\vec{r}) = \varepsilon_0 (\varepsilon_r - 1) \vec{E}_1(\vec{r}) \quad \Rightarrow \quad \vec{P}_1(\vec{r}) = \varepsilon_0 (6-1)(2\hat{a}_x + 3\hat{a}_y)$$

$$\therefore \vec{P}_1(\vec{r}) = 10\varepsilon_0 \hat{a}_x + 15\varepsilon_0 \hat{a}_y$$

- Thus, the volume charge density of bound charge is again zero:

$$\rho_{vp1}(\vec{r}) = -\nabla \cdot \vec{P}_1(\vec{r}) \quad \Rightarrow \quad \rho_{vp1}(\vec{r}) = -\nabla \cdot (10\varepsilon_0 \hat{a}_x + 15\varepsilon_0 \hat{a}_y) \quad \therefore \rho_{vp1}(\vec{r}) = 0$$

However, we again find that the surface bound charge density is not zero!
Example: Boundary Conditions (contd.)

- Note that the unit vector normal to the bottom surface of the top dielectric slab points downward, i.e., $\hat{a}_{n1} = -\hat{a}_y$:

- Since the polarization vector is constant, we know that its value at the dielectric interface is likewise equal to:
  \[ \vec{P}_1(\vec{r}) = 10\varepsilon_0\hat{a}_x + 15\varepsilon_0\hat{a}_y \]

- Thus, the equivalent polarization (i.e., bound) surface charge density on the bottom of region 1 (at the dielectric interface) is:
  \[ \rho_{sp1}(\vec{r}_b) = \vec{P}_1(\vec{r}_b) \cdot \hat{a}_{n1} \]
  \[ \rho_{sp1}(\vec{r}_b) = \left(10\varepsilon_0\hat{a}_x + 15\varepsilon_0\hat{a}_y\right) \cdot (-\hat{a}_y) \]
  \[ \therefore \rho_{sp1}(\vec{r}_b) = -15\varepsilon_0 \]

- Now, we can determine the net surface charge density of bound charge that is lying on the dielectric interface:
  \[ \rho_{sp}(\vec{r}_b) = \rho_{sp1}(\vec{r}_b) + \rho_{sp2}(\vec{r}_b) \]
  \[ \rho_{sp}(\vec{r}_b) = -15\varepsilon_0 + 12\varepsilon_0 = -3\varepsilon_0 \]