Lecture – 14

• Energy Density in Electrostatic Field
• Conduction and Convection Current
• Conductors
• Ohm’s Law
• Resistor

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Energy Density in Electrostatic Field

- To determine the energy in an assembly of charges, let us first determine the amount of work needed to assemble them.
- Suppose, 3 point charges $Q_1$, $Q_2$ and $Q_3$ need to be assembled in empty space.

- No work is required to transfer $Q_1$ from infinity to $P_1$ as the space is free from any charge and thus without any electric field.
- The work done in transferring $Q_2$ from infinity to $P_2$ is $Q_2 V_{21}$.
- The work done in bringing $Q_3$ from infinity to $P_3$ is $Q_3 (V_{32} + V_{31})$.
- Therefore:

$$W_E = W_1 + W_2 + W_3$$

$$W_E = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$
Energy Density in Electrostatic Field (contd.)

- If the charges were positioned in reverse order, then:

\[
W_E = W_3 + W_2 + W_1 \\
W_E = 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13})
\]

- Let's combine the two expressions to get:

\[
2W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32}) \\
2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3
\]

\[
\therefore W_E = \frac{1}{2} (Q_1 V_1 + Q_2 V_2 + Q_3 V_3)
\]

where \(V_1, V_2\) and \(V_3\) are the total potentials at \(P_1, P_2\) and \(P_3\) respectively. In general, if there are \(n\) point charges then:

\[
\therefore W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k
\]

potential energy stored by all the charges
Energy Density in Electrostatic Field (contd.)

- For continuous charge distributions:
  \[ W_E = \frac{1}{2} \int \rho_i V dl \]
  \[ W_E = \frac{1}{2} \int \rho_s V dS \]
  \[ W_E = \frac{1}{2} \int \rho_v V dv \]

- We know from Maxwell’s equation for electrostatics:
  \[ \rho_v = \nabla \cdot \vec{D} \]

- Therefore:
  \[ W_E = \frac{1}{2} \int \rho_v V dv = \frac{1}{2} \int (\nabla \cdot \vec{D}) V dv \]

- We also know the relationship:
  \[ \nabla \cdot \vec{D} = \vec{D} \cdot \nabla V + V \left( \nabla \cdot \vec{D} \right) \]

- Therefore:
  \[ \Rightarrow V \left( \nabla \cdot \vec{D} \right) = \nabla \cdot \vec{D} - \vec{D} \cdot \nabla V \]

- Thus:
  \[ W_E = \frac{1}{2} \int_{v} \left( \nabla \cdot \vec{D} \right) dv - \frac{1}{2} \int_{v} \left( \vec{D} \cdot \nabla V \right) dv \]
Energy Density in Electrostatic Field (contd.)

- Application of Divergence Theorem leads to:

\[ W_E = \frac{1}{2} \oint_S (V \mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2} \int_V (D \cdot \nabla V) \, dv \]

For large surface \( \frac{1}{2} \oint_S (V \mathbf{D}) \cdot d\mathbf{S} \rightarrow 0 \)

- Thus:

\[ W_E = -\frac{1}{2} \int_V (\mathbf{D} \cdot \nabla V) \, dv \quad \mathbf{E} = -\nabla V \]

\[ W_E = \frac{1}{2} \int_V (\mathbf{D} \cdot \mathbf{E}) \, dv \]

- We know that \( \mathbf{D} = \varepsilon_0 \mathbf{E} \):

\[ \therefore W_E = \frac{1}{2} \int_V \varepsilon_0 E^2 \, dv \]

- Therefore energy density \( w_E \) [in J/m\(^3\)] is:

\[ w_E = \frac{dW_E}{dv} = \frac{1}{2} \mathbf{D} \cdot \mathbf{E} = \frac{1}{2} \varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0} \]
Example – 1

- If $V = \rho^2 z \sin \phi$, calculate the energy within the region defined by
  
  $1 < \rho < 4, -2 < z < 2, 0 < \phi < \frac{\pi}{3}$

**Start:**

$$\bar{E} = -\nabla V$$

$$\Rightarrow \bar{E} = -\left( \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z \right)$$

$$\therefore \bar{E} = -\left( 2\rho z \sin \phi \hat{a}_\rho + \rho \sin \phi \hat{a}_\phi + \rho^2 \sin \phi \hat{a}_z \right)$$

**Therefore:**

$$W_E = \frac{1}{2} \varepsilon_0 \int \left| \bar{E} \right|^2 dv$$

$$\frac{2W_E}{\varepsilon_0} = \iiint \left( 4\rho^2 z^2 \sin^2 \phi \hat{a}_\rho + \rho^2 z^2 \cos^2 \phi \hat{a}_\phi + \rho^4 \sin^2 \phi \hat{a}_z \right) \rho d\phi dz d\rho$$

$$\therefore W_E = \frac{1507.67 \times 10^{-9}}{2 \times 36\pi}$$
Energy Density in Electrostatic Field (contd.)

\[ W_E = \frac{1}{2} \int \overrightarrow{D} \cdot \overrightarrow{E} \, dv \]

\[ W_E = \frac{1}{2} \int \rho_v V \, dv \]

What these expressions mean is that it takes energy to assemble a charge distribution \( \rho_v(\vec{r}) \), or equivalently, an electric field \( \vec{E}(\vec{r}) \). This energy is stored until it is released— the charge density returns to zero.

Q: Is this energy stored in the fields \( \vec{E}(\vec{r}) \) and \( \vec{D}(\vec{r}) \), or by the charge \( \rho_v(\vec{r}) \)?

A: One equation for \( W_E \) would suggest that the energy is stored by the fields, while the other by the charge.

It turns out, either interpretation is correct! The fields \( \vec{E}(\vec{r}) \) and \( \vec{D}(\vec{r}) \) cannot exist without a charge density \( \rho_v(\vec{r}) \), and knowledge of the fields allow us to determine completely the charge density.

In other words, charges and the fields they create are “inseparable pairs”, since both must be present, we can attribute the stored energy to either quantity.
Electrostatic Discharge (ESD)

- It refers to the sudden transfer of static charge between objects at different electrostatic potential.
- For example, the “zap” you feel while walking on a synthetic carpet and then touching a metal doorknob.
- Design of mechanism to protect electronic devices, systems, and equipments against the static electricity is extremely important.

Please go through the additional materials posted on course URL to know about ESD, its impact, and the associated issues and solutions.
New Beginning

- We have been studying the electrostatics of free space (i.e., a vacuum).

But, the universe is full of stuff!

Q: Does stuff (material) affect our electrostatics knowledge?
A: ???
**Convection and Conduction Current**

- The current through a given area is the electric charge passing through the area per unit time.
  \[ I = \frac{dQ}{dt} \]

- Now, if the current \( \Delta I \) flows through a planar surface \( \Delta S \) then:
  \[ \frac{\Delta I}{\Delta S} = J \]
  \[ \Rightarrow \Delta I = J \Delta S \]
  When current density is perpendicular to the surface

- For the case when current density is not normal to the surface:
  \[ \Delta I = \vec{J} \cdot \Delta \vec{S} \]
  Total current flowing through the surface
  \[ I = \int_S \vec{J} \cdot d\vec{S} \]
  current “I” through S is the flux of current density \( \vec{J} \)
Convection and Conduction Current (contd.)

• “I” can be produced in three ways and therefore three kinds of current density exist: Convection Current Density, Conduction Current Density, and Displacement Current Density.
• The derived expression for current density is valid for any type of current.
• Convection current doesn’t involve conductors and as a consequence doesn’t satisfy Ohm’s Law.
• It occurs when current flows through an insulating medium such as liquid, rarefied gas, or a vacuum.
• A beam of electrons in a vacuum tube, for example, is a convection current.
• For example, if there is a charge flow, of density \( \rho_v \), at velocity \( \vec{u} = u y \hat{a}_y \) then:

\[
\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t}
\]
Convection and Conduction Current (contd.)

\[ \Delta I = \rho_v \Delta S \frac{\Delta y}{\Delta t} = \Delta S \rho_v u_y \]

- **Conduction current** requires conductor.
- A conductor is characterized by a large number of free electrons that provide conduction current due to an applied electric field.
- The force due to an electric field \( \vec{E} \) on an electron with charge \(-e\) is:
  \[ \vec{F} = -e \vec{E} \]
Convection and Conduction Current (contd.)

- Since the electron isn’t in free space, it will not experience an average acceleration under the influence of electric field.
- Instead, it suffers constant collisions with the atomic lattice and drifts from one atom to another.
- If electron of mass $m$ is moving in an electric field $\vec{E}$ with an average drift velocity $\vec{u}$ then:

$$\frac{m\bar{u}}{\tau} = -e\bar{E}$$

$$\bar{u} = -\frac{e\tau}{m} \bar{E}$$

$\tau$ is average time between collisions

- If there are $n$ electrons per unit volume:

$$\rho_v = -ne$$

$$\vec{J} = \rho_v \vec{u} = \frac{ne^2\tau}{m} \bar{E}$$

$$\vec{J} = \sigma \bar{E}$$

Point form of Ohm’s Law

Conductivity of Conductor

$$\frac{ne^2\tau}{m} = \sigma$$
Example – 2

- For the current density $\mathbf{j} = 10z\sin^2\phi \hat{\rho} A/m^2$, find the current through the cylindrical surface $\rho = 2, 1 \leq z \leq 5 \text{ m}$.

$$\mathbf{dS} = \rho d\phi dz \hat{\rho} \quad \Rightarrow \quad I = \int_{S} \mathbf{j} \cdot d\mathbf{S}$$

$$I = \int_{\phi=0}^{2\pi} \int_{z=1}^{5} 10z\sin^2\phi \rho d\phi dz \bigg|_{\rho=2} = 10(2) \int_{0}^{2\pi} \int_{0}^{2} (1 - \cos 2\phi) d\phi$$

$$\therefore \quad I = 240\pi = 754 \text{ A}$$
Example – 3

- A typical example of convective charge transport is found in the Van der Graaf generator where charge is transported on a moving belt from the base to the dome as shown in Figure.

- If a surface charge density $10^{-7} \, C/m^2$ is transported by the belt at a velocity of $2 \, m/s$, calculate the charge collected in $5s$. Take the width of the belt as $10 \, cm$.

\[
I = (\rho_s w)u \quad \Rightarrow \quad Q = It = (\rho_s w)ut \\
\Rightarrow Q = \left(10^{-7} \times 0.1\right) \times 2 \times 5
\]

\[\therefore Q = 100nC\]
Consider a very **simple** model of an **atom**:  

- ** electron (negative charge)  
- ** nucleus (positive charge)

Say an **electric field** is applied to this atom.  
- Note the field will apply a **force** on both the positively charged nucleus and the negatively charged electron.  
- However, these forces will move these particles in **opposite** directions!  
- This will lead to two situations.
Dielectrics and Conductors (contd.)

- In the **first** case, the atom may **stretch**, but the electron will remain **bound** to the atom:

\[
\hat{p} \quad \text{and} \quad \vec{E}(\vec{r})
\]

Note, an **electric dipole** has been created!
Dielectrics and Conductors (contd.)

- For the **second** case, the electron may **break free** from the atom, creating a positive ion and a **free electron**. We call these free charges, and the electric field will cause them to **move** in opposite directions.

\[ \vec{u}^- \quad \vec{E}(\vec{r}) \quad + \quad \vec{u}^+ \]

- **Moving charge**! We know what moving charge is.

Moving charge is **electric current** \( \vec{J}(\vec{r}) \).

These two examples provide a simple demonstration of what occurs when an electric field is applied to some **material** (e.g., plastic, copper, water, oxygen).
Dielectrics and Conductors (contd.)

1. Materials where the charges remain bound (and thus dipoles are created) are called **insulator** (or **dielectric**) materials.
2. Materials where the electrons are free to move are called **conductors**.

- Of course, materials consists of molecules with **many electrons**, and in general some electrons are **bound** and some are **free**. As a result, there are no **perfect** conductors or **perfect** insulators, although some materials are **very** close!
- Additionally, some materials lie between being a good conductor or a good insulator. We call these materials **semiconductors** (e.g., Silicon).
Ohm’s Law

• Recall that a positively charged particle will move in the direction of an electric field, whereas a negative charge will move in the opposite direction. Both types of charge, however, result in current moving in the same direction as the electric field:
Ohm’s Law (contd.)

Q: So, the direction of current density \( \vec{J}(\vec{r}) \) and electric field \( \vec{E}(\vec{r}) \) are the same. The question then is, how are their magnitudes related?

A: They are related by Ohm’s Law:

\[
\vec{J}(\vec{r}) = \sigma(\vec{r}) \vec{E}(\vec{r})
\]

The scalar value \( \sigma(\vec{r}) \) is called the material’s conductivity.

**Note:** the units of conductivity are:

\[
\sigma(\vec{r}) = \frac{\vec{J}(\vec{r})}{\vec{E}(\vec{r})} \left( \frac{\text{Ampere}}{m^2} \right) \left( \frac{\text{Volts}}{m} \right)^{-1}
\]

\[
= \frac{\vec{J}(\vec{r})}{\vec{E}(\vec{r})} \left( \frac{\text{Ampere}}{\text{Volts} \ast m} \right)
\]

\[
= \frac{\vec{J}(\vec{r})}{\vec{E}(\vec{r})} \left( \frac{1}{\text{Ohm} \ast m} \right)
\]

In other words, the unit of conductivity is conductance/unit length.
Ohm’s Law (contd.)

- We emphasize that conductivity $\sigma(\vec{r})$ is a material parameter. For example, the conductivity of **copper** is:

  $$\sigma_{\text{copper}} = 5.8 \times 10^7 \left[ \frac{1}{\Omega m} \right]$$

- and the conductivity of **polyethylene** (a plastic) is:

  $$\sigma_{\text{polyethylene}} = 1.5 \times 10^{-12} \left[ \frac{1}{\Omega m} \right]$$

Note the vast difference in conductivity between these two materials. Copper is a **conductor** and polyethylene is an **insulator**.

A **perfect insulator** (i.e., dielectric) is a material with $\sigma = 0$. In contrast, a **perfect conductor** is a material with $\sigma = \infty$.

Alternatively, we can say: For a perfect dielectric $\vec{J} = 0$, whereas for a perfect conductor $\vec{E} = 0$. 
**Georg Simon Ohm** (1789-1854) was the German physicist who in 1827 discovered the law that the current flowing through a conductor is proportional to the voltage and inversely proportional to the resistance. Ohm was then a professor of mathematics in Cologne. His work was **coldly** received! The Prussian minister of education announced that "a professor who preached such heresies was unworthy to teach science." Ohm resigned his post, went into academic exile for several years, and then left Prussia and became a professor in Bavaria.
Resistors

- Consider a **uniform** cylinder of material with mediocre to poor to pathetic **conductivity** $\sigma(\vec{r}) = \sigma$.
- This cylinder is centered on the $z$-axis, and has **length** $l$. The **surface area** of the ends of the cylinder is $S$.

![Diagram of a cylinder with current flowing through it](image)

- Say the cylinder has **current** “$I$” flowing into it (and thus out of it), producing a current **density** $\vec{J}(\vec{r})$.
- By the way, we can refer such a cylinder is commonly as a **resistor**!
Resistors (contd.)

**Q:** What is the **resistance** $R$ of this resistor, given length $l$, cross-section area $S$, and conductivity $\sigma$?

**A:** Let’s first begin with the circuit form of Ohm’s Law:

$$R = \frac{V}{I}$$

where $V$ is the potential difference between the two ends of the resistor (i.e., the voltage across the resistor), and $I$ is the current through the resistor.

- From **electrostatics**, we know that the potential difference $V$ is:

$$V = V_{ab} = \int_{a}^{b} E(\vec{r}).d\vec{l}$$

- and the current “$I$” is:

$$I = \int_{S} J(\vec{r}).d\vec{S}$$
Resistors (contd.)

• Thus, we can combine these expressions and find resistance $R$, expressed in terms of electric field $\vec{E}(\vec{r})$ within the resistor, and the current density $\vec{J}(\vec{r})$ within the resistor:

$$R = \frac{V}{I} = \frac{\int_{a}^{b} \overline{E(\vec{r}).dl}}{\int_{s} \overline{J(\vec{r}).dS}}$$

• Lets evaluate each integral in this expression to determine the resistance $R$ of the device described earlier!

1. The voltage $V$ is the potential difference $V_{ab}$ between point $a$ and point $b$:

$$V = V_{ab} = \int_{a}^{b} \overline{E(\vec{r}).dl}$$

Q: But, what is the electric field $\vec{E}(\vec{r})$?
A: The electric field within the resistor can be determined from Ohm’s Law:

$$\overline{E(\vec{r})} = \frac{\overline{J(\vec{r})}}{\sigma(\vec{r})}$$
Resistors (contd.)

• We can assume that the current density is approximately constant across the cross section of the cylinder:

\[
\overrightarrow{J}(\overrightarrow{r}) = J\hat{a}_z
\]

• Likewise, we know that the conductivity of the resistive material is a constant:

\[
\sigma(\overrightarrow{r}) = \sigma
\]

• As a result, the electric field within the resistor is:

\[
\overrightarrow{E}(\overrightarrow{r}) = \frac{\overrightarrow{J}(\overrightarrow{r})}{\sigma(\overrightarrow{r})} = \frac{J}{\sigma} \hat{a}_z
\]

• Therefore, integrating in a straight line along the z-axis from point a to point b, we find the potential difference V to be:

\[
V = \int_a^b \overrightarrow{E}(\overrightarrow{r}).d\overrightarrow{l} = \frac{J}{\sigma} \int_{Z_a}^{Z_b} \hat{a}_z.\hat{a}_z \, dz = \frac{J}{\sigma} \int_{Z_a}^{Z_b} dz = \frac{Jl}{\sigma}
\]
Resistors (contd.)

2. We likewise know that the current $I$ through the resistor is found by evaluating the surface integral:

$$I = \iiint_S J\hat{a}_z\hat{a}_z \, ds = J\iint_S ds = JS$$

• Therefore, the resistance $R$ of this particular resistor is:

$$R = \frac{V}{I} = \left(\frac{Jl}{\sigma}\right)\left(\frac{1}{JS}\right) = \frac{l}{\sigma S}$$

• An interesting result! Consider a resistor as sort of a “clogged pipe”. Increasing the cross-sectional area $S$ makes the pipe bigger, allowing for more current flow. In other words, the resistance of the pipe decreases, as predicted by the above equation.

• Likewise, increasing the length $l$ simply increases the length of the “clog”. The current encounters resistance for a longer distance, thus the value of $R$ increases with increasing length $l$. Again, this behavior is predicted by the equation shown above.
Resistors (contd.)

• For example, consider the case where we add two resistors together:

\[
R_1 = \frac{l_1}{\sigma S}
\]

\[
R_2 = \frac{l_2}{\sigma S}
\]

\[l_1\quad \quad \quad l_2\]

• We can view this case as a single resistor with a length \(l_1 + l_2\), resulting in a total resistance of:

\[R_{total} = \frac{l_1 + l_2}{\sigma S}\]

\[R_{total} = \frac{l_1}{\sigma S} + \frac{l_2}{\sigma S}\]

\[\therefore R_{total} = R_1 + R_2\]

But, this result is not the least bit surprising, as the two resistors are connected in series!
Resistors (contd.)

• Now let’s consider the case where two resistors are connected in a different manner:

\[ R_1 = \frac{l}{\sigma S_1} \]

\[ R_2 = \frac{l}{\sigma S_2} \]

• We can view this as a single resistor with a total cross sectional area of \( S_1 + S_2 \). Thus, its total resistance is:

\[ R_{total} = \frac{l}{\sigma (S_1 + S_2)} \]

\[ = \left[ \frac{\sigma (S_1 + S_2)}{l} \right]^{-1} = \left[ \frac{\sigma S_1}{l} + \frac{\sigma S_2}{l} \right]^{-1} \]

\[ \therefore R_{total} = \left[ \frac{1}{R_1} + \frac{1}{R_2} \right]^{-1} \]

Again, this should be no surprise, as these two resistors are connected in parallel.
Resistors (contd.)

**IMPORTANT NOTE:** The result $R = l/\sigma S$ is valid **only** for the resistor whose conductivity is a **constant** ($\sigma(\vec{r}) = \sigma$).

- If the conductivity is **not** a constant, then we **must** evaluate the potential difference across the resistor with the more **general** expression:

$$V_{ab} = \int_a^b \vec{E}(\vec{r}) \cdot d\vec{l} = \int_a^b \frac{\vec{J}(\vec{r})}{\sigma(\vec{r})} \cdot d\vec{l}$$
Example – 4

A lead \((\sigma = 5 \times 10^6 \, S/m)\) bar of square cross section has a hole bored along its length of 4m so that the cross section becomes as shown below. Find the resistance between the square ends.

- Since the cross section of the bar is uniform:
  \[ R = \frac{l}{\sigma S} \]

- Where:
  \[ S = d^2 - \pi r^2 \]
  \[ S = (3)^2 - \pi \left(\frac{1}{2}\right)^2 \]
  \[ S = \left(9 - \frac{\pi}{4}\right) \, cm^2 \]

- Therefore:
  \[ R = \frac{4}{5 \times 10^6 \times \left(9 - \frac{\pi}{4}\right) \times 10^{-4}} = 974 \, \mu\Omega \]