Lecture – 23

- Multi-port networks
- Impedance and Admittance Matrix
- Lossless and Reciprocal Networks
Introduction

- A pair of terminals through which a current may enter or leave a network is known as a *port*.
- Two-terminal devices or elements (such as resistors, capacitors, and inductors) result in one-port networks. Most of the circuits we have dealt with so far are two-terminal or one-port circuits.
- So far, considered the voltage across or current through a single pair of terminals—such as the two terminals of a resistor, capacitor, inductor.
- We have also studied four-terminal or two-port circuits involving op amps, and transformers.

A **two-port network** is an electrical network with two separate ports for input and output.
2-port Networks

- **Requirement of Matrix Formulation**

  In principle, N by N impedance matrix completely characterizes a linear N-port device. Effectively, the impedance matrix defines a multi-port device the way a $Z_L$ describes a single port device (e.g., a resistor).

  Linear networks can be completely characterized by parameters measured at the network ports without knowing the content of the networks.
Multiport Networks

• Networks can have any number of ports – however, analysis of a 2-port, 3-port or 4-port network is sufficient to explain the theory and the associated concepts.

- The ports can be characterized with many parameters (Z, Y, h, S, ABCD). Each has a specific advantage.
- For 2-port Network, each parameter set is related to 4 variables:
  o 2 independent variables for excitation
  o 2 dependent variables for response
The Impedance Matrix

- Let us consider the following 4-port network:

This could be a simple linear device or a large/complex linear system.

Either way, the network can be fully described by its impedance matrix.

The arbitrary locations are known as ports of the network.

Each cable has specific location that defines input impedances to the network.

Four identical cables used to connect this network to the outside world.

4-port Linear Microwave Network

- $I_1(z_1)$
- $V_1(z_1)$
- $I_2(z_2)$
- $V_2(z_2)$
- $I_3(z_3)$
- $V_3(z_3)$
- $I_4(z_4)$
- $V_4(z_4)$

$z_1 = z_{1P}$
$z_2 = z_{2P}$
$z_3 = z_{3P}$
$z_4 = z_{4P}$

$P_{zz}$
The Impedance Matrix (contd.)

- In principle, the current and voltages at the port-n of networks are given as:
  \[ V_n(z_n = z_{nP}) \quad I_n(z_n = z_{nP}) \]

- the simplified formulations are:
  \[ V_n = V_n(z_n = z_{nP}) \quad I_n = I_n(z_n = z_{nP}) \]

- If we want to say that there exists a non-zero current at port-1 and zero current at all other ports then we can write as:
  \[ I_1 \neq 0 \quad I_2 = I_3 = I_4 = 0 \]

- In order to define the elements of impedance matrix, there will be need to measure/determine the associated voltages and currents at the respective ports. Suppose, if we measure/determine current at port-1 and then voltage at port-2 then we can define:
  \[ Z_{21} = \frac{V_2}{I_1} \]

- Similarly, the trans-impedance parameters \( Z_{31} \) and \( Z_{41} \) are:
  \[ Z_{31} = \frac{V_3}{I_1} \quad Z_{41} = \frac{V_4}{I_1} \]

- We can define other trans-impedance parameters such as \( Z_{34} \) as the ratio between the complex values \( I_4 \) (into port-4) and \( V_3 \) (at port-3), given that the currents at all other ports (1, 2, and 3) are zero.
The Impedance Matrix (contd.)

• Therefore, the more generic form of trans-impedance is:

$$Z_{mn} = \frac{V_m}{I_n}$$

(given that $I_k = 0$ for all $k \neq n$)

How do we ensure that all but one port current is zero?

• Open the ports where the current needs to be zero

The ports should be opened! not the cables connected to the ports

• We can then define the respective trans-impedances as:

$$Z_{mn} = \frac{V_m}{I_n}$$

(given that all ports $k \neq n$ are open)
The Impedance Matrix (contd.)

- Once we have defined the trans-impedance terms by opening various ports, it is time to formulate the impedance matrix.
- Since the network is linear, the voltage at any port due to all the port currents is simply the coherent sum of the voltage at that port due to each of the currents.
- For example, the voltage at port-3 is: 
  \[ V_3 = Z_{34} I_4 + Z_{33} I_3 + Z_{32} I_2 + Z_{31} I_1 \]
- Therefore we can generalize the voltage for \( N \)-port network as: 
  \[ V_m = \sum_{n=1}^{N} Z_{mn} I_n \quad \Rightarrow \quad V = ZI \]
- Where \( I \) and \( V \) are vectors given as:
  \[ V = [V_1, V_2, V_3, \ldots, V_N]^T \quad I = [I_1, I_2, I_3, \ldots, I_N]^T \]
- The term \( Z \) is matrix given by: 
  \[ Z = \begin{bmatrix} Z_{11} & Z_{12} & \ldots & Z_{1n} \\ Z_{21} & \ddots & \vdots \\ \vdots & & \ddots \\ Z_{n1} & Z_{n2} & \ldots & Z_{nn} \end{bmatrix} \]
The Admittance Matrix

- Let us consider the 4-port network again:

\[ I_2(z_2) \quad + \quad - \quad V_2(z_2) \quad Z_2 = Z_{2P} \]

\[ I_1(z_1) \quad + \quad V_1(z_1) \quad Z_1 = Z_{1P} \]

\[ I_3(z_3) \quad + \quad - \quad V_3(z_3) \quad Z_3 = Z_{3P} \]

\[ I_4(z_4) \quad + \quad - \quad V_4(z_4) \quad Z_4 = Z_{4P} \]

This can be characterized using admittance matrix – if currents are taken as dependent variables instead of voltages.

The elements of admittance matrix are called trans-admittance parameters \( Y_{mn} \).
The Admittance Matrix (contd.)

• The trans-admittances \( Y_{mn} \) are defined as:
\[
Y_{mn} = \frac{I_m}{V_n}
\]
(given that \( V_k = 0 \) for all \( k \neq n \))

• It is apparent that the voltage at all but one port must be equal to zero. This can be ensured by short-circuiting the voltage ports.

The ports should be short-circuited! not the cable connected to the ports

• Now, since the network is linear, the current at any one port due to all the port voltages is simply the coherent sum of the currents at that port due to each of the port voltages.

Important
\[
Y_{mn} \neq \frac{1}{Z_{mn}}
\]
The Admittance Matrix (contd.)

- For example, the current at port-3 is:

- Therefore we can generalize the current for N-port network as:

- Where I and V are vectors given as:

  \[
  V = \begin{bmatrix} V_1, V_2, V_3, \ldots, V_N \end{bmatrix}^T 
  \]

  \[
  I = \begin{bmatrix} I_1, I_2, I_3, \ldots, I_N \end{bmatrix}^T 
  \]

- The term Y is matrix given by:

  \[
  Y = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mn} \end{bmatrix} 
  \]

  Admittance Matrix

- The values of elements in the admittance matrix are frequency dependents and often it is advisable to describe admittance matrix as:

  \[
  Y(\omega) = \begin{bmatrix} Y_{11}(\omega) & Y_{12}(\omega) & \cdots & Y_{1n}(\omega) \\ Y_{21}(\omega) & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ Y_{m1}(\omega) & Y_{m2}(\omega) & \cdots & Y_{mn}(\omega) \end{bmatrix} 
  \]

\[
I_3 = Y_{34}V_4 + Y_{33}V_3 + Y_{32}V_2 + Y_{31}V_1 
\]

\[
I_m = \sum_{n=1}^{N} Y_{mn}V_n \quad \Rightarrow I = YV
\]
The Admittance Matrix (contd.)

You said that: \[ Y_{mn} \neq \frac{1}{Z_{mn}} \]

Is there any relationship between admittance and impedance matrix of a given device?

**Answer:** Let us see if we can figure it out!

- Recall that we can determine the inverse of a matrix. Denoting the matrix inverse of the admittance matrix as \( Y^{-1} \), we find:
  \[ I = YV \]
  \[ \Rightarrow Y^{-1}I = Y^{-1}(YV) \]
  \[ \Rightarrow Y^{-1}I = (Y^{-1}Y)V \]
  \[ \Rightarrow Y^{-1}I = V \]

- We also know: \( V = ZI \)

\[ Z = Y^{-1} \quad \text{OR} \quad Y = Z^{-1} \]
Reciprocal and Lossless Networks

- We can classify multi-port devices or networks as either lossless or lossy; reciprocal or non-reciprocal. Let’s look at each classification individually.

**Lossless Network**

- A lossless network or device is simply one that cannot absorb power. This does not mean that the delivered power at every port is zero; rather, it means the total power flowing into the device must equal the total power exiting the device.

- A lossless device exhibits an impedance matrix with an interesting property. Perhaps not surprisingly, we find for a lossless device that the elements of its impedance matrix will be purely reactive:

  \[ \text{Re}(Z_{mn}) = 0 \]

  For a lossless device

- If the device is lossy, then the elements of the impedance matrix must have at least one element with a real (i.e., resistive) component.

- Furthermore, we can similarly say that if the elements of an admittance matrix are all purely imaginary (i.e., \( \text{Re}(Y_{mn}) = 0 \)), then the device is lossless.
Reciprocal and Lossless Networks (contd.)

Reciprocal Network

• Ideally, most **passive, linear** components will turn out to be **reciprocal**—regardless of whether the designer **intended** it to be or not!
• Reciprocity is a tremendously important characteristic, as it greatly **simplifies** an impedance or admittance matrix!
• Specifically, we find that a reciprocal device will result in a **symmetric** impedance and admittance **matrix**, meaning that:

\[
Z_{mn} = Z_{nm} \quad Y_{mn} = Y_{nm}
\]

For a reciprocal device

• **For example**, we find for a reciprocal device that \(Z_{23} = Z_{32}\), and \(Y_{12} = Y_{21}\). 
Reciprocal and Lossless Networks (contd.)

<table>
<thead>
<tr>
<th>$Z$</th>
<th>neither lossless nor reciprocal</th>
<th>lossless, but not reciprocal</th>
<th>lossless and reciprocal</th>
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<td>$\begin{bmatrix} j2 &amp; j0.1 &amp; j3 \ -j &amp; -j1 &amp; j1 \ j4 &amp; -j2 &amp; j0.5 \end{bmatrix}$</td>
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**Example – 1**

- determine the $Y$ matrix of this two-port device.

$$
\begin{bmatrix} \beta, Z_0 \\ V_1 \\ - \\ \beta, Z_0 \end{bmatrix}
\begin{bmatrix} 2R \\ R \end{bmatrix}
\begin{bmatrix} I_1 \\ \beta, Z_0 \\ I_2 \\ \beta, Z_0 \end{bmatrix}
= 
\begin{bmatrix} V_1 \\ + \\ - \end{bmatrix}
$$
Example – 1 (contd.)

**Step-1:** Place a short at port 2

\[
\beta, Z_0 \quad V_1 \quad 2R \quad R \quad V_2 = 0
\]

**Step-2:** Determine currents \( I_1 \) and \( I_2 \)

- Note that after the short was placed at port 2, both resistors are in parallel, with a potential \( V_1 \) across each.
  
  **Therefore current \( I_1 \) is**

\[
I_1 = \frac{V_1}{2R} + \frac{V_1}{R} = \frac{3V_1}{2R}
\]

- The current \( I_2 \) equals the portion of current \( I_1 \) through \( R \) but with opposite sign

\[
I_2 = -\frac{V_1}{R}
\]
Example – 1 (contd.)

**Step-3:** Determine the trans-admittances $Y_{11}$ and $Y_{21}$

\[
Y_{11} = \frac{I_1}{V_1} = \frac{3}{2R}
\]

\[
Y_{21} = \frac{I_2}{V_1} = -\frac{1}{R}
\]

Note that $Y_{21}$ is real and negative.

This is still a valid physical result, although you will find that the diagonal terms of an impedance or admittance matrix (e.g., $Y_{22}$, $Z_{11}$, $Y_{44}$) will always have a real component that is positive.

To find the other two trans-admittance parameters, we must move the short and then repeat each of our previous steps!
Example – 1 (contd.)

**Step-1:**
Place a short at port 1

\[ V_1 = 0 \]

\[ R \]

\[ 2R \]

\[ V_2 \]

**Step-2:** Determine currents \( I_1 \) and \( I_2 \)

- Note that after a short was placed at port 1, resistor 2R has zero voltage across it—and thus zero current through it!

**Therefore:**

\[ I_2 = \frac{V_2}{R} \]

\[ I_1 = -I_2 = -\frac{V_2}{R} \]

**Step-3:** Determine the trans-admittances \( Y_{12} \) and \( Y_{22} \)

\[ Y_{12} = \frac{I_1}{V_2} = -\frac{1}{R} \]

\[ Y_{22} = \frac{I_2}{V_2} = \frac{1}{R} \]

Therefore the admittance matrix is:

\[ Y = \begin{bmatrix} \frac{3}{2R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} \end{bmatrix} \]

Is it lossless or reciprocal?
Example – 2

- Consider this circuit:

- Where the 3-port device is characterized by the impedance matrix:

\[
\begin{bmatrix}
2 & 1 & 2 \\
1 & 1 & 4 \\
2 & 4 & 1
\end{bmatrix}
\]

- determine all port voltages \( V_1, V_2, V_3 \) and all currents \( I_1, I_2, I_3 \).