Lecture – 20

Date: 03.11.2016

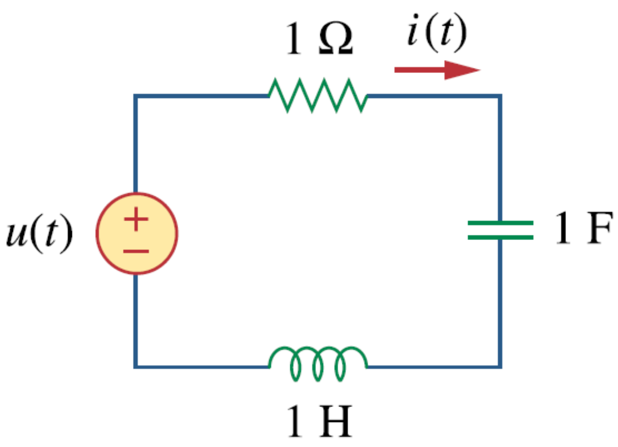
- Circuit Analysis in s-domain
- State Variables
- Network Stability

Circuit Analysis in s-domain

- Circuit analysis is relatively easy in the *s*-domain.
- Just need to transform a complicated set of mathematical relationships in the time domain into the *s*-domain where convert operators (derivatives and integrals) into simple multipliers of s and $\frac{1}{s}$.
- The exciting thing about it is that *all* of the circuit theorems and relationships are perfectly valid in the *s*-domain.

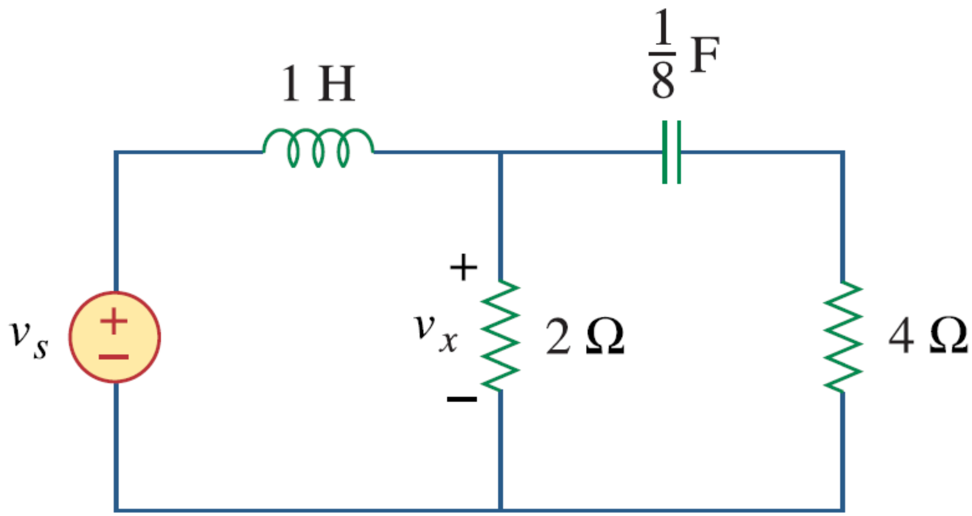
Example – 1

Determine $i(t)$ using Laplace Transform



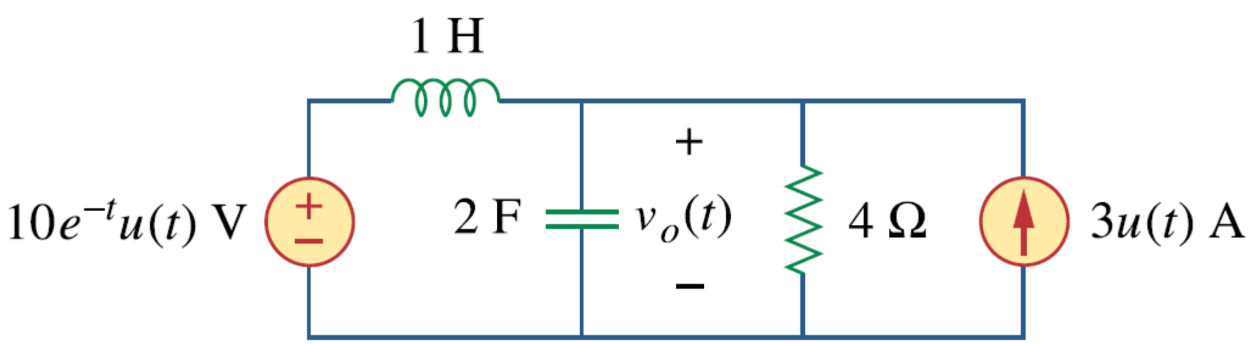
Example – 2

Find $v(x)$ in this circuit given $v(s) = 4u(t)$V



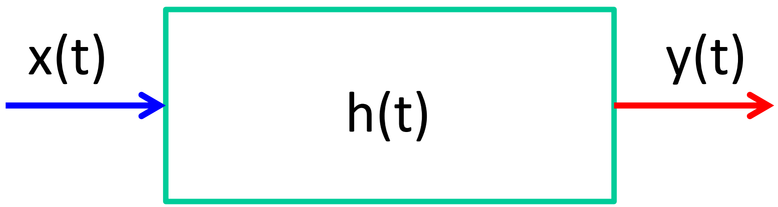
Example – 3

Find $v_0(t)$in this circuit.



Transfer Functions

- The transfer function $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.



$$H(s) = \frac{Y(s)}{X(s)}$$

Determination of Transfer Functions:

First method: assume any convenient input $X(s)$ and then employ any circuit analysis technique (such as current or voltage division, nodal or mesh analysis) to find the output $Y(s)$, and then obtain the ratio of the two.

Second method: assume that the output is 1 V or 1 A as appropriate and use the basic laws of Ohm and Kirchhoff (KCL only) to obtain the input. The transfer function becomes unity divided by the input.

Both these methods rely on the linearity property

Transfer Functions (contd.)

- Sometimes, the input $X(s)$ and the transfer function $H(s)$ is known and then:

$$Y(s) = H(s)X(s)$$

take the inverse transform to get $y(t)$.

- A special case: when the input is the unit impulse function $x(t) = \delta(t)$ i.e., $X(s) = 1$

$$Y(s) = H(s)$$

$$\Rightarrow y(t) = h(t)$$

h(*t*) represents the *unit impulse response*—it is the time-domain response of the network to a unit impulse.

Once the impulse response *h*(*t*) of a network is known, the response of the network to *any* input signal can be obtained using the above expression in the *s*-domain or using the convolution integral in the time domain.

Example – 4

- The transfer function of a system is $\frac{s^2}{3s+1}$. Find the output when the system has an input of $4e^{-\frac{t}{3}}u(t)$.

Example – 5

When the input to a system is a unit step function, the response is $10cos2tu(t)$. Obtain the transfer function of the system.

Example – 6

A circuit is known to have its transfer function as $H(s) = \frac{s+3}{s^2+4s+5}$

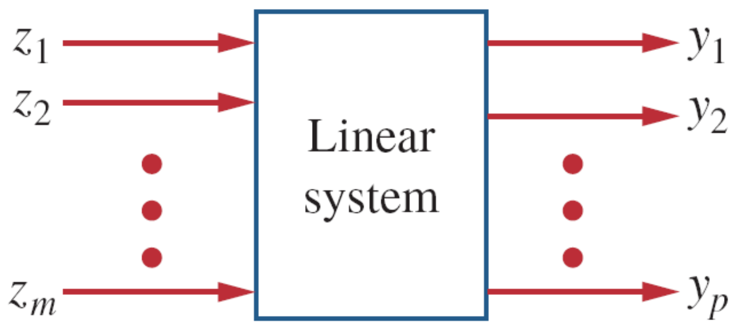
Find its output when:

(a) the input is a unit step function.

(b) the input is $6te^{-2t}\,u(t)$.

State Variables

- Thus far we have considered techniques for analyzing systems with only one input and only one output.
- Many engineering systems have many inputs and many outputs, e..g, multiple current and voltage inputs and outputs.
- The state variable method is a very important tool in analyzing systems and understanding such highly complex systems.
- Thus, the state variable model is more general than the single-input, single-output model, such as a transfer function.



- In the state variable model, a collection of variables that describe the internal behavior of the system are specified. These variables are known as the *state variables* of the system.
- These variables determine the future of a system when the present state of the system and the input signals are known.

State Variables (contd.)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

state vector representing *n* state vectors. the dot represents the first derivative with respect to time.

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_m(t) \end{bmatrix}$$

input vector representing *m* inputs

- **A** and **B** are nXn and nXm respectively and matrices.

- The complete state model or state space is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{z}$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

the output vector representing *p* outputs

- **C** and **D** are p X n and p X m respectively and matrices

State Variables (contd.)

- Assuming zero initial conditions:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{Z}(s) \Rightarrow (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{Z}(s)$$

$$\boxed{\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{Z}(s)}$$

- Similarly: $\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{Z}(s)$

- Then: $\mathbf{H}(s) = \dfrac{\mathbf{Y}(s)}{\mathbf{Z}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

$\mathbf{A}$ = system matrix

$\mathbf{B}$ = input coupling matrix

$\mathbf{C}$ = output matrix

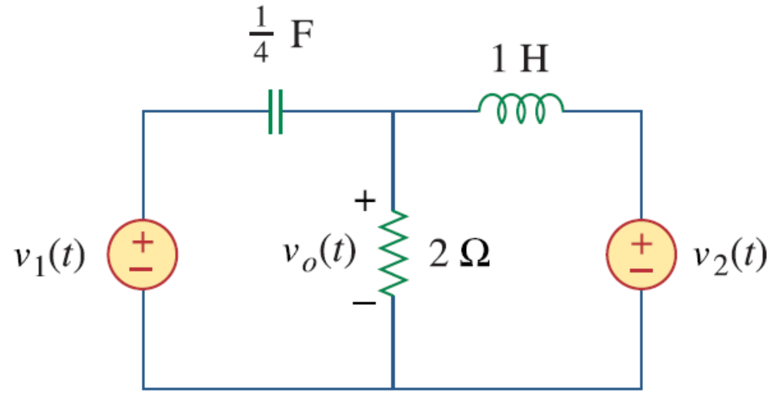
$\mathbf{D}$ = feedforward matrix

In most cases, **D =0**

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Example – 7

Develop the state equations for this circuit.



Example – 8

Develop the state equations for this differential equation.

$$\frac{d^2y(t)}{dt^2} + \frac{4dy(t)}{dt} + 3y(t) = z(t)$$

Example – 9

Given this state equation, solve for $y_1(t)$.

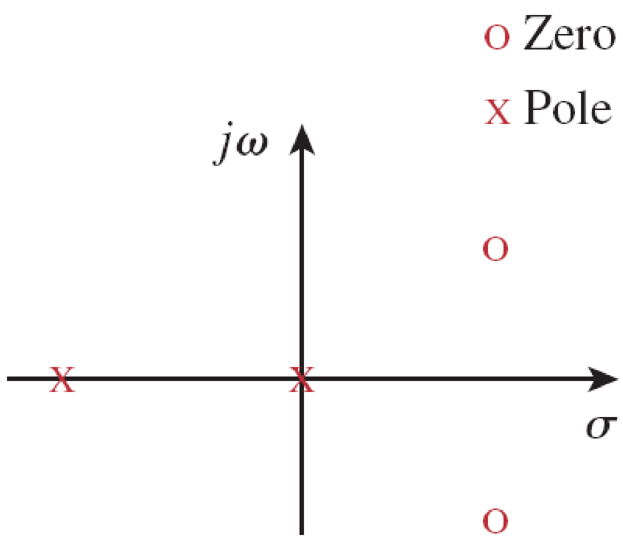
$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & -1 \\ 2 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ 2u(t) \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -2 & -2 \\ 1 & -0 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u(t) \\ 2u(t) \end{bmatrix}$$

Network Stability

- A circuit is *stable* if its impulse response *h*(*t*) is bounded (i.e., *h*(*t*) converges to a finite value) as $t \to \infty$.
- Let us take:

$$H(s) = \frac{N(s)}{D(s)} \Rightarrow H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$



H(*s*) must meet two requirements for the circuit to be stable.

Network Stability (contd.)

- **First,** the degree of $N(s)$ must be less than the degree of $D(s)$; otherwise, long division would produce:

$$H(s) = k_n s^n + k_{n-1} s^{n-1} + \cdots + k_1 s + k_0 + \frac{R(s)}{D(s)}$$

where the degree of $R(s)$, the remainder of the long division, is less than the degree of $D(s)$. → this will lead to unbounded h(t) as $t \to \infty$

- **Second,** all the poles of $H(s)$ (i.e., all the roots of D(s)=0) must have negative real parts; in other words, all the poles must lie in the left half of the *s* plane.