Lecture – 18 Date: 27.10.2016

- Introduction to Laplace Transform
- Properties of Laplace Transform

Introduction

- *Laplace transformation (LT)* turns differential equations into *algebraic equations* and thus facilitate the solution process of circuits and systems.
- When using phasors for the analysis of circuits, we transform the circuit from the time domain to the frequency or phasor domain. Once we obtain the phasor result, we transform it back to the time domain.
- The LT method follows the same process: use the LT to transform the circuit from the time domain to the frequency domain, obtain the solution, and apply the inverse LT to the result to transform it back to the time domain.
- The LT is significant for a number of reasons.
  - First, it can be applied to a wider variety of inputs than phasor analysis.
  - Second, it provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations.
  - Third, the Laplace transform is capable of providing us, in one single operation, the total response of the circuit comprising both the natural and forced responses.

Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

s is a complex variable:

$$s = \sigma + j\omega$$

the lower limit is specified as 0^- to indicate a time just before t=0. We use 0^- as the lower limit to include the origin and capture any discontinuity of $f(t)$ at t=0.

The Laplace transform is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.

When the LT is applied to circuit analysis, the differential equations represent the circuit in the time domain. The terms in the differential equations take the place of $f(t)$. Their LT, which corresponds to $F(s)$, constitutes algebraic equations representing the circuit in the frequency domain.

It is assumed that $f(t)$ is ignored for $t < 0$. To ensure that this is the case, a function is often multiplied by the unit step. Thus, $f(t)$ is written as $f(t)u(t)$ or $f(t), t \geq 0$

Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

In order for *f* (*t*) to have a LT, the integral must converge to a finite value.

$$\int_{0^-}^{\infty} e^{-\sigma t}|f(t)|\,dt < \infty$$

Example – 1

Determine the Laplace transform of each of these functions:
$(a)u(t),\ \ (b)\ e^{-at}u(t),\ \ (c)\delta(t)$

Example – 2

Determine the Laplace transform of: $(a)\cosh(at),\ \ (b)\ \sinh(at)$

Example – 3

Determine the Laplace transform of $\sinh(\omega t)\,u(t)$

Example – 4

Determine the Laplace transform of: $(a)\cos(\omega t + \theta),\ \ (b)\ \sinh(\omega t + \theta)$

Properties of Laplace Transform

Linearity $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$

where a_1 and a_2 are constants.

Scaling $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Time Shift $\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$

if a function is delayed in time by *a*, the result in the *s*-domain is found by multiplying the Laplace transform of the function (without the delay) by e^{-as}. This is called the *time-delay* or *time-shift property* of the LT.

Frequency Shift $\mathcal{L}[e^{-at} f(t) u(t)] = F(s+a)$

the Laplace transform of $e^{-at} f(t)$ can be obtained from the LT of *f* (*t*) by replacing every s with $s + a$. This is known as *frequency shift* or *frequency translation.*

Properties of Laplace Transform

Time Differentiation:

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - s^0 f^{(n-1)}(0^-)$$

$\Rightarrow$ $\mathcal{L}[f'(t)] = sF(s) - f(0^-)$

Time Integration:

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s)$$

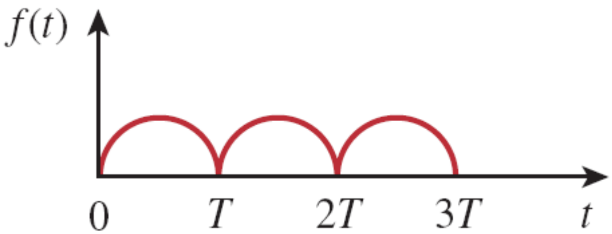
Frequency Differentiation:

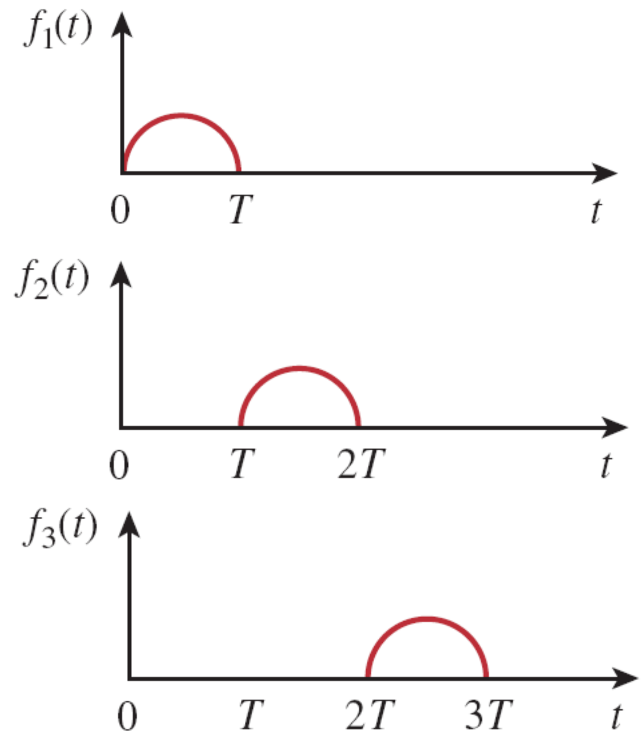
$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

$\Rightarrow$ $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$

Properties of Laplace Transform

Time Periodicity:





$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) + \cdots \\
&= f_1(t) + f_1(t-T)u(t-T) \\
&\quad + f_1(t-2T)u(t-2T) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} + \cdots \\
&= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots]
\end{aligned}
$$

$$
F(s) = \frac{F_1(s)}{1 - e^{-Ts}}
$$

LT of a periodic function is the transform of the first period of the function divided by $1 - e^{-TS}$

Properties of Laplace Transform

Initial and Final Values:

$$f(0) = \lim_{s\to\infty} sF(s)$$ *initial-value theorem*

$$f(\infty) = \lim_{s\to 0} sF(s)$$ *final-value theorem*

Caution: In general, the final value theorem does not apply in finding the final values of sinusoidal functions—these functions oscillate forever and do not have final values.

Example – 5

Determine the Laplace transform of:

(a) $e^{-2t}\cos 3t\,u(t)$ (b) $e^{-2t}\sin 4t\,u(t)$

(c) $e^{-3t}\cosh 2t\,u(t)$ (d) $e^{-4t}\sinh t\,u(t)$

(e) $te^{-t}\sin 2t\,u(t)$

Example – 6

Determine the Laplace transform of: $g(t) = 6\cos(4t-1)$

Example – 7

Determine the Laplace transform of: $t^2 \cos(2t + 30°)u(t)$

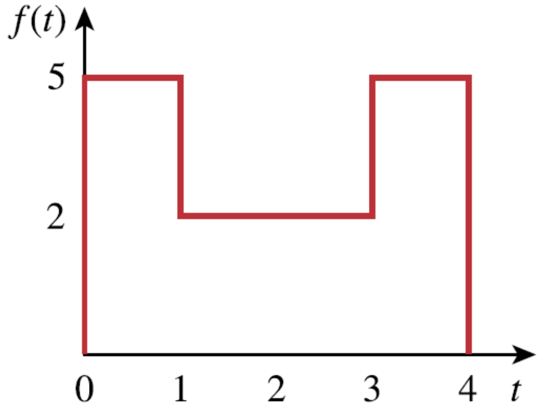
Example – 8

In two different ways determine the Laplace transform of:

$$g(t) = \frac{d}{dt}\left(te^{-t}\cos t\right)$$

Example – 9

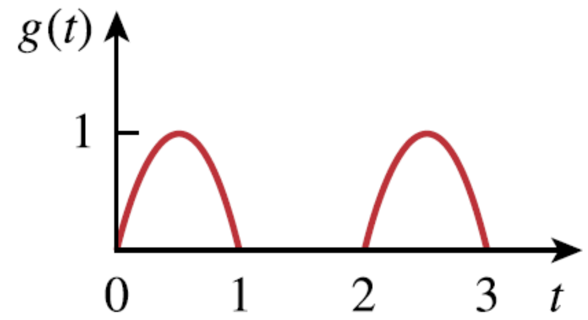
Determine the Laplace transform of f(t)



Example – 10

The periodic function shown is defined over its period as:

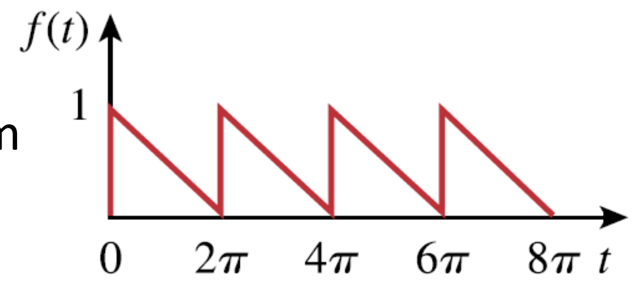
$$g(t)\begin{cases} \sin \pi t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$



Find G(s)

Example – 11

Obtain the Laplace transform of this periodic waveform



Example – 12

Given that:

$$F(s) = \frac{s^2 + 10s + 6}{s(s+1)^2(s+3)}$$

Evaluate $f(0)$ and $f(\infty)$ if they exist