Circuit Design Research Lab

Lecture – 15

Date: 17.10.2016

- Frequency Response
- Bode Plot

Introduction

- In sinusoidal circuit analysis, we learnt how to find voltages and currents in a circuit with a constant frequency source.
- However, if the amplitude of the sinusoidal source remain constant and the frequency is varied then one can obtain the circuit's *frequency response*.
- The frequency response may be regarded as a complete description of the sinusoidal steady-state behavior of a circuit as a function of frequency.
- The sinusoidal steady-state frequency responses of circuits are of significance in many applications, especially in communications and control systems.
- A specific application is in electric filters that block out or eliminate signals with unwanted frequencies and pass signals of the desired frequencies.
- Filters are used in radio, TV, and telephone systems to separate one broadcast frequency from another.

Transfer Function

- The transfer function (also called the *network function*) is a useful analytical tool for finding the frequency response of a circuit.
- It is represented by $H(\omega)$.
- Circuit's frequency response is essentially the plot of $H(\omega)$ when ω varies between 0 and ∞.

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

It is the frequency-dependent ratio of a phasor output **Y**(ω) to a phasor input **X**(ω).

- Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

Transfer Function (contd.)

- Being a complex quantity, $\boldsymbol{H}(\omega)$ has a magnitude $H(\omega)$ and a phase φ.
- The transfer function of a circuit can be obtained by first converting it to frequency-domain equivalent by replacing resistors, inductors, and capacitors with their impedances R, $j\omega L$ and $^1/_{j\omega C}$.
- One can then use any circuit technique(s) to obtain the appropriate expressions.
- Can be simplified to:

$$\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}$$

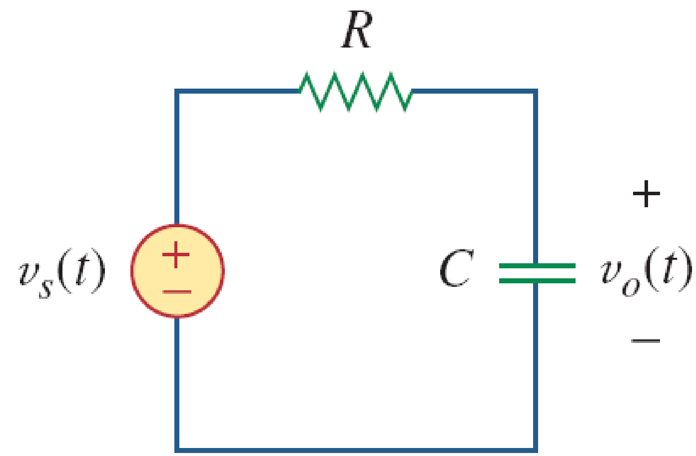
The roots of $\boldsymbol{N}(\omega)$ are called the *zeros* and are usually represented as $j\omega = z_1, z_2, \ldots$ ...Similarly, the roots of $\boldsymbol{D}(\omega)$ are the *poles* and are represented as $j\omega = p_1, p_2, \ldots \ldots$

A zero is a value that results in a zero value of the function. A pole is a value for which the function is infinite.

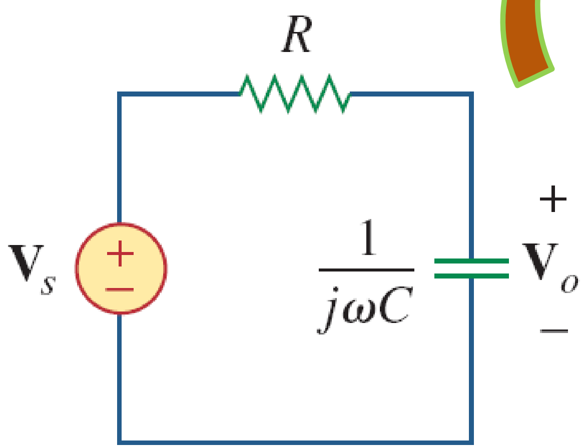
To avoid complex algebra, it is expedient to replace temporarily $j\omega$ with s when working with **H**(ω) and replace s with $j\omega$ at the end.

Example – 1

Find the transfer function $\frac{V_0}{V_s}$ and the corresponding frequency response of this *RC* circuit. Assume, $v_s = V_m cos\omega t$.



The frequency-domain equivalent of the circuit:



the transfer function

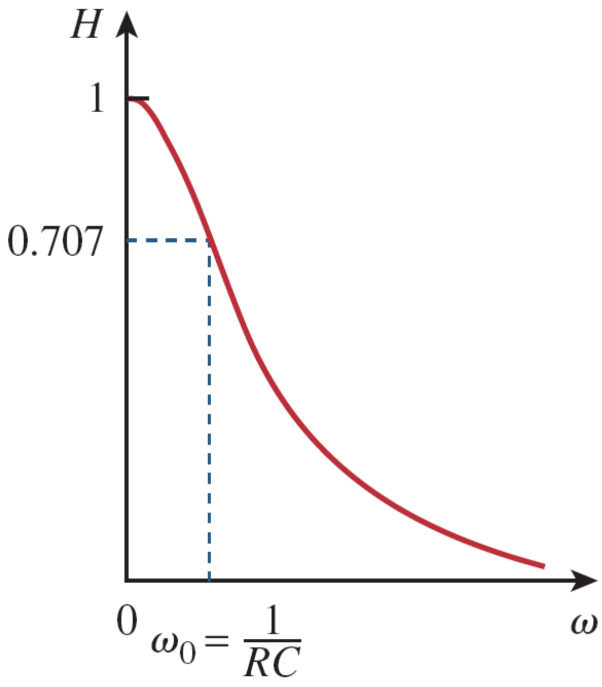
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

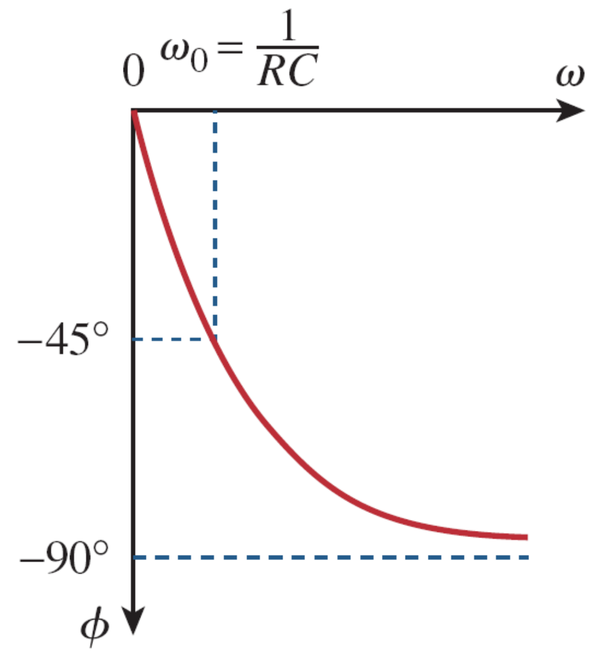
$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\omega_0 = \frac{1}{RC}$. For plotting *H* and φ for $0 < \omega < \infty$, we need values at some critical points.

Example – 1 (contd.

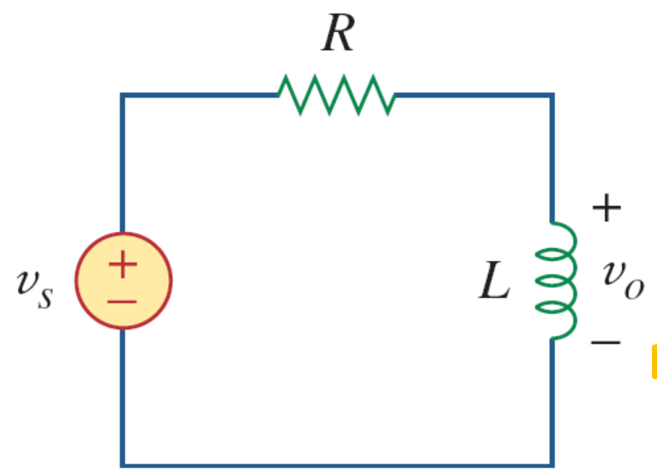
ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	-84°
1	0.71	-45°	20	0.05	-87°
2	0.45	-63°	100	0.01	-89°
3	0.32	-72°	∞	0	-90°



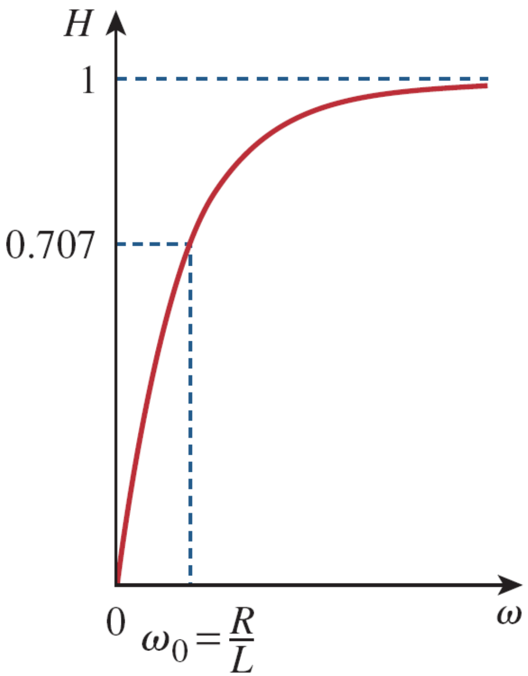


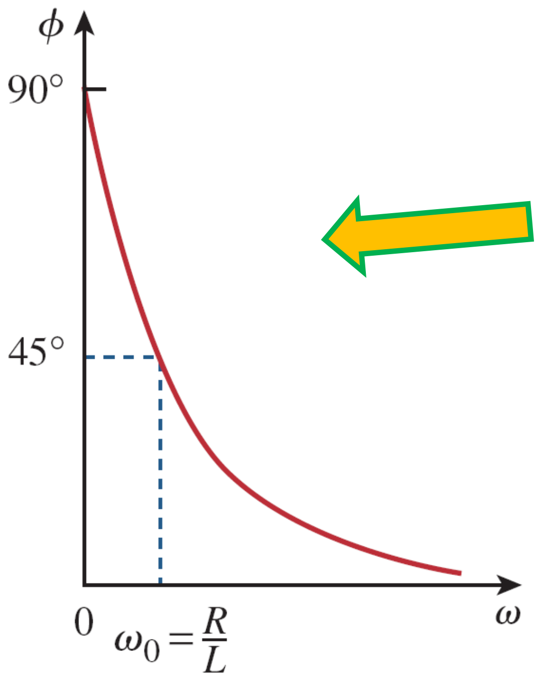
Example – 2

Find the transfer function $\frac{V_0}{V_s}$ and the corresponding frequency response of this *RL* circuit. Assume, $v_s = V_m cos\omega t$.



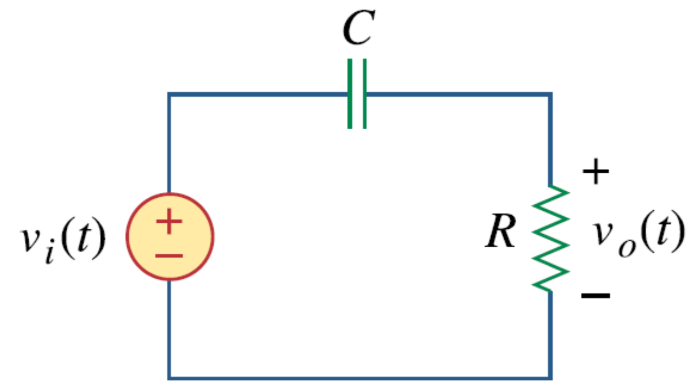
$$H(\omega) = \frac{V_0}{V_S} = \frac{j\omega L}{R + j\omega L}$$





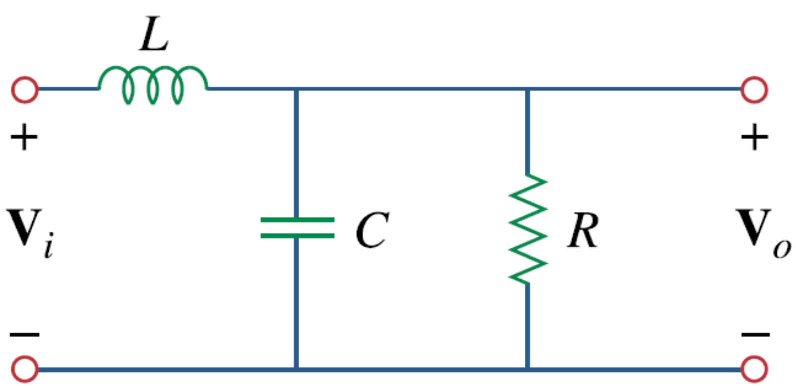
Example – 3

Find the transfer function $\frac{V_0}{V_s}$ and the corresponding frequency response of this *RC* circuit. Assume, $\omega_0 = \frac{1}{RC}$.

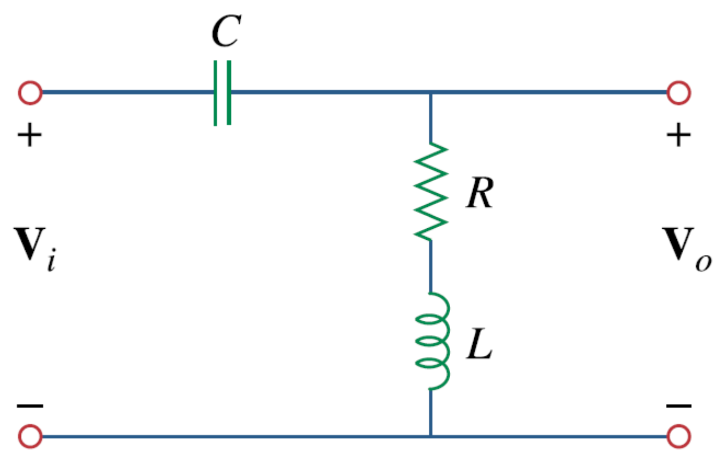


Example – 4

Find the transfer function $\frac{V_0}{V_i}$ of the following circuits.



(a)

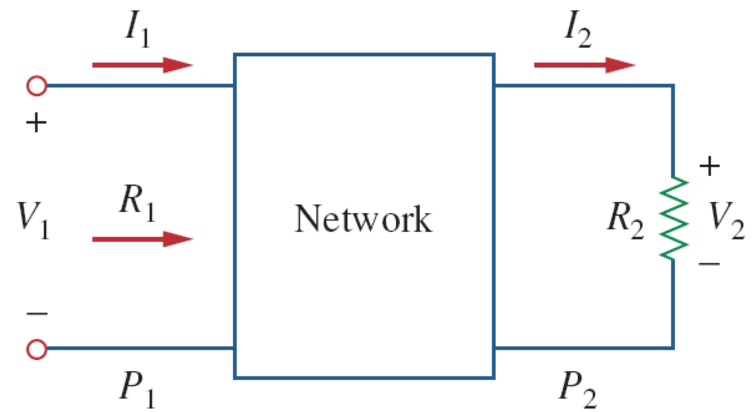


(b)

Decibel Scale

- It is not always easy to get a quick plot of the magnitude and phase of the transfer function.
- A more systematic way of obtaining the frequency response is to use Bode plots (based on logarithms).
- The bel is used to measure the ratio of two levels of power or power gain *G:*

G = number of bels = $log_{10} \frac{P_2}{P_1}$



- The *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a bel:

$$G_{dB} = 10log_{10} \frac{P_2}{P_1}$$

For $P_2 = P_1$, the gain is 0dB. If $P_2 = 2P_1$ then gain = 3dB while for $P_2 = 0.5P_1$ the gain is -3dB

Decibel Scale (contd.)

- In terms of voltage or current ratio:

$$G_{dB} = 20log_{10}\frac{V_2}{V_1} \qquad\qquad G_{dB} = 20log_{10}\frac{I_2}{I_1}$$

- We use $\mathbf{10log_{10}}$ for power and $\mathbf{20log_{10}}$ for voltage or current, because of the square relationship between them.
- The ***dB*** value is a logarithmic measurement of the *ratio* of one variable to another *of the same type*. Therefore, it can only be applied when the transfer function *H* is expressed as ratio of same quantities.
- It is important to note that so far we only used voltage and current magnitudes in above equations. Negative signs and angles will be can also be handled independently.

With this in mind, we can apply the concepts of logarithms and decibels to construct Bode plots.

Example – 5

Calculate $|\mathbf{H}(\omega)|$ if H_{dB} equals: (a) $0.05\ dB$ (b) $-6.2\ dB$ (c) $104.7\ dB$

Example – 6

Determine the magnitude (in dB) and the phase (in degrees) of $\mathbf{H}(\omega)$ at $\omega = 1$ if $\mathbf{H}(\omega)$ equals:

(a) 0.05 (b) 125 (c) $\frac{10j\omega}{2+j\omega}$ (d) $\frac{3}{1+j\omega} + \frac{6}{2+j\omega}$

Bode Plot

- Obtaining the frequency response from the transfer function is extremely tedious as the frequency range required in frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis.
- Furthermore, there is a more systematic way of locating the important features of the magnitude and phase plots of the transfer function using semilogarithmic plots known as ***Bode Plot***.
- In the ***Bode Plot***, the *magnitude in decibels* is plotted against the *logarithm of the frequency*; on a separate plot, the *phase in degrees* is plotted against the *logarithm of the frequency*.
- Bode plots contain the same information as the non logarithmic but are much easier to construct.

$$\mathbf{H} = H\underline{/\phi} = He^{j\phi}$$

$$\ln \mathbf{H} = \ln H + \ln e^{j\phi} = \ln H + j\phi$$

the real part is a function of the magnitude while the imaginary part is the phase.

Bode Plot (contd.)

- In a Bode plot, magnitude is plotted in decibels (dB) vs frequency while the phase is plotted in degrees vs frequency on a semilog scale.
- A generic transfer function:

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

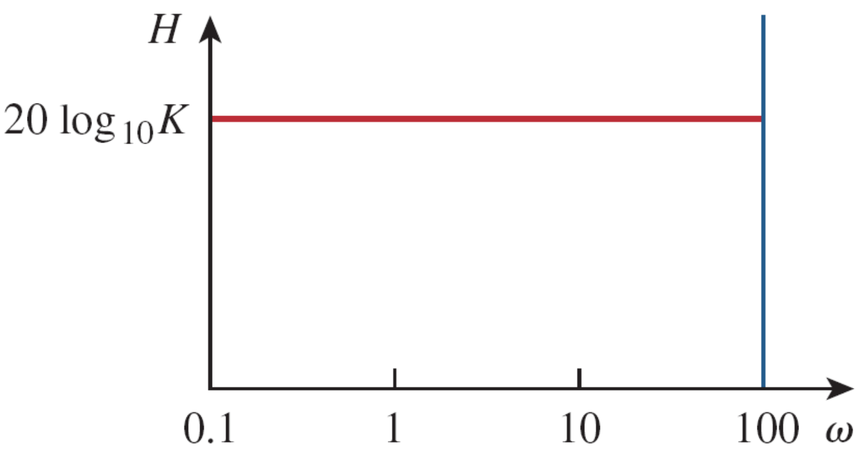
This generic formulation may include up to seven types of different factors that can appear in various combinations.

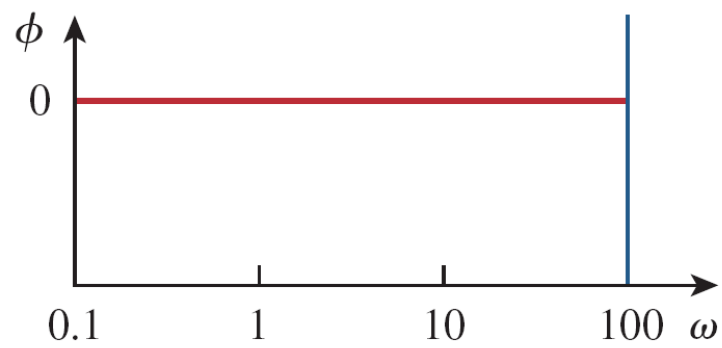
- A gain K
- A pole $(j\omega)^{-1}$ or zero $(j\omega)$ at the origin
- A simple pole $1/\left(1+j\omega/p_1\right)$ or zero $\left(1 + j\omega/z_1\right)$
- A quadratic pole $1/\left[1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]$ or zero $\left[1 + \frac{j2\zeta_1\omega}{\omega_k} + \left(\frac{j\omega}{\omega_k}\right)^2\right]$

Bode Plot (contd.)

- The Bode plot is constructed by plotting each factor separately and then by subsequent addition.

<u>Constant term:</u> For K, the magnitude is $20log_{10}K$ and the phase angle is 0°





For negative K, the magnitude remains $20log_{10}|K|$ but the phase angle is $\pm 180°$

Bode Plot (contd.)

<u>Pole/zero at the origin:</u> For the zero $(j\omega)$ at the origin, the magnitude is $20log_{10}(\omega)$ and the phase is 90°.

- **slope of the magnitude plot is 20dB/decade, while the phase is constant with frequency.**

The Bode plots for the pole $(j\omega)^{-1}$ are similar except that the slope of the magnitude plot is -20dB/decade while the phase is $-90°$. In general, for $(j\omega)^N$ where *N* is an integer, the magnitude plot will have a slope of 20*N* dB/decade, while the phase is 90*N* degrees.

Bode Plot (contd.)

Simple pole/zero: $\left(1 + {}^{j\omega}/_{z_1}\right)$

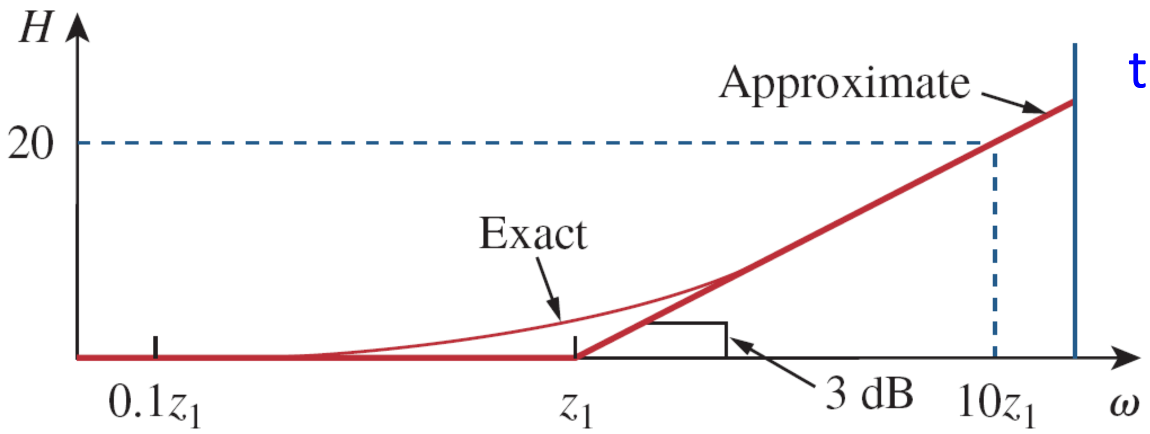
$$H_{\mathrm{dB}} = 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right| \rightarrow = 0 \quad \text{as } \omega \to 0$$

$$\rightarrow = 20log_{10}\frac{\omega}{z_1} \quad \text{as } \omega \to \infty$$

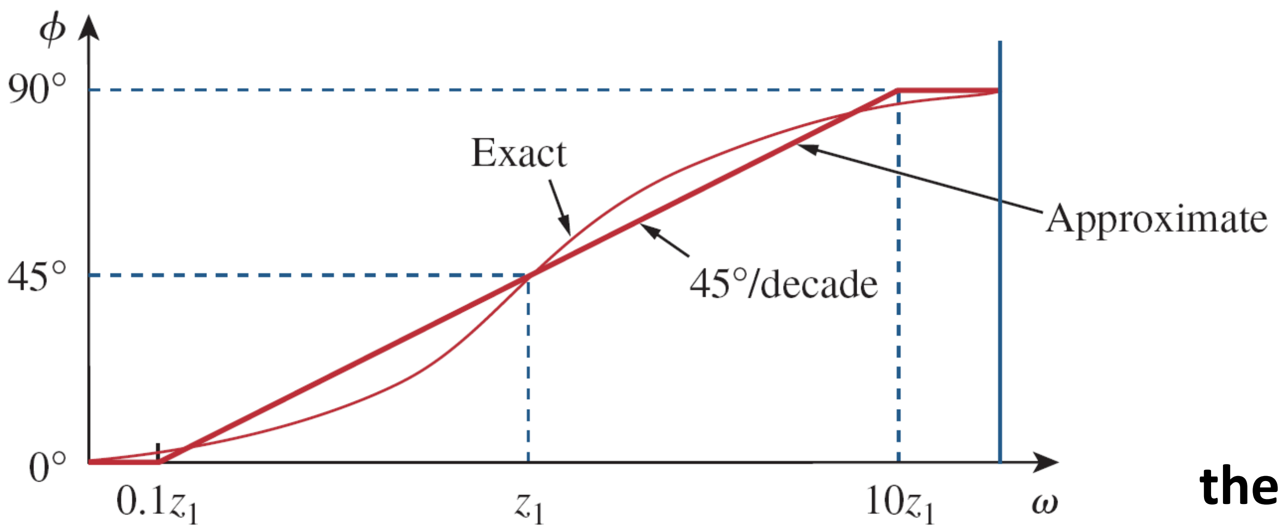
- It shows that we can approximate the magnitude as zero (a straight line with zero slope) for small values of ω and by a straight line with slope 20 dB/decade for large values of ω.
- The frequency $\omega = z_1$ where the two asymptotic lines meet is called the *corner frequency* or *break frequency*.

$$\phi = \tan^{-1}\left(\frac{\omega}{z_1}\right) = \begin{cases} 0, & \omega = 0 \\ 45^\circ, & \omega = z_1 \\ 90^\circ, & \omega \to \infty \end{cases}$$

Bode Plot (contd.)



the approximate plot is close to the actual plot except at the break frequency ($\omega = z_1$), where the deviation is $20log_{10}|1 + j1| = 20log_{10}|\sqrt{2}| \approx 3dB$.



As a straight-line approximation we let:

$\varphi \cong 0°$ for $\omega \leq \frac{z_1}{10}$

$\varphi \cong 45°$ for $\omega = z_1$

$\varphi \cong 90°$ for $\omega \geq 10z_1$

the straight-line plot has a slope of 45° per decade.

Bode Plot (contd.)

- The Bode plots for the pole $\mathbf{1}/\left(\mathbf{1}+{}^{\boldsymbol{j\omega}}/_{\boldsymbol{p_1}}\right)$ are similar except that the corner frequency is at $\omega = p_1$, the magnitude has a slope of –20 dB/decade and the phase has a slope of $-45°$ per decade.

Quadratic pole/zero:

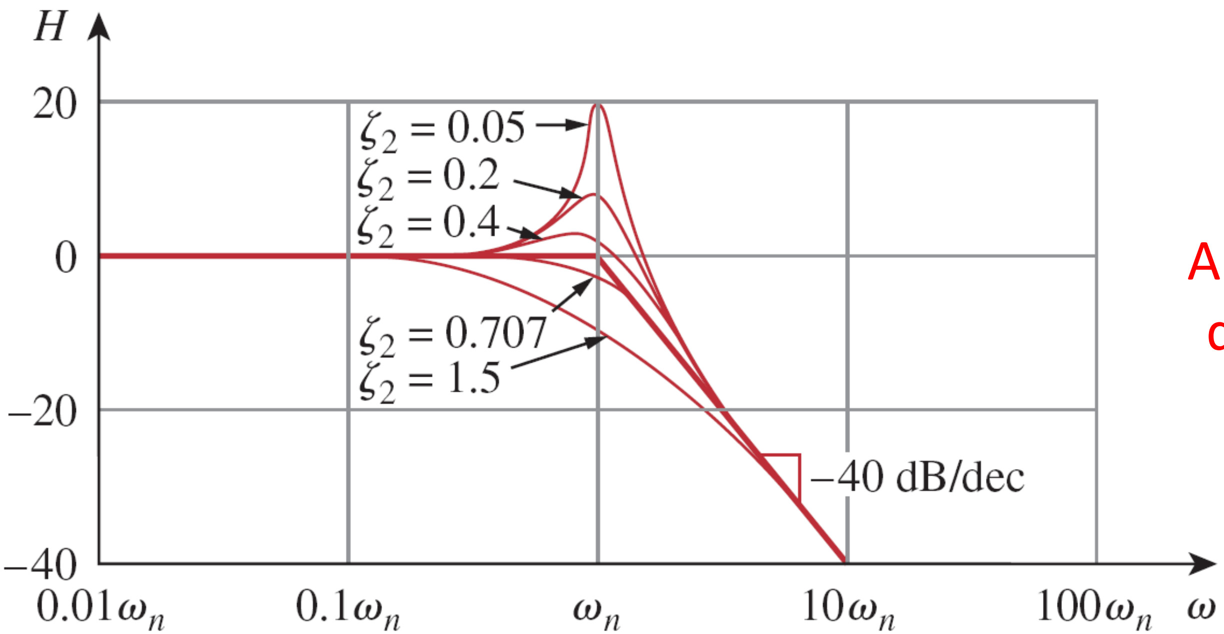
$$1\bigg/\left[1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]$$

$$H_{\text{dB}} = -20\log_{10}\left|1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad 0 \quad \text{as} \quad \omega \to 0$$

$$H_{\text{dB}} = -20\log_{10}\left|1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad -40\log_{10}\frac{\omega}{\omega_n} \quad \text{as} \quad \omega \to \infty$$

Clearly, the amplitude plot consists of two straight asymptotic lines: one with zero slope for $\omega < \omega_n$ and the other with slope $-40\ dB$/decade for $\omega > \omega_n$ with ω_n as the corner frequency.

Bode Plot (contd.)



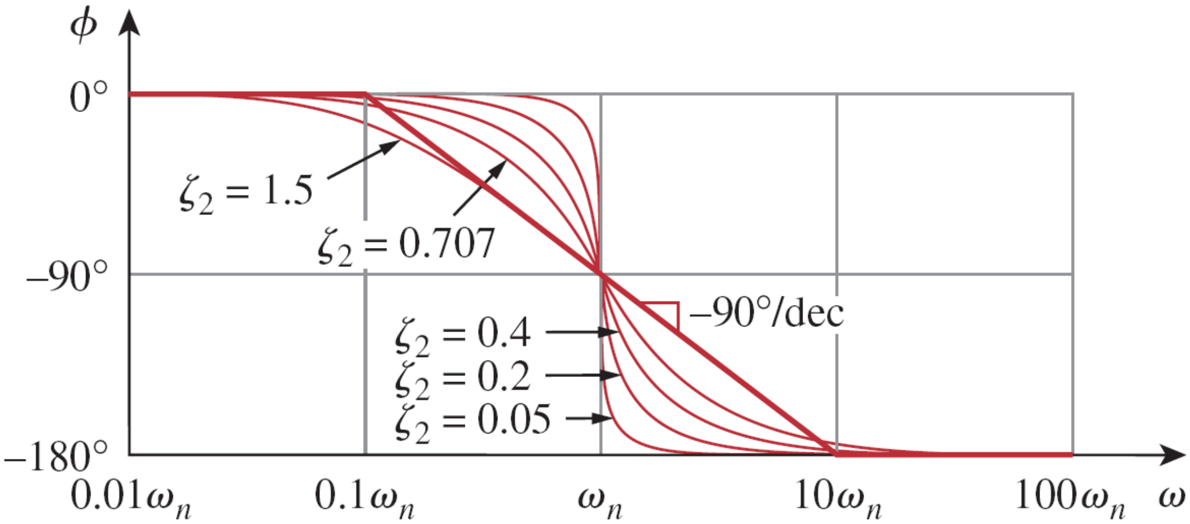
Apparently, the actual plot depends on the damping factor ζ_2 as well

- For precise frequency response, the significant peaking in the neighborhood of the corner frequency should be added to the straight-line approximation.
- However, for generic analysis the straight-line approximation is sufficient.

Bode Plot (contd.)

Quadratic pole/zero:

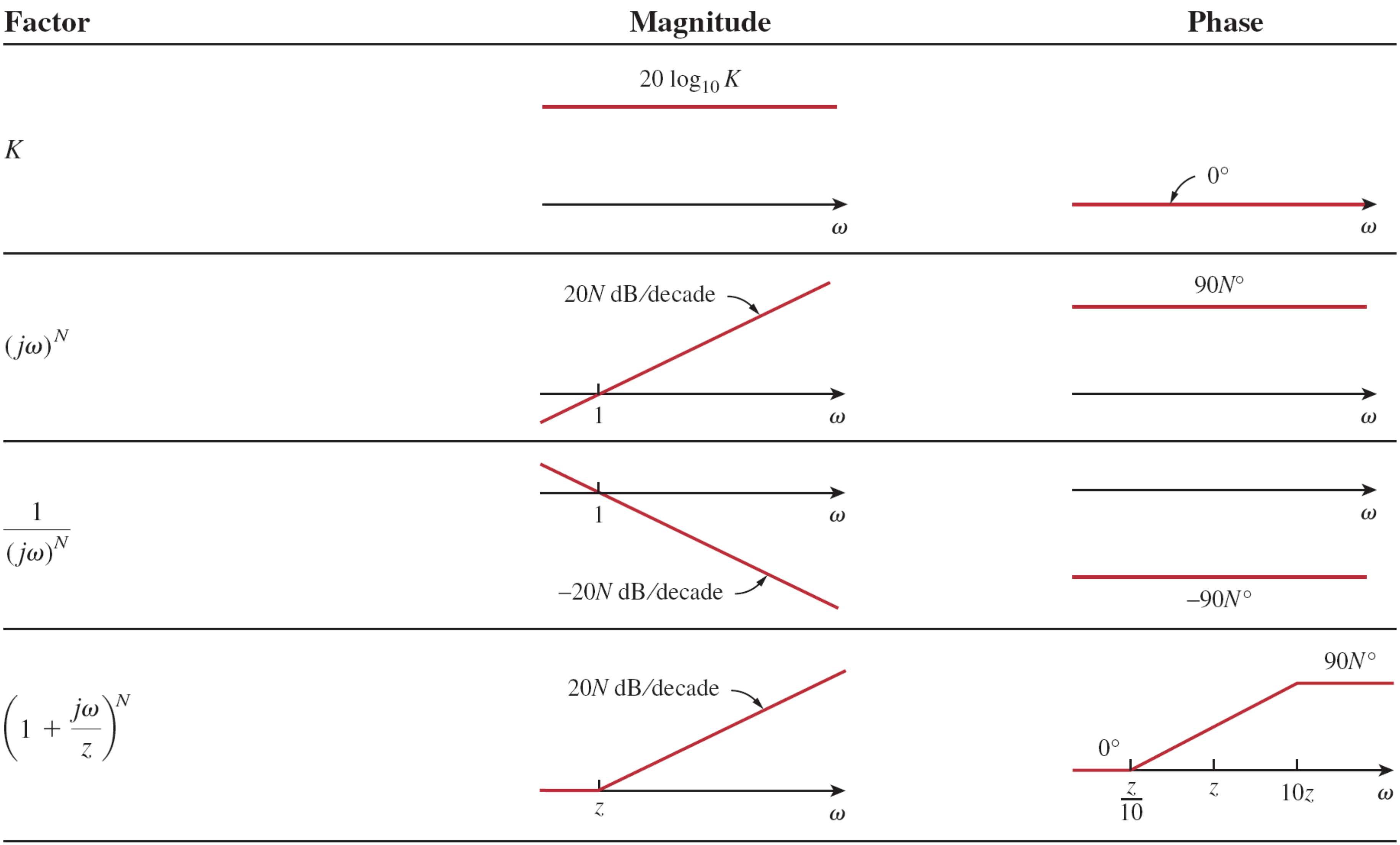
$$1\Big/\left[1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right] \Rightarrow \phi = -\tan^{-1}\frac{2\zeta_2\omega/\omega_n}{1-\omega^2/\omega_n^2} = \begin{cases} 0, & \omega = 0 \\ -90^\circ, & \omega = \omega_n \\ -180^\circ, & \omega \to \infty \end{cases}$$



The phase plot is a straight line with a slope of $-90°$ per decade starting at $\frac{\omega_n}{10}$ and ending at $10\omega_n$

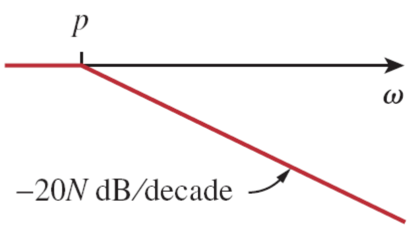
- For the quadratic zero, the magnitude and phase plots are inverted because the magnitude plot has a slope of 40 dB/decade while the phase plot has a slope of 90° per decade .

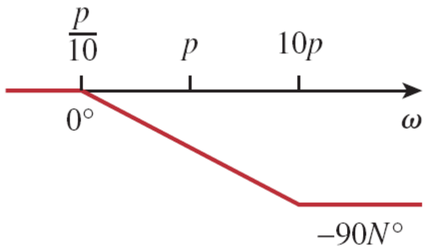
Bode Plot (contd.)



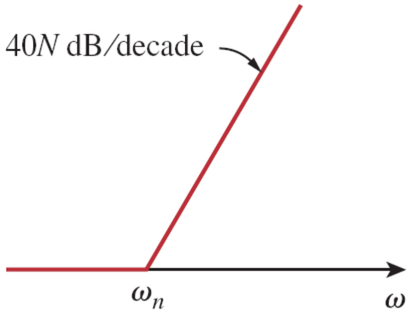
Bode Plot (contd.)

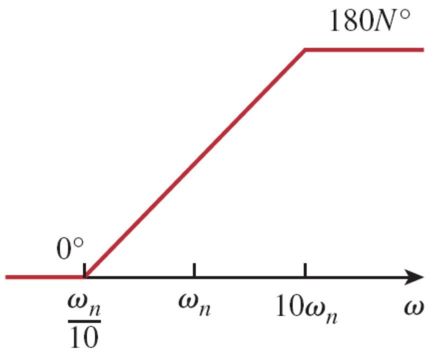
$$\frac{1}{(1 + j\omega/p)^N}$$



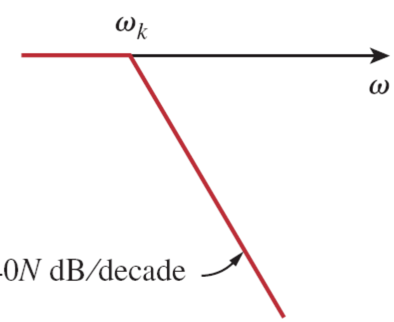


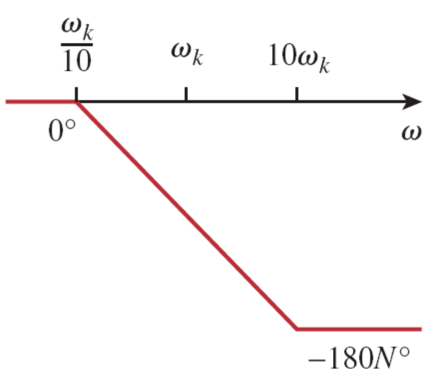
$$\left[1 + \frac{2j\omega\zeta}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right]^N$$





$$\frac{1}{[1 + 2j\omega\zeta/\omega_k + (j\omega/\omega_k)^2]^N}$$





Bode Plot (contd.)

- not every transfer function has all seven factors.
- To sketch the Bode plots for a generic function H(ω), we first record the corner frequencies on the semilog graph paper, and sketch the factors one at a time as discussed.
- Then combine additively the graphs of the factors.
- The combined graph is usually drawn from left to right, changing the slopes appropriately each time a corner frequency is encountered.

Example – 7

A ladder network has a voltage gain of $\mathbf{H}(\omega) = \dfrac{10}{(1 + j\omega)(10 + j\omega)}$

Sketch the Bode plots for the gain.