CSE622/622A IQC: Homework 3

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It is mandatory to use bra-ket notation. You may need to perform linear-algebraic calculations either by hand (submit calculations with the homework) or use numpy (submit python or ipynb/Jupyter notebook file as attachment on Classroom).

Q1 [3 points]

(a) Let E be a single-qubit projection operator. Prove the following

$$\langle 0|E|0\rangle + \langle 1|E|1\rangle = 1.$$

Hint: Start by writing $E = |b\rangle\langle b|$ *and express* $|b\rangle$ *in the standard basis.*

(b) Suppose we measure the first qubit of $|\beta_{00}\rangle$ using the $\{E, I - E\}$ projective measurement. Prove that both the outcomes appear with equal probability.

(c) Let $\{|b\rangle, |b^{\perp}\rangle\}$ be a basis, and $E = |b\rangle\langle b|$. Derive conditions on $|b\rangle$ (equivalently, conditions on the amplitudes of $|b\rangle$ when expressed in the standard basis) such that if the first qubit is measured using $\{E, I - E\}$, the state collapses to either $|b\rangle|b\rangle$ or $|b^{\perp}\rangle|b^{\perp}\rangle$.

Q2 [2+1+1+0+4+6+2=16 points] This question will teach you to implement oracles from Boolean functions.

(a) Let $f : \{0,1\}^n \to \{0,1\}^n$ be a bijective function. Prove that any operator that maps $|x\rangle \mapsto |f(x)\rangle$ must be unitary.

(b) Let $g : \{0,1\}^n \to \{0,1\}$ and let U_g be an operator with the following mapping on standard basis states: $|x\rangle|b\rangle \mapsto |x\rangle|b \oplus g(x)\rangle$. Here, $x \in \{0,1\}^n$ and $b \in \{0,1\}$. Prove that U_g must be unitary. *Hint: Use (a).*

(c) For any $x \in \{0,1\}^n$, define operator

$$V_x^g = |x\rangle\langle x| \otimes \left|\overline{g(x)}\right\rangle\langle 1| + |x\rangle\langle x| \otimes |g(x)\rangle\langle 0| + (I - |x\rangle\langle x|) \otimes I.$$

Here, $\overline{g(x)}$ denotes $1 - g(x)$. Describe the action of V_x^g on the standard basis states.

(d) (0 credit, do not submit) U_g can be implemented as $\Pi_{x\in\{0,1\}^n} V_x^g$. Draw a schematic diagram illustration how U_g can be implemented using the V_x^g operators. Prove that the above formulation of U_g is correct mathematically (this requires induction; however, you can verify the implementation for $n = 1$).

(e) Consider a binary mapping $g()$ on 4-bit strings that maps $0, 7, D, E$ to 1 and rest to 0. Design and implement quantum circuit for U_g using Qiskit; you are allowed to use single-qubit and multi-controlled NOT/X gates. Explain your design and submit a PDF/notebook showing the code and circuit; use "barriers" in your code for better visual clarity.

(f) Implement and run Grover's algorithm using Qiskit to find any input that $g()$ maps to 1. You should use the `grover_iterator` API in Qiskit:

```
grover_operator(oracle, state_preparation, insert_barriers=True)
```

and use appropriate `oracle` (for marking) and `state_preparation` circuit. You have to solve these tasks first.

- Grover's algorithm has to search among all the 16 strings of $\{0,1\}^4$, and so, will require a 4 qubit register initialised to $|0000\rangle$.
- `state_preparation` is a subcircuit to create a uniform superposition of all the 16 4-bit strings from $|0000\rangle$ – design this subcircuit.
- `oracle` should be a phase oracle used to mark the "good" strings (0,7,D,E); use U_g to implement this oracle.
- The circuit for the Grover's algorithm is: first apply the sampling oracle and then apply the `grover_iterator` several times.
- Upon measurement of the register in the standard basis, the observed 4-bit string is recorded as the solution returned by Grover's algorithm.

For this homework, the number of times to apply `grover_iterator` should be iterated from 0 to 16; plot the success probability of observing a good solution against the number of iterations. Also, submit the entire code as PDF/notebook file.
Careful! Qiskit prints the observed strings in the reverse order.

(g) Consider a $g()$ on n-bit strings. Suppose we want to find any two distinct $i \neq j$ such that $g(i) = g(j)$. Explain how Amplitude Amplification can be used to solve this task; essentially, what should be the sampling oracle and the marking oracle.

Q3 [2 points] Draw the entire quantum circuit to add two 3-bit integers. You can use single-qubit and two-qubit gates. Show each gate clearly. For rotation gates, specify their angles.

Q4 [2+2=4 points] This question is about teleportation.

(a) Suppose I have two qubits that I want to measure in the Bell basis. But my QPU only allows standard basis measurement. Explain how I can simulate a Bell basis measurement. Essentially, after the measurement, you should output one of the 4 outcomes corresponding to the 4 Bell basis states, and the probabilities of the outcomes should match that of the Bell basis measurements.

(b) For this question, suppose that there are three parties: Alice, Bob and Charlie. Alice and Bob share an entangled state $|\beta_{00}\rangle$. Bob and Charlie also share an entangled state $|\beta_{00}\rangle$. Now, Bob measures the two qubits he has in the Bell basis. Explain the measurement outcomes: the states observed by Bob, the corresponding probabilities, and the corresponding collapsed state of Alice and Charlie's qubits.
Hint: The calculations will become less messy if you cleverly use Bell basis instead of the standard basis.

You will observe that Alice and Charlie's qubits are now entangled even though those qubits never came from a common source. This approach is quite common in quantum networks to create entanglement among unrelated parties since most quantum communication protocols requires parties to share entangled qubits.