Mid-sem Exam: IQC 2026

50 points, 2 hours

24th Feb 2026

Note: I will not grade a solution that involves any matrix or vector.

1. **(2 + 3 = 5 points)** QFT_k indicates a k-dimensional QFT operator. Take any $m \geq 2$. (a) Is $QFT_{2^m} |0^m\rangle = (QFT_2)^{\otimes m} |0^m\rangle$? (b) Is $QFT_{2^m} = (QFT_2)^{\otimes m}$; either prove or give a counter-example.

2. **(4 points)** What is the expected outcome of measuring $|\psi\rangle = \frac{\sqrt{2}}{\sqrt{3}}|01\rangle + \frac{1+i}{\sqrt{6}}|10\rangle$ using the observable $Z \otimes X$?

3. **(3 points)** Design a 3-qubit phase oracle $\hat{U}$ that marks the state $|101\rangle$ by flipping its phase ($|101\rangle \mapsto -|101\rangle$) and acts as identity on the other standard basis states. Draw the circuit using single-qubit gates and multi-controlled NOT (MCX) gates. If you use any ancillæ, you must perform clean computation.

4. **(3 points)** Recall that $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}[|01\rangle + |10\rangle]$. Explain how this state can be converted to each of $|\beta_{00}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$ by applying appropriate operators <u>only on the first qubit of $|\beta_{01}\rangle$</u>.

5. **(3 + 2 = 5 points)** (a) Prove that $H\sqrt{Z}H = \sqrt{X}$. (b) Show how to implement the $c - Z$ (controlled-Z) operator using single-qubit gates and CNOT gate. Briefly explain why your implementation is correct.

6. **(3 + 2 = 5 points)** What is the ground state energy and the ground states for $H = -5I + 5Z_1 - Z_2 + Z_1Z_2$? Write down an equivalent QUBO whose optimal solution matches the ground state energy.

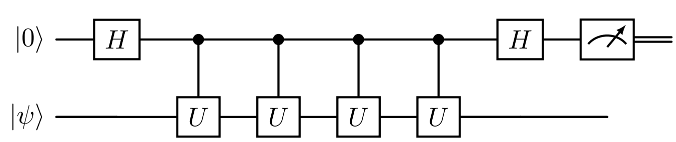
7. **(7 points)** Design a quantum circuit to implement $e^{iH/2}$ where $H = X \otimes I + I \otimes Z$, preferably using no ancillæ. Explain why your construction is correct.
   If you are unable to do this, you can construct a circuit for $H = X \otimes Z$ instead, for a reduced total of 4 points. If you are unable to do any of the above, you can construct a circuit for $H = X \otimes I$ instead, for a reduced total of 2 points.

8. **(8 points)** Consider the following operator U_{+2}:

   $$U_{+2} : |a\rangle \mapsto |a + 2 \mod 8\rangle, \quad a \in \{0, 1\}^3$$

   Design a QFT-based quantum circuit for the same that does not use an ancillæ. Since the circuit for QFT has not been taught until now, you can represent QFT and its inverse by a box on the required qubits. Explain why your construction is correct.
   Hint: Do not worry about $a + 2$ becoming 0 or 1 due to rounding after $\mod 8$.

9. **(5+5=10 points)** Let U be some unitary operator such that $U|\psi\rangle = e^{2\pi i \cdot t}|\psi\rangle$ for some $t \in [0, 1]$; furthermore, it is known that t is 3-bits, i.e., $t = 0.t_1t_2t_3$. (a) Trace the states of the following quantum circuit after each gate. (b) Explain how you can infer t_3 by observing the first qubit. Analyse the error of your strategy.



Operators

Pauli Matrices:

- $X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $Y = i|1\rangle\langle 0| - i|0\rangle\langle 1| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $S = \sqrt{Z}$, $T = \sqrt{S}$

Hadamard (H): $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Rotation Gates General rotation about axis $\hat{n}$: $R_{\hat{n}}(\theta) = \exp\left(-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}\right)$

- $R_P(\theta) = \cos(\theta/2)I - i\sin(\theta/2)P$ for $P = \{X, Y, Z\}$ and any other single-qubit P such that $P^2 = I$.
- $R_z(\theta) = \mathrm{diag}(e^{-i\theta/2}, e^{i\theta/2})$

Phase Gate (P): $P(\phi) = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|$. Note $S = P(\pi/2), T = P(\pi/4)$.

Bloch Sphere $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$.
θ: angle from Z-axis ($0 \to \pi$), ϕ: phase in X-Y plane ($0 \to 2\pi$).

Pauli Identities $X^2 = Y^2 = Z^2 = I$, $XY = iZ$, $YZ = iX$, $ZX = iY$.

Bell States

$$\beta_{00} = \tfrac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$
$$\beta_{01} = \tfrac{1}{\sqrt{2}}[|01\rangle + |10\rangle]$$
$$\beta_{10} = \tfrac{1}{\sqrt{2}}[|00\rangle - |11\rangle]$$
$$\beta_{11} = \tfrac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$$

QFT Quantum Fourier Transform

$\mathrm{QFT}|j\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi ijk/N}|k\rangle$

$\mathrm{QFT}^\dagger|k\rangle = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{-2\pi ijk/N}|j\rangle$

Binary Representation: $|j\rangle = |j_1 j_2 \dots j_n\rangle$ where $j = \sum j_l 2^{n-l}$.

$$\mathrm{QFT}|j_1 \dots j_n\rangle = \frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i 0.j_1 \dots j_n}|1\rangle)}{\sqrt{2^n}}$$

Amplitude Amplification Generalizes Grover for any algorithm A where $A|0\rangle = |\psi\rangle$.

- Let $|\psi\rangle = \sin(\theta)|\psi_{good}\rangle + \cos(\theta)|\psi_{bad}\rangle$.
- Grover operator $Q = -AS_0A^\dagger S_\chi$, where S_0 reflects about $|0\rangle$ and S_χ reflects about "good" states.
- Number of iterations of Q required for constant error: $O(1/\sin\theta)$.

QUBO to Ising Mapping QUBO: Minimize $f(x) = x^T Q x$ where $x_i \in \{0, 1\}$.

$$H_{Ising} = \sum_i h_i Z_i + \sum_{i<j} J_{ij} Z_i Z_j + \mathrm{const}$$

Transformation: Substitute $x_i = \frac{1-Z_i}{2}$ where Z_i is the Pauli-Z op.