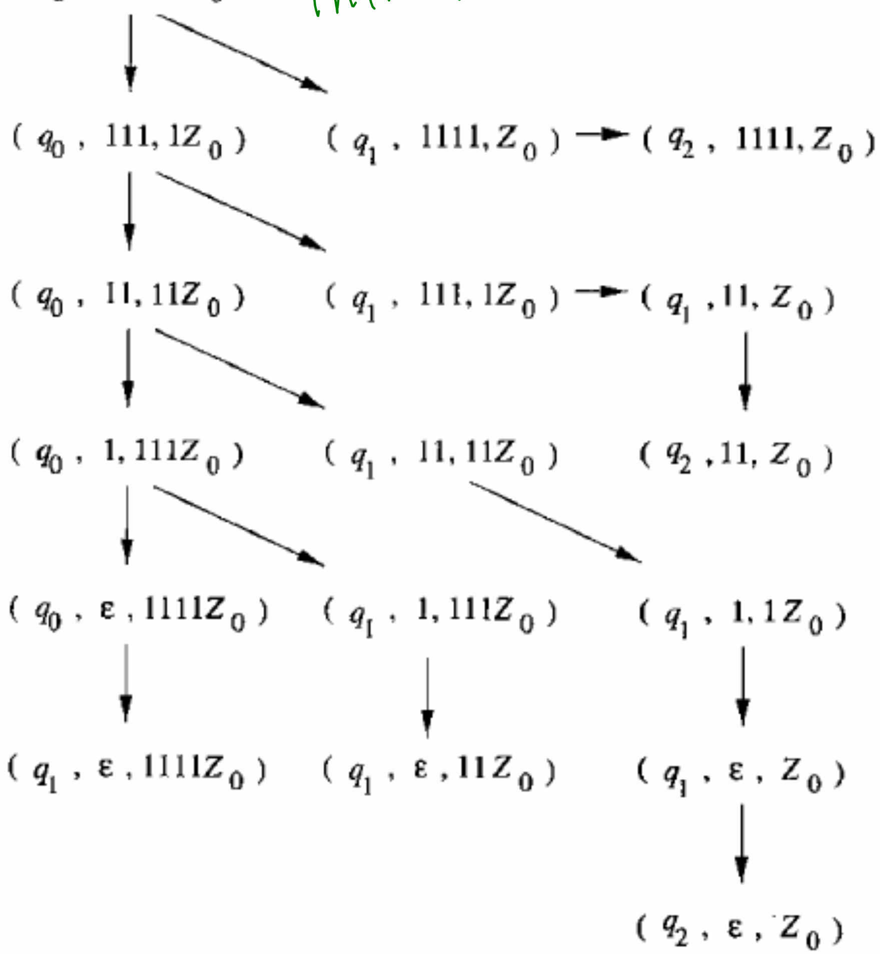
CSE322 Theory of Computation (L14)

Today

PDA to CFG



initial ID

L = { w.rev(w) }

accepting ID: State ∈ F

no input should be left

ID of a PDA :

(state, remaining input, stack contents)

(q , aw , sZ) |- (q', w, tZ)

means d(q,a,s) = {(q',t), ...}

Notation: |-* denotes multiple moves

$\boxed{\text{(q0,1111,e) |-* (qa, e, e)}}$

$(q1,\ w1\,y,\ S1)\ \vdash$

$(q2,\ w2\,y,\ S2)\ \vdash$

...

$(qk,\ wk\,y,\ Sk)\ \vdash$

1

$(q1,\ w1.\cancel{y},\ S1)\ \vdash$

$(q2,\ w2.\cancel{y},\ S2)\ \vdash$

...

$(qk,\ wk.\cancel{y},\ Sk)\ \vdash$

2

$(q1,\ w1,\ S1\,T)\ \vdash$

$(q2,\ w2,\ S2\,T)\ \vdash$

...

T

T

T

T

$(qk,\ wk,\ Sk\,T)\ \vdash$

3

$(q1,\ w1,\ \cancel{\$}\ )\ \vdash$

$(q2,\ w2,\ ...\cancel{\$}\ )\ \vdash$

$\cancel{\$}$

...

$\cancel{\$}$

stack always has \$, never empty

\$

$(qk,\ wk,\ ...\cancel{\$}\ )\ \vdash$

Can we remove \$?

What if no step can both pop & push?

Consider a sequence of transitions: $(p, w, S) \vdash^* (q, y, T)$.

1 $(p, wz, S) \vdash^* (q, yz, T)$

2 if $w = w'z$, $y = y'z$, then $(p, w', S) \vdash^* (q, y', T)$

3 $(p, w, SV) \vdash^* (q, y, TV)$

4 * if $S = S'\$$, $T = T'\$$, under special conditions, $(p, w, S') \vdash^* (q, y, T')$

PDA to CFG

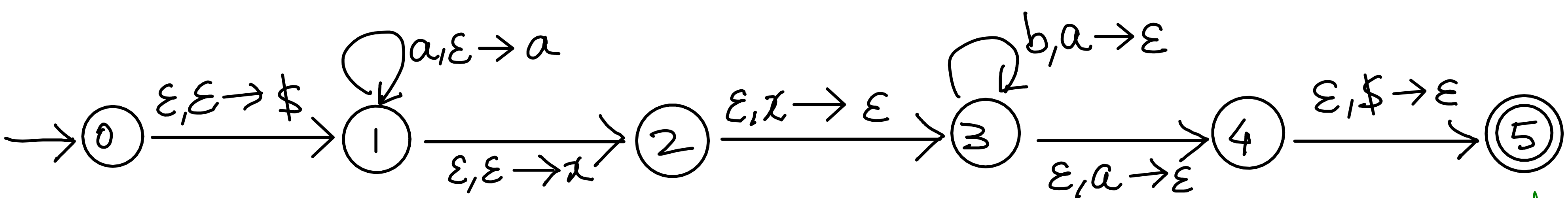
Construct G from PDA: PDA accets w iff G generates w.

Modify PDA:

$|Q| = q$

* One accepting state qa.
* Stack is empty at beginning and at end.
* Each transition either pushes or pops but not both.

$a/\varepsilon \to x$ $\quad a/x \to \varepsilon$ $\quad \cancel{a/x \to y}$



Variables

A_{00} A_{01} ... A_{05} start var.

... A_{50} ... A_{55}

A_{pq} $\forall p, q \in Q$

...

Rules

q rules

(A) $A_{00} \to \varepsilon$ $\quad A_{11} \to \varepsilon$ ... $A_{55} \to \varepsilon$ $\quad A_{qq} \to \varepsilon \ \forall q$

$A_{05} \to A_{01} A_{15}$ $\quad A_{05} \to A_{02} A_{25}$ $\quad A_{pq} \to A_{pr} A_{rq}$

(B) q^3 rules ... $A_{55} \to A_{50} A_{05}$ $\quad \forall p, q, r$

$A_{05} \to \varepsilon A_{14} \varepsilon$ $\quad$ (p', x) in $d(p, a, e)$ & (C) $\leq q^4 \cdot |\Sigma+1| \cdot x|\Sigma+1| \cdot x|\Gamma+1|$

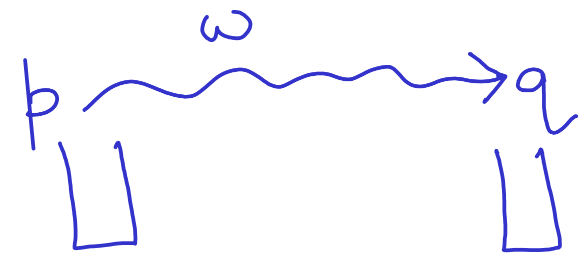
$A_{13} \to a A_{13} b$ $\quad$ (q, e) in $d(q', b, x)$ for some x

$A_{pq} \to a A_{p'q'} b$ if

$p \xrightarrow{a,\varepsilon \to x} p' \dashrightarrow q' \xrightarrow{b,x \to \varepsilon} q$

PDA to CFG

Lemma: $A(p,q) \Rightarrow^* w$ $\equiv$ For any $w \in \Sigma^*$ w s.t. $(p,w,e) \vdash^* (q,e,e)$



$L = \{ w \mid (q0, w, e) \vdash^* (qa, e, e) \} \equiv \{ w \text{ derived from } A(q0,qa) \}$

starting ID final ID

Idea

Take w in L. First move must be push and last move must be pop.

Either $w = w1.w2$ and for some intermediate r,

$$(q0, w1w2, e) \vdash^* (r, w2, e) \vdash^* (qa, e, e)$$

empty stack

$\therefore (q0, w1, \epsilon) \vdash^* (r, \epsilon, \epsilon)$

$$A_{q0,qa} \rightarrow A_{q0,r} A_{r,qa} \Rightarrow^* w_1 w_2$$

or, $w = a.w'.b$ and $(q0, aw'b, e) \vdash (p', w'b, c) \underbrace{\vdash^* (q', b, c)}$ $\vdash (qa, e, e)$

first symbol pushed popped at last.

Stack not empty, a move can't pop + push

$\therefore$ No intermediate transition can depend on e

$\Rightarrow (p', w'b, \epsilon) \vdash^* (q', b, \epsilon)$

$$A_{q0,qa} \rightarrow a A_{p'q'} b$$

$\Downarrow^*$ w'

If $A(p,q) \Rightarrow^* x$, then $(p,x,e) \vdash^* (q,e,e)$.

$k=1$ single step derivation

$A_{pq} \Rightarrow x$ only class A rule

$A_{qq} \to \varepsilon$

$\therefore p=q$ & $x=\varepsilon$

$I \vdash^* I$

Proof by induction on k=length of derivation of x from $A(p,q)$.

Base case: $k=1$. $A(p,p) \Rightarrow e$. $\underline{(p,e,e) \vdash^* (p,e,e)}$. ✓

Ind. Hyp.: Stmt true for $k=1\ldots n$

Ind. Step: To prove stmt for $k=n+1$.

$A(p,q) \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots$ n+1 times $\Rightarrow x$

Case analysis on the first step of derivation

class B rule

$A_{pq} \Rightarrow A_{pr} A_{rq} \Rightarrow^* x = x_1 x_2$

So, $A_{pr} \Rightarrow^* x1$ and $A_{rq} \Rightarrow^* x2$ & $x=x1\ x2$

$\leq k$ $\leq k$

By IH,

$\underline{(p,x1,e) \vdash^* (r,e,e)}$ & $(r,x2,e) \vdash^* (q,e,e)$.

By the previous lemma,

$(p,x1\ x2,e) \vdash^* (r,x2,e)$.

Combining,

$(p, \underbrace{x1\ x2}_{x}, e) \vdash^* (q,e,e)$.

class C rule

$\boxed{A_{pq} \Rightarrow a A_{p'q'} b} \Rightarrow^* x = a x' b$

where $A_{p'q'} \Rightarrow^* x'$

$\leq k$ steps

$A_{pq} \to a A_{p'q'} b$

$p \xrightarrow{a,\varepsilon \to x} p'$

$q' \xrightarrow{b,x \to \varepsilon} q$

If $(p,x,e) \vdash^* (q,e,e)$, then $A(p,q) \Rightarrow^* x$.

$\vdash^*$ $\quad I \vdash^* I$

Induction on k = number of transitions.

Base case: $k=0$, so $p=q$, $x=e$.

RHS $A_{pp} \Rightarrow^* \varepsilon$. ($\because A_{pp} \to \varepsilon$)

Ind Hyp.: True for $k=0 \ldots n$

Ind Step: $(p,x,e) \vdash \ldots \vdash \ldots (n+1)$ times $\vdash (q,e,e)$?

Case analysis

Stack is empty only at beginning and at end

First symbol pushed (c) must be popped at last.

$(p, \underbrace{a\, x'\, b}_{x}, e) \vdash (p', x'\, b, c) \vdash^*$

$(q', b, c) \vdash (q, e, e)$

(p', c) in $d(p, a, e)$ & (q, e) in $d(q', b, c) \Rightarrow$

G has rule: $A(p,q) \Rightarrow a\, A(p'q')\, b$.

steps $\leq k$

Since $(p', x'b, c) \vdash^* (q', b, c)$ without emptying stack, none of its transitions can depend on the c on the stack. So, the following is also a valid transition: $(p', x', e) \vdash^* (q', e, e)$. # steps. $\leq k$

Stack is empty in the middle too

$x = ax'$

$(p, x, \varepsilon) \vdash (p', x', c)$ $\quad a \in \Sigma_\varepsilon$, $b \in \Sigma_\varepsilon$

& $(q', y', c) \vdash (q, y, \varepsilon)$

$y' = by$ $\quad x = ax''b$

Exercise

$A_{pq} \to a\, A_{p'q'}\, b$

$\delta(p, a, \varepsilon) \ni (p', c)$

$\delta(q', b, c) \ni (q, \varepsilon)$

By IH, $A(p'q') \Rightarrow^* x'$.

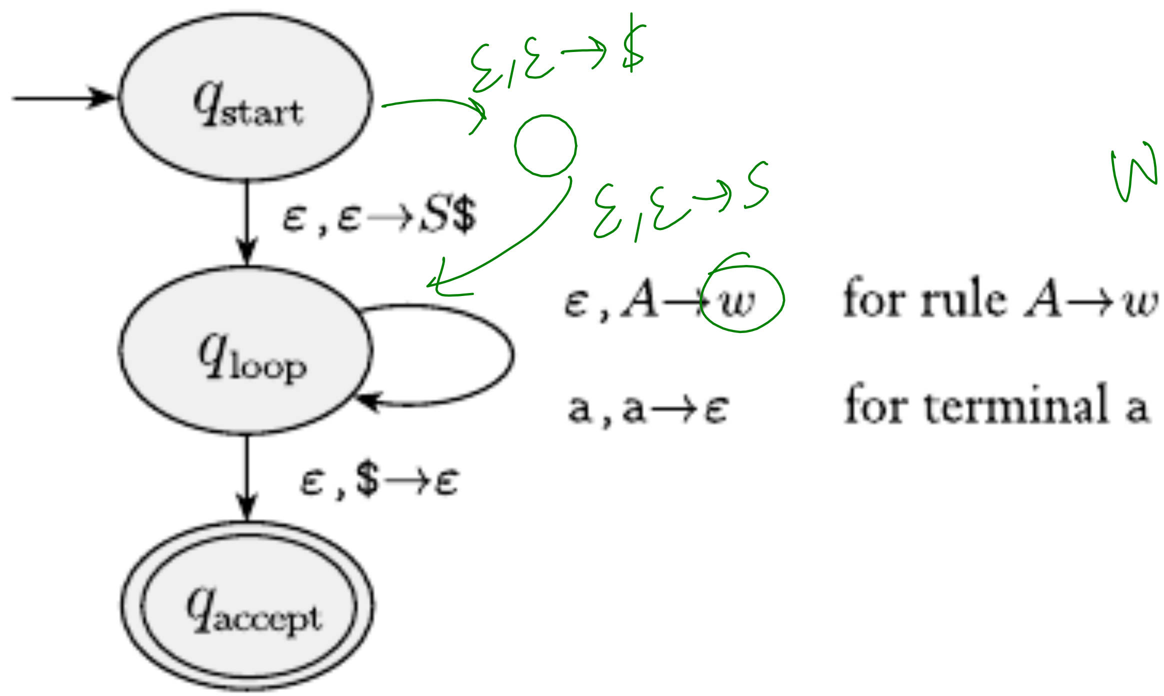
Therefore, $A(p,q) \Rightarrow a\, A(p',q')\, b \Rightarrow \underbrace{a\, x'\, b}_{x}$.

CFG to PDA

Construct PDA from G: G generates w iff PDA accepts w.

PDA: Non-deterministically guess the derivation/parse tree

$\varepsilon, \varepsilon \to S\$$



S
$

$w = aAbB$

a
A
b
B

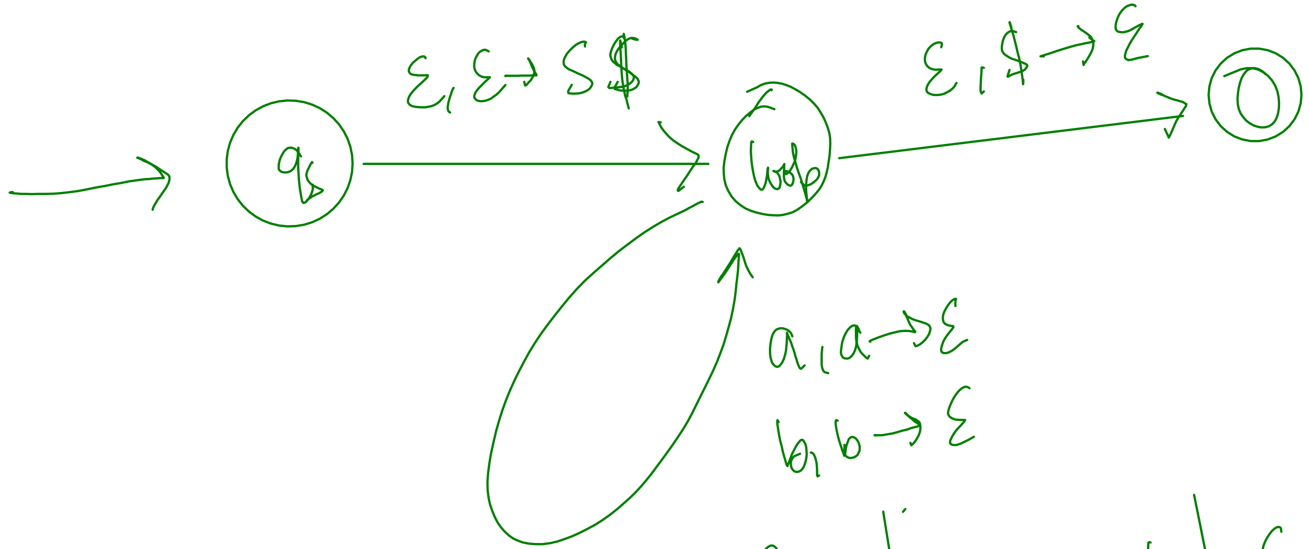
Ex. Generate PDA for

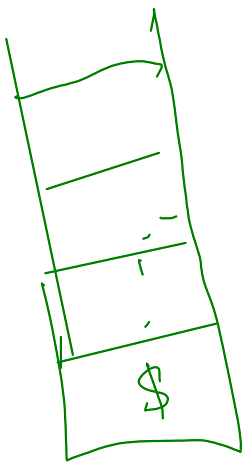
S -> aTb | b, T -> Ta | e

$S \overset{*}{\Rightarrow} aab$

$S \Rightarrow aTb \Rightarrow aTab$

$\Rightarrow aab$





$\varepsilon, S \to aTb \;\big|\; \varepsilon, S \to b \;\big|\; \varepsilon, T \to Ta \;\big|\; \varepsilon, T \to \varepsilon$

Show that ...

$\{w \text{ over } \{a,b,c\} : \#a(w) = \#b(w) = \#c(w)\}$ is not CFL

L = above language

L1 = $a^*b^*c^*$

Prove that L intersect L1 is not CFL.

Then prove that L is not CFL.

Closure under NOPREFIX

Not closed under NOP

$NOP(L) = \{ w \text{ in } L \text{ s.t. no proper prefix of } w \text{ is in } L \}$

$L_1 = \{ a^n b^m c : n \neq m,\ n \geq 1,\ m \geq 1 \}$ $= \{a^n b^m : n \neq m\} \cdot \{c\}$ $\rightarrow$ CFL

$L_2 = \{ a^n b^m c^m : n \geq 1,\ m \geq 1 \}$ $\rightarrow$ CFL

$L = L_1 \cup L_2$

What is NOP(L1) = ?

What is NOP(L2) = ?

What is NOP(L1 U L2) = ?

Prove that L1 is CFL, L2 is CFL and L is CFL.

Q: How to prove NOP(L) is NOT CFL? $= L1 \cup \{a^n b^n c^n : n \geq 0\}$

$\{a^n b^m c : \cancel{n \neq m}\}$

Earlier not CFL.

Let $L3 = a^* b^* cc\, c^*$

What is $L4 = NOP(L)$ intersect $L3$? $= \{a^n b^n c^n : n \geq 2\} \cup \{\varepsilon, abc\}$

How to prove that L4 is not CFL?

Prove $L = \{ w=a^* b^* : \#(a,w) \,!= \#(b,w) \text{ and } \#(a,w) \,!= 2\,\#(b,w) \}$ is CFL.

Divide $L = L1 \cup L2 \cup L3$ and show that each is CFL.

Exercise: Show that $L1 = \{ w=a^*b^* : \#(a,w) < \#(b,w) \}$ is CFL.

Exercise: Show that $L2 = \{ w=a^*b^* : \#(a,w) > 2\,\#(b,w) \}$ is CFL.

Show that $L3 = \{ w=a^*b^* : \#(b,w) < \#(a,w) < 2\,\#(b,w) \}$

Let $i = \#(a,w)$, $j=\#(b,w)$

Show that $i=k+2h$ and $j=k+h$.