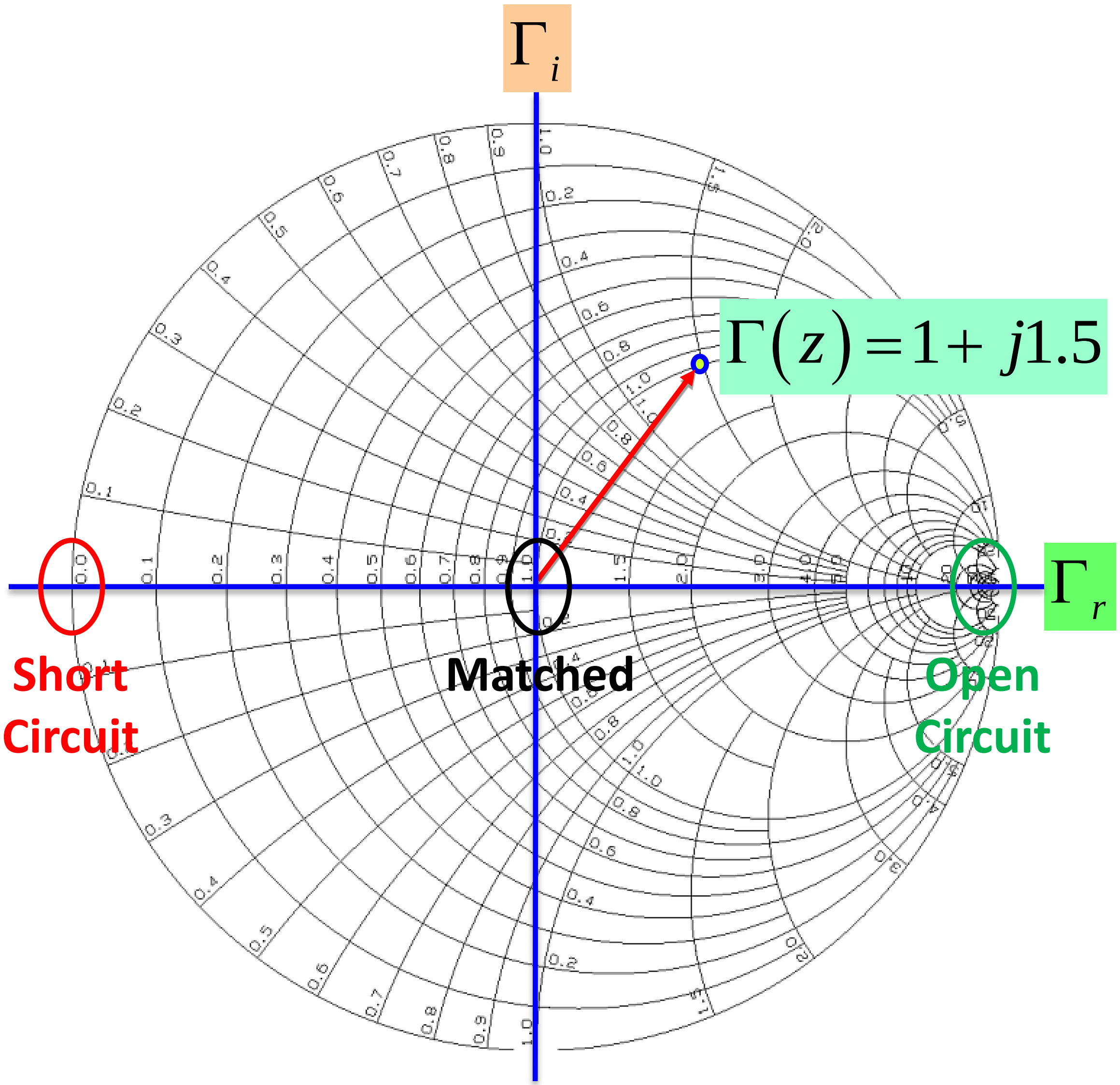
Lecture – 7

Date: 26.08.2014

- Smith Chart
- Smith Chart – Geography
- Smith Chart – Outer Scales
- Examples

The Smith Chart

Actual Smith chart



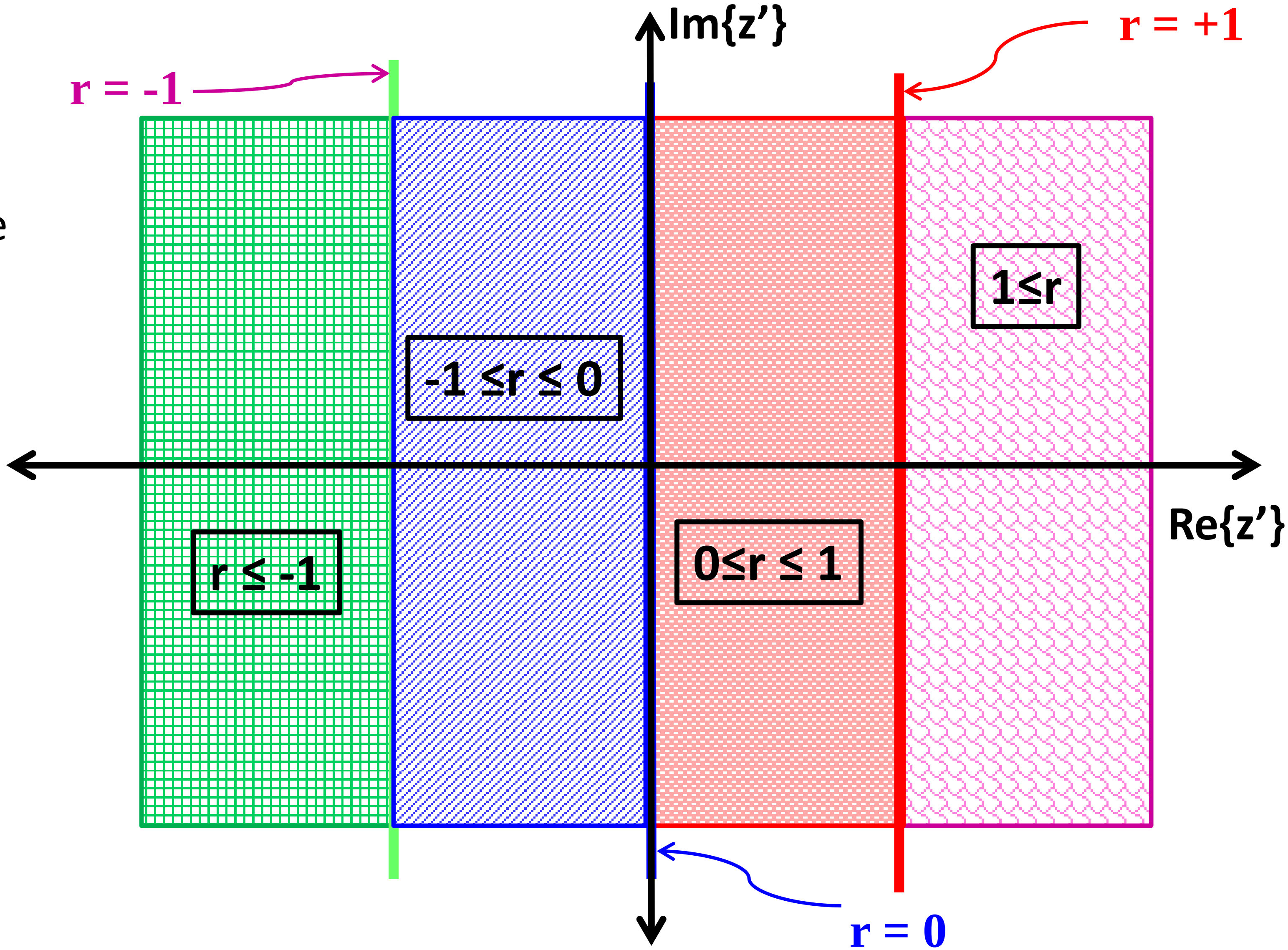
The Smith Chart – Geography

- We have located specific **points** on the complex impedance plane, such as a **short circuit** or a **matched load**
- We've also identified **contours**, such as r =1 or x =1.5

We can likewise identify **whole regions (!)** of the complex **impedance** plane, providing a bit of a **geography lesson** of the complex impedance plane

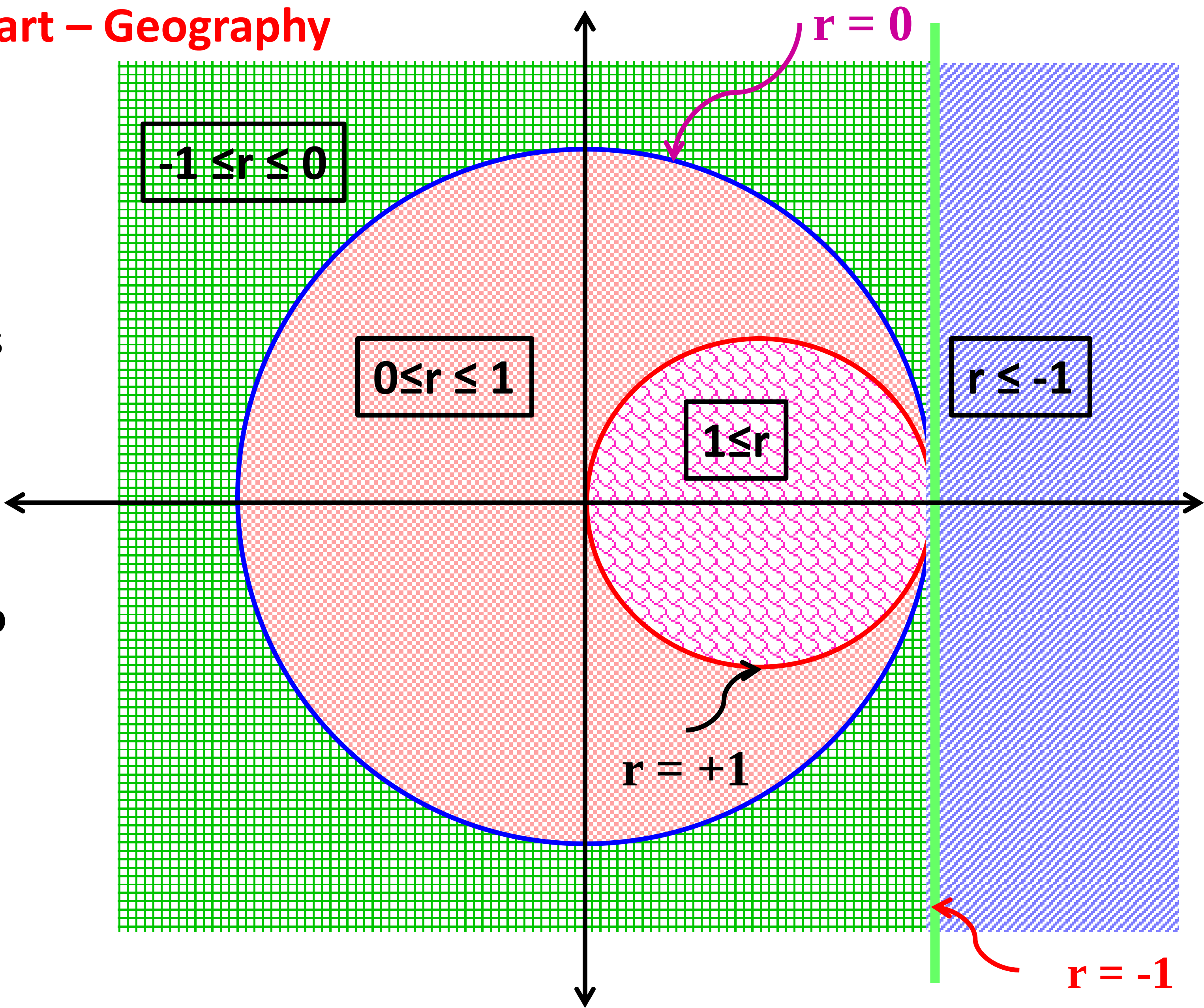
The Smith Chart – Geography

For example, we can divide the complex impedance plane into **four** regions based on normalized **resistance** value **r**:



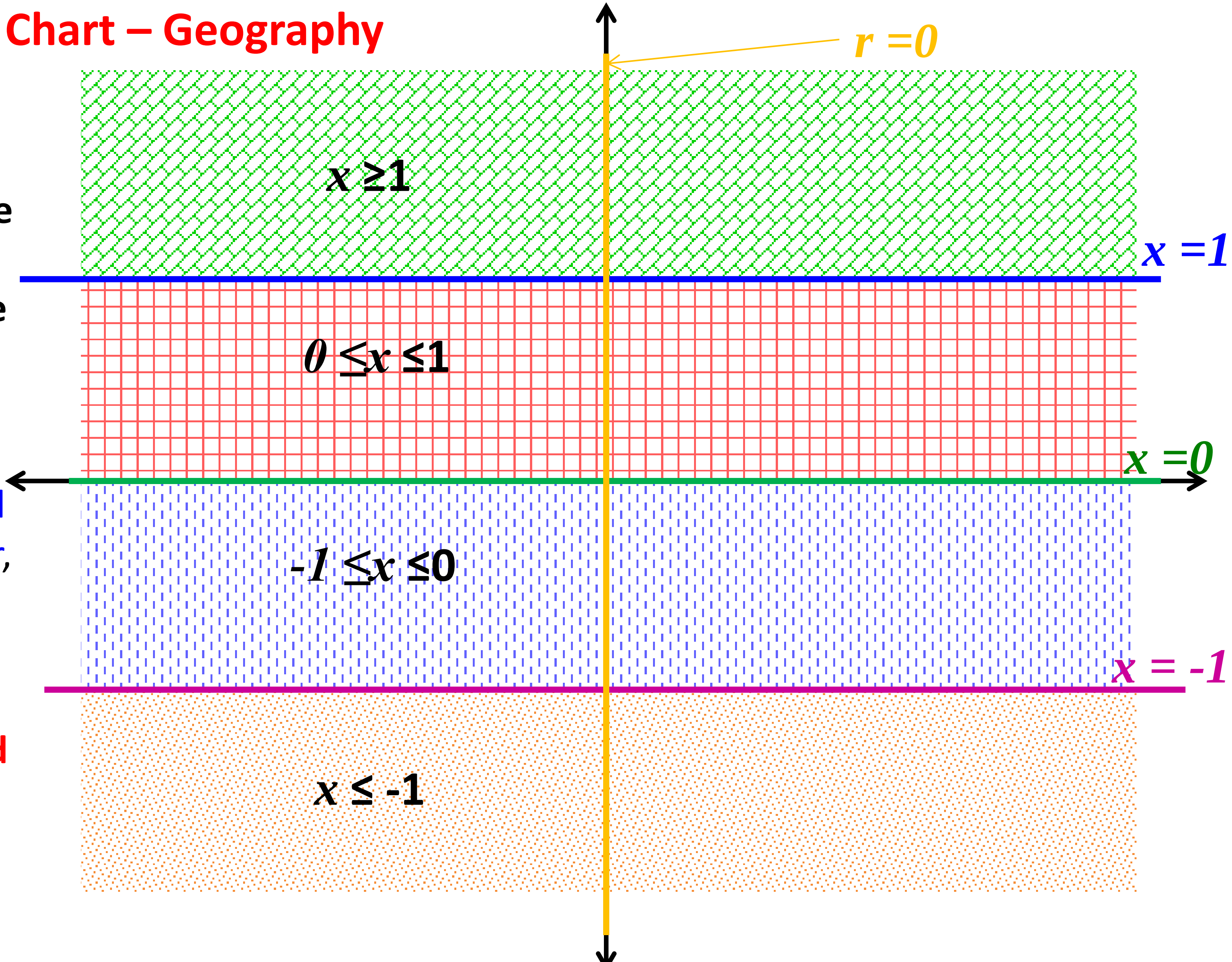
The Smith Chart – Geography

Just like points and contours, these regions of the complex impedance plane can be **mapped onto the complex gamma plane!**



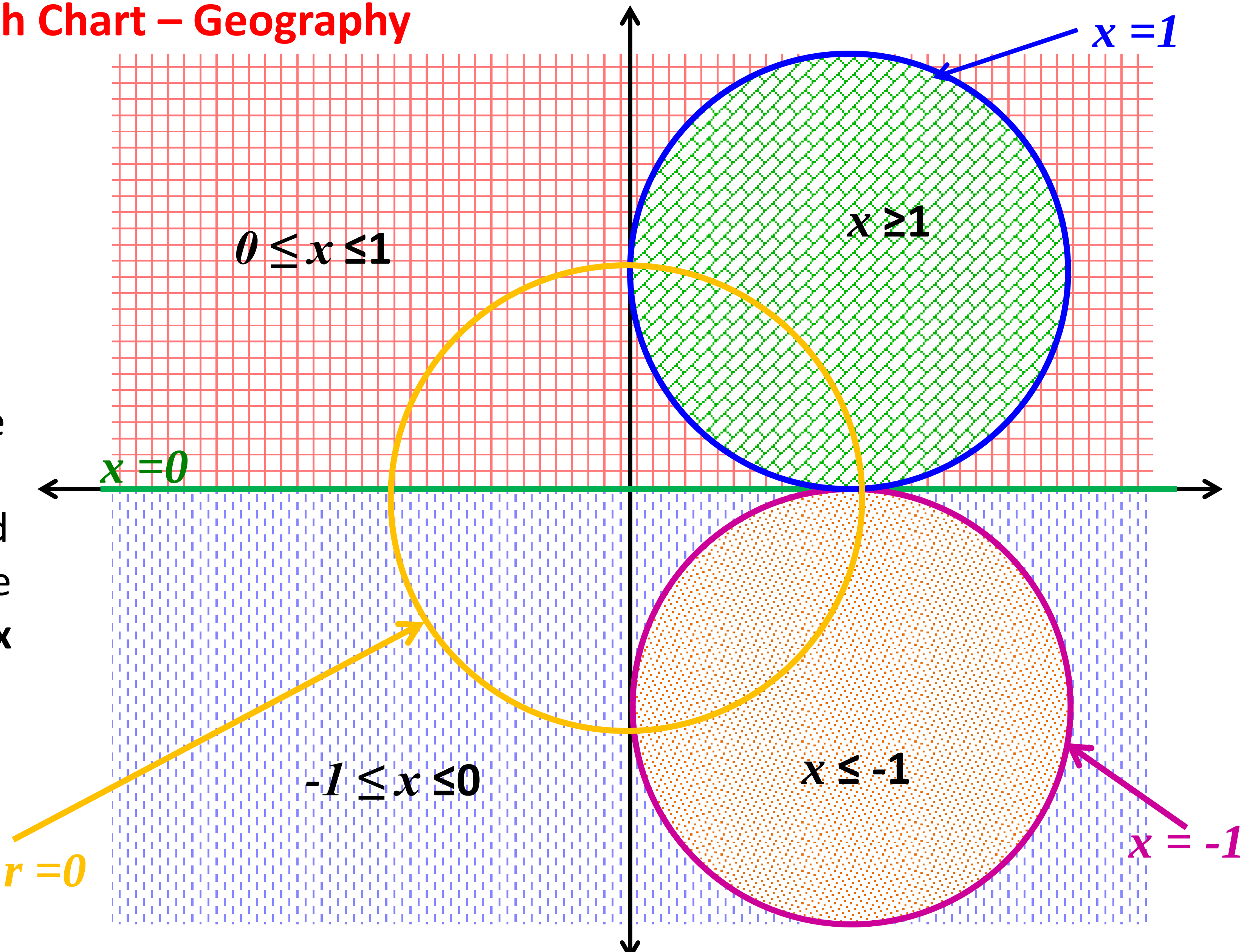
The Smith Chart – Geography

Instead of dividing the complex impedance plane into regions based on **normalized resistance r**, we could divide it based on **normalized reactance x**:

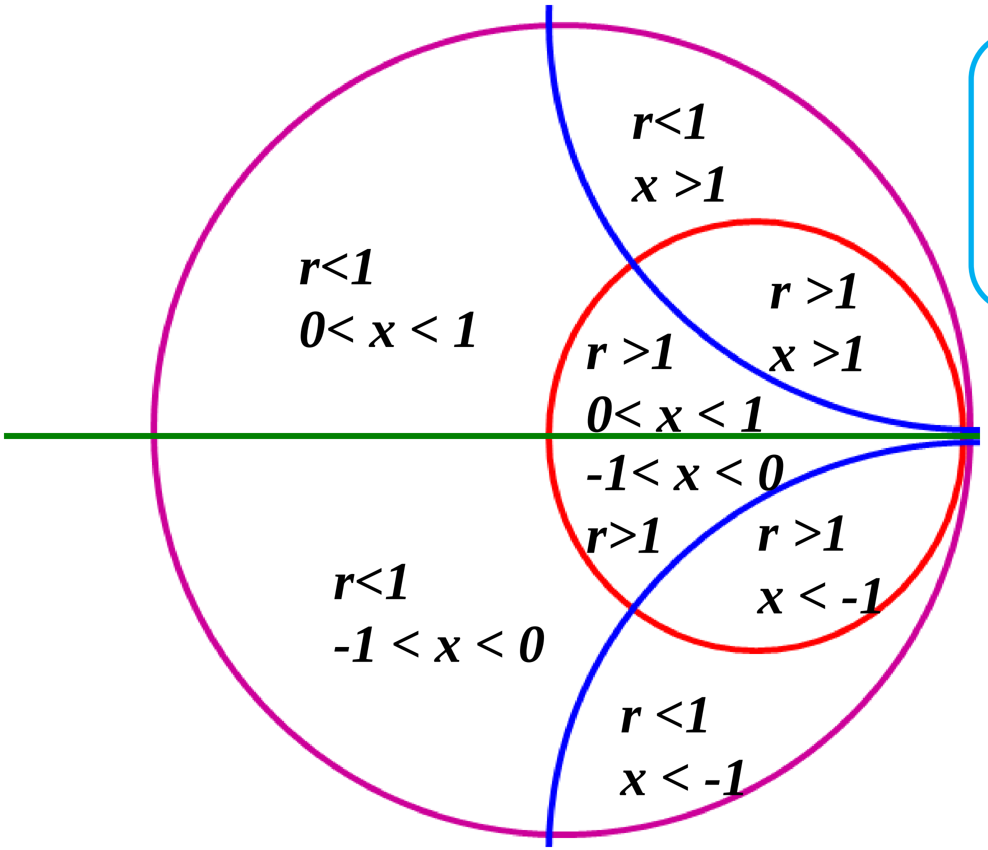


The Smith Chart – Geography

These four regions can likewise be mapped onto the **complex gamma plane:**



The Smith Chart – Geography



Note the four resistance regions and the four reactance regions combine to from **16 separate regions on the complex impedance** and complex gamma planes!

Eight of these sixteen regions lie in the valid region (i.e., r > 0), while the other eight lie entirely in the invalid region.

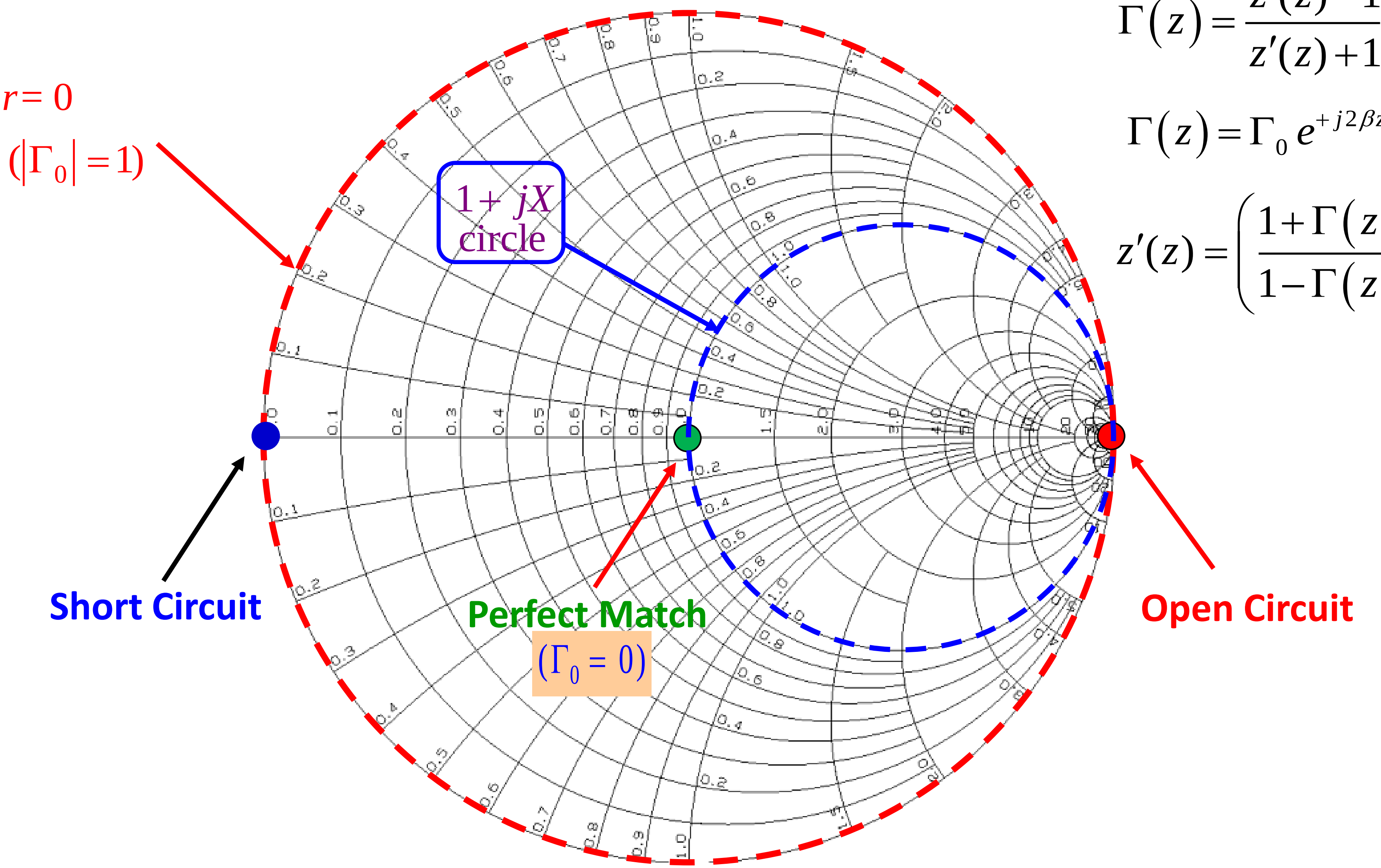
Make sure **you can locate the eight impedance regions on a Smith Chart—this understanding of Smith Chart geography** will help you understand your design and analysis results!

The Smith Chart – Important Points

$$\Gamma(z) = \frac{z'(z)-1}{z'(z)+1}$$

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

$$z'(z) = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

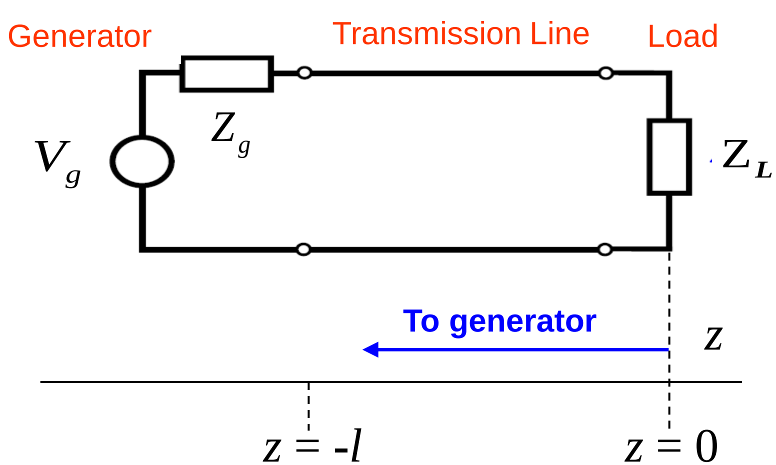


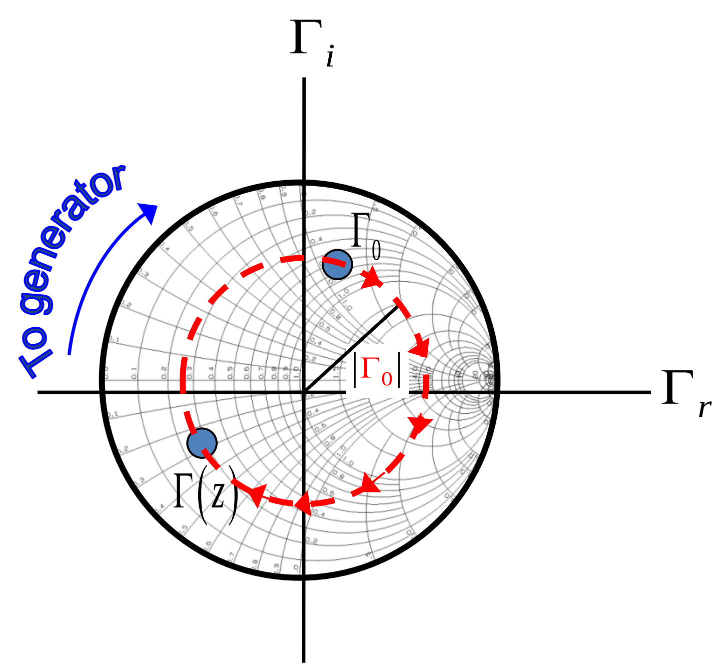
The Smith Chart (contd.)

$$z'(z) = \frac{1+\Gamma_0 e^{+2j\beta z}}{1-\Gamma_0 e^{+2j\beta z}}$$

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

movement in negative z direction (toward generator) $\Leftrightarrow$ clockwise motion on circle of constant $|\Gamma_0|$





angle change = $2\beta z$

The Smith Chart (contd.)

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

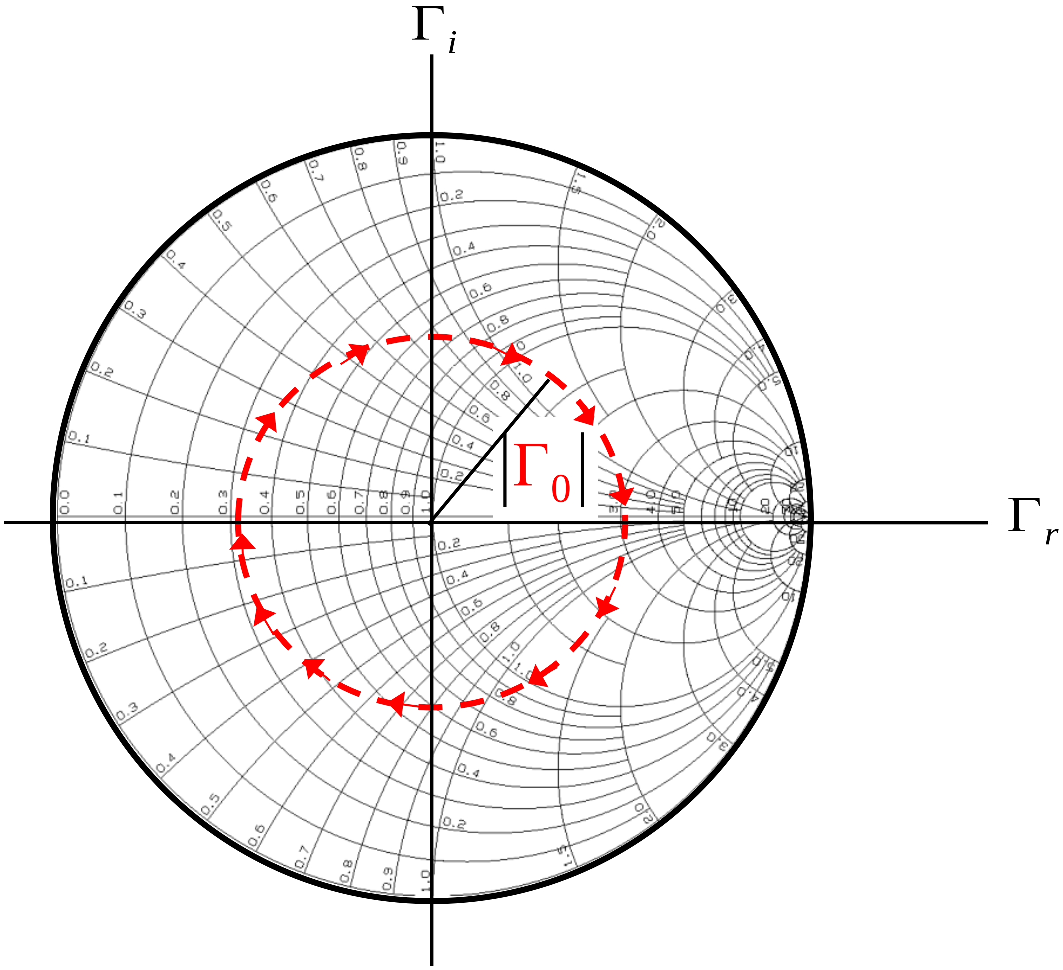
We go completely around the Smith chart when

$$z = \lambda / 2$$

$$\beta z = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{2}\right) = \pi$$

In general:

$$2\beta z = 2\left(\frac{2\pi}{\lambda}\right)(z) = 4\pi\left(\frac{z}{\lambda}\right)$$



Note: the Smith chart already has wavelength scales on the perimeter for your convenience.

The Smith Chart (contd.)

Reciprocal Property

$$z'(z) = \left(\frac{1+\Gamma_0 e^{+j2\beta z}}{1-\Gamma_0 e^{+j2\beta z}} \right)$$

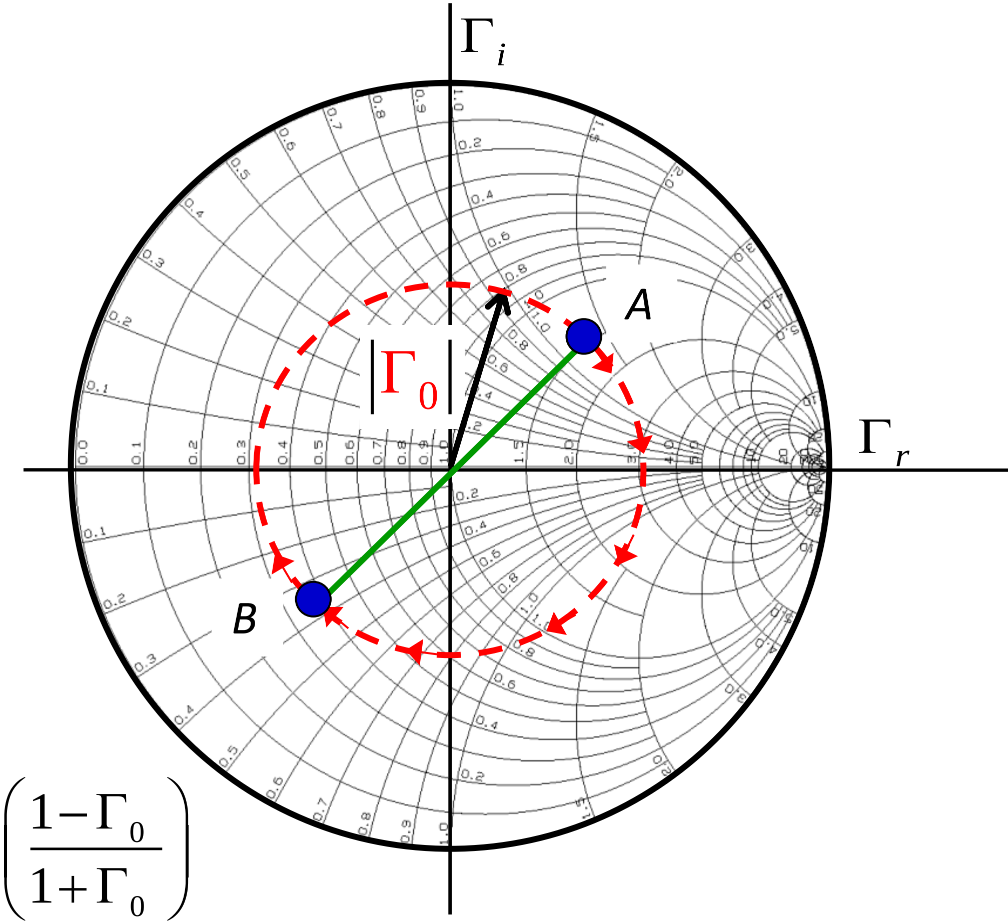
- Go half-way around the Smith chart:

$-l = \lambda / 4$

$$2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\left(-\frac{\lambda}{4}\right) = -\pi$$

$$z'(z=0) = \left(\frac{1+\Gamma_0}{1-\Gamma_0}\right) \qquad z'(z=-l) = \left(\frac{1-\Gamma_0}{1+\Gamma_0}\right)$$

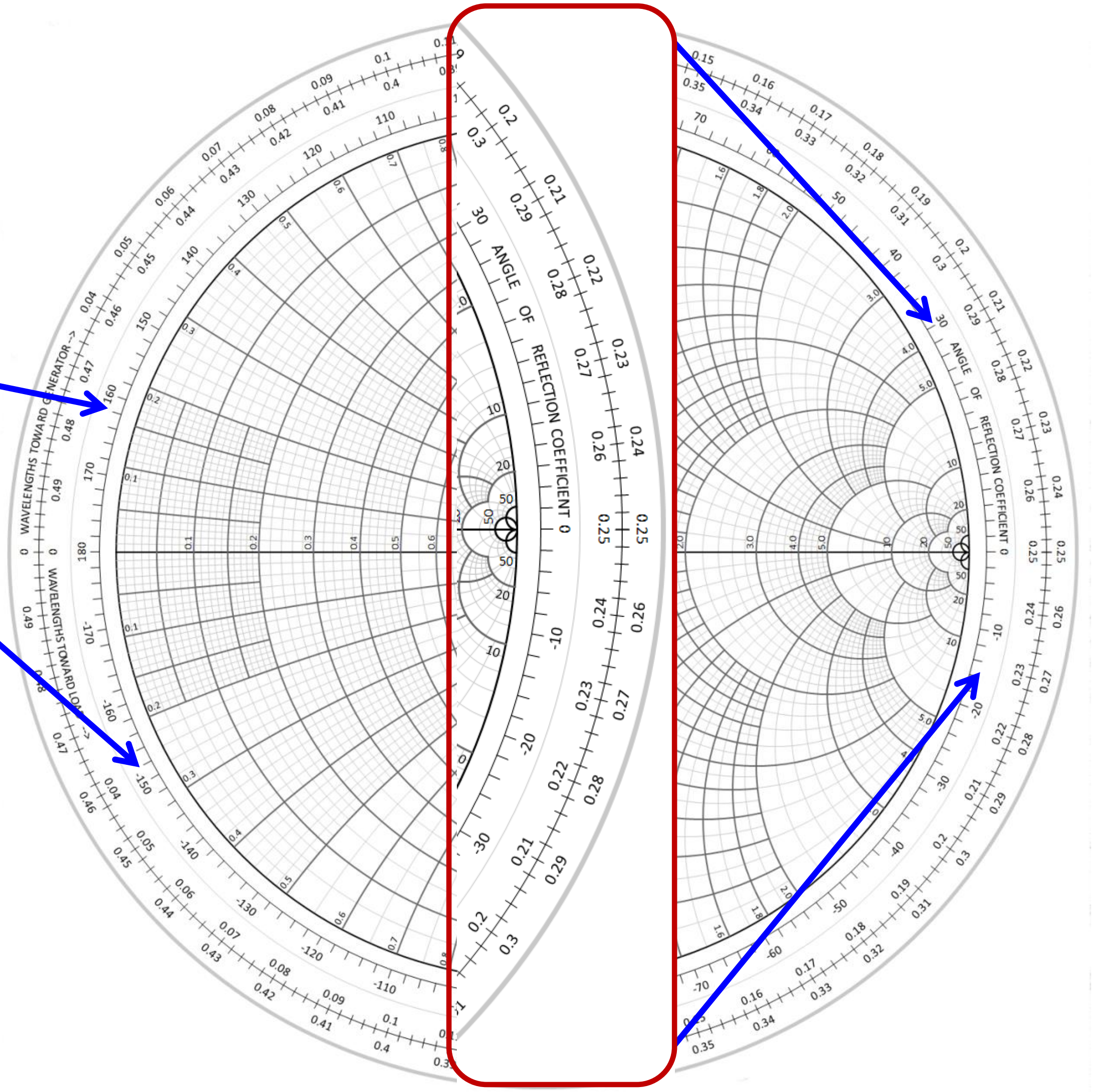
$$z'(z=0) = \frac{1}{z'(z=-l)}$$



$$z'(A) = \frac{1}{z'(B)} \Rightarrow z'(A) = y'(B)$$

The Smith Chart – Outer Scale

Note that around the **outside** of the Smith Chart there is a scale indicating the **phase angle**, from 180^{o} to -180^{o}.



The Smith Chart – Outer Scale (contd.)

- Recall however, for a **terminated** transmission line, the reflection coefficient function is:

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z} = |\Gamma_0| e^{+j(2\beta z + \theta_0)}$$

- Thus, the **phase** of the reflection coefficient function depends on transmission line **position z** as:

$$\theta_\Gamma(z) = 2\beta z + \theta_0 = 2\left(\frac{2\pi}{\lambda}\right) z + \theta_0 = 4\pi\left(\frac{z}{\lambda}\right) + \theta_0$$

- As a result, a **change** in line position z (i.e., Δz) results in a **change** in reflection coefficient phase θ_Γ (i.e., $\Delta\theta_\Gamma$):

$$\Delta\theta_\Gamma = 4\pi\left(\frac{\Delta z}{\lambda}\right)$$

- E.g., a change of position equal to one-quarter wavelength **$\Delta z = \lambda/4$** results in a phase change of π radians—we rotate **half-way** around the complex Γ-plane (otherwise known as the Smith Chart).

The Smith Chart – Outer Scale (contd.)

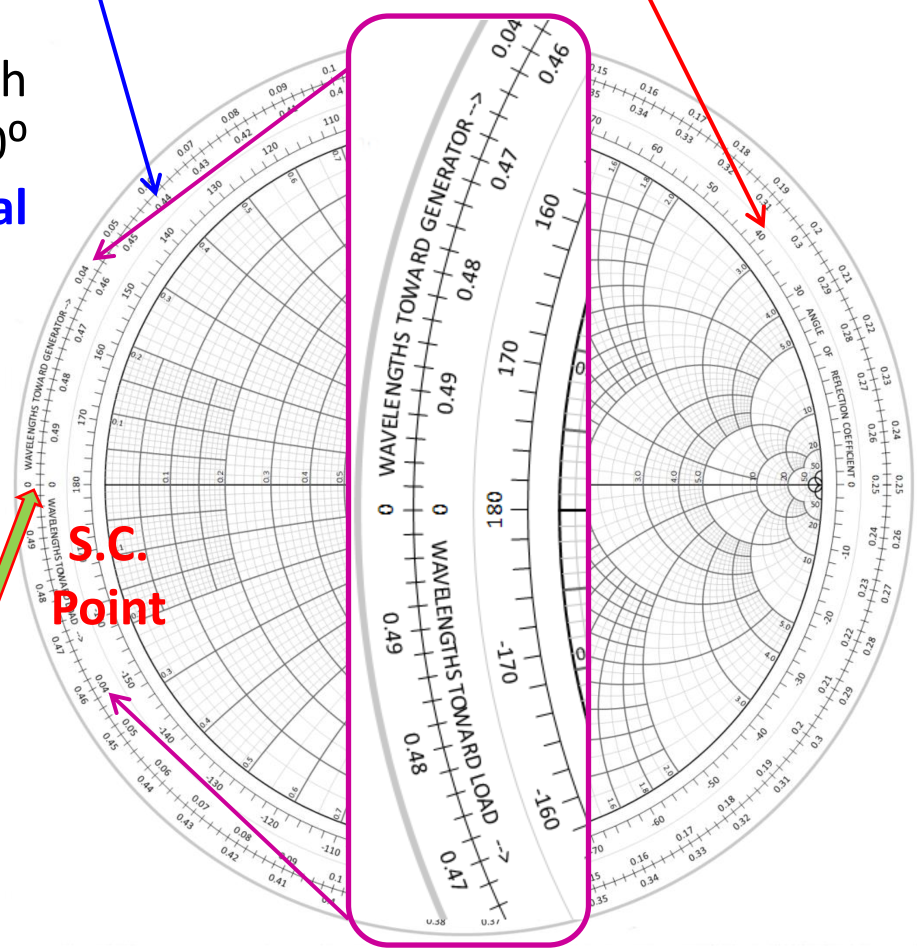
- The Smith Chart then has a **second scale** (besides θ_Γ) that surrounds it—one that relates TL position in **wavelengths** $(\Delta z/\lambda)$ to the θ_Γ:

- Since the phase scale on the Smith Chart extends from $-180^0 < \theta_\Gamma < 180^0$ (i.e., $-\pi < \theta_\Gamma < \pi$), this **electrical length scale** extends from:

$$0 < z/\lambda < 0.5$$

- Note, for this mapping the reflection coefficient phase at location z = 0 is $\theta_\Gamma = -\pi$. Therefore, $\theta_0 = -\pi$, and we find that:

$$\Gamma_0 = |\Gamma_0| e^{+j\theta_0} = |\Gamma_0| e^{-j\pi} = -|\Gamma_0|$$



The Smith Chart – Outer Scale (contd.)

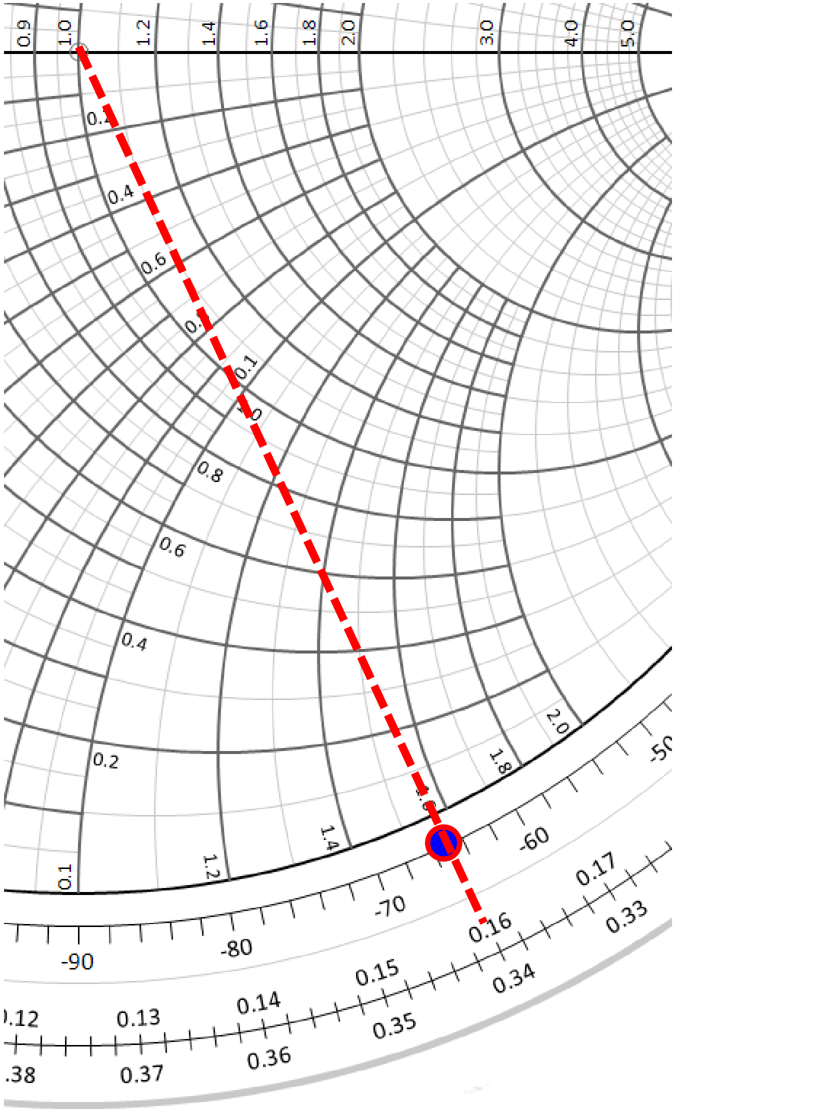
- **Example:** say you're at some location z = z_1 along a TL. The value of the **reflection coefficient** at that point happens to be:

$$\Gamma(z = z_1) = 0.685 e^{-j65^\circ}$$

- Finding the **phase angle** of θ_Γ = -65° on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

$$0.160\lambda$$

Note: this tells us **nothing** about the location z = z_1. This does **not** mean that z_1 =0.160λ , for example!



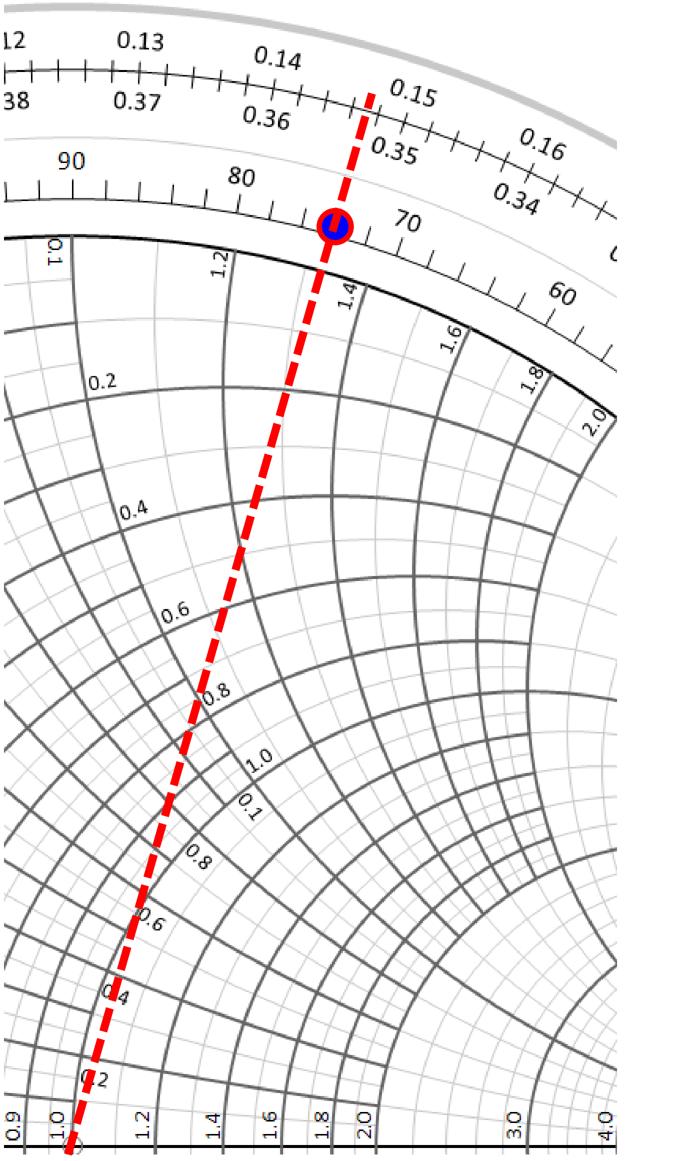
The Smith Chart – Outer Scale (contd.)

- Now, say we **move a short distance** Δz (i.e., a distance less than λ/2) along the transmission line, to a **new location** denoted as $z = z_2$ and find that the **reflection coefficient** has a value of:

$$\Gamma(z = z_2) = 0.685 e^{j74^\circ}$$

- Now finding the **phase angle** of $\theta_\Gamma = 74^o$ on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

$$0.353\lambda$$



Note: this tells us **nothing** about the location $z = z_2$. This does **not** mean that $z_1 = 0.353\lambda$, for example!

The Smith Chart – Outer Scale (contd.)

Q: So what do the values 0.160λ and 0.353λ tell us?

A: They allow us to determine the **distance between** points z_2 and z_1 on the transmission line.

$$\Delta z = z_2 - z_1 = 0.353\lambda - 0.160\lambda = 0.193\lambda$$

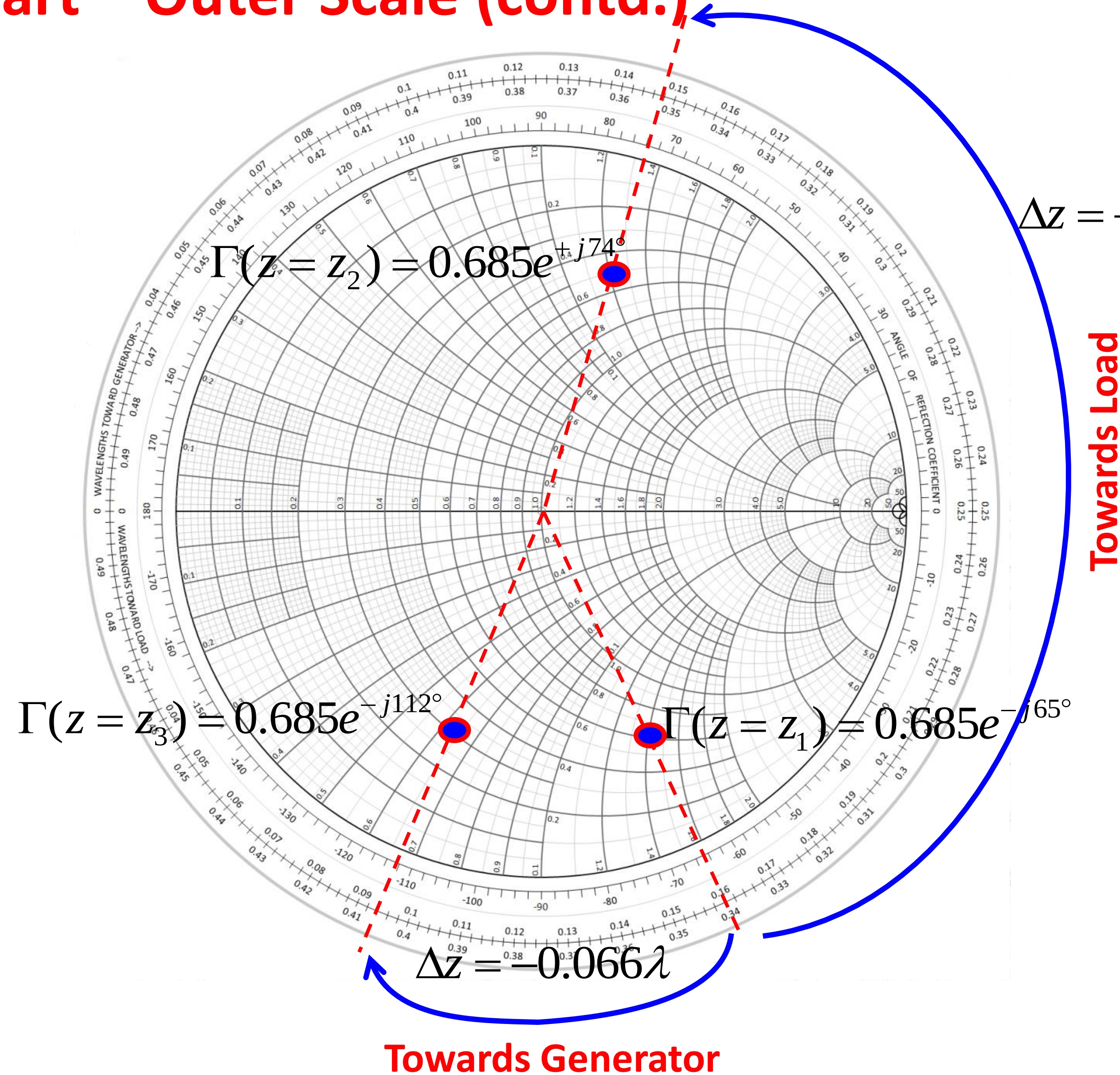
The transmission line location z_2 is a distance of 0.193λ from location z_1!

Q: But, say the reflection coefficient at some point z_3 has a phase value of $\theta_\Gamma = -112^o$, which maps to a value of 0.094λ on the outer scale of Smith chart. It gives $\Delta z = z_3 - z_1 = 0.094\lambda - 0.160\lambda = -0.066\lambda$. What does the **–ve** value mean?

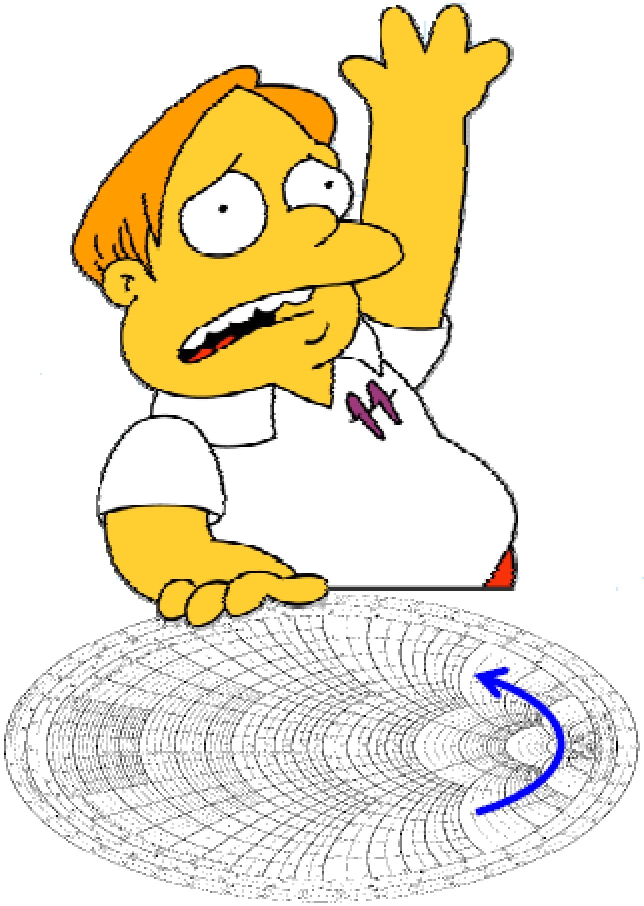
The Smith Chart – Outer Scale (contd.)

- **In the first example,** $\Delta z > 0$, meaning $z_2 > z_1$ → the location z_2 is closer to the load than is location z_1
  - the **positive** value Δz maps to a phase change of $74^o - (-65^o) = 139^o$
  - In other words, as we **move toward the load** from location z_1 to location z_2, we rotate **counter-clockwise** around the Smith chart
- **In the second example,** $\Delta z < 0$, meaning $z_3 < z_1$ → the location z_3 is closer to the beginning of the TL (i.e., farther from the load) than is location z_1
  - the **negative** value Δz maps to a phase change of $-112^o - (-65^o) = -47^o$
  - In other words, as we **move away from the load (i.e, towards the generator)** from location z_1 to location z_3, we rotate **clockwise** around the Smith chart

The Smith Chart – Outer Scale (contd.)



The Smith Chart – Outer Scale (contd.)



Q: Wait! I just used a Smith Chart to analyze a TL problem in the manner you have just explained. At one point on my transmission line the phase of the reflection coefficient is $\theta_\Gamma = +170^o$, which is denoted as 0.486λ on the **"wavelengths toward load"** scale.

I then moved a short distance along the line **toward the load**, and found that the reflection coefficient phase was $\theta_\Gamma = -144^o$, which is denoted as 0.050λ on the **"wavelengths toward load"** scale.

According to **your** "instruction", the distance between these two points is:

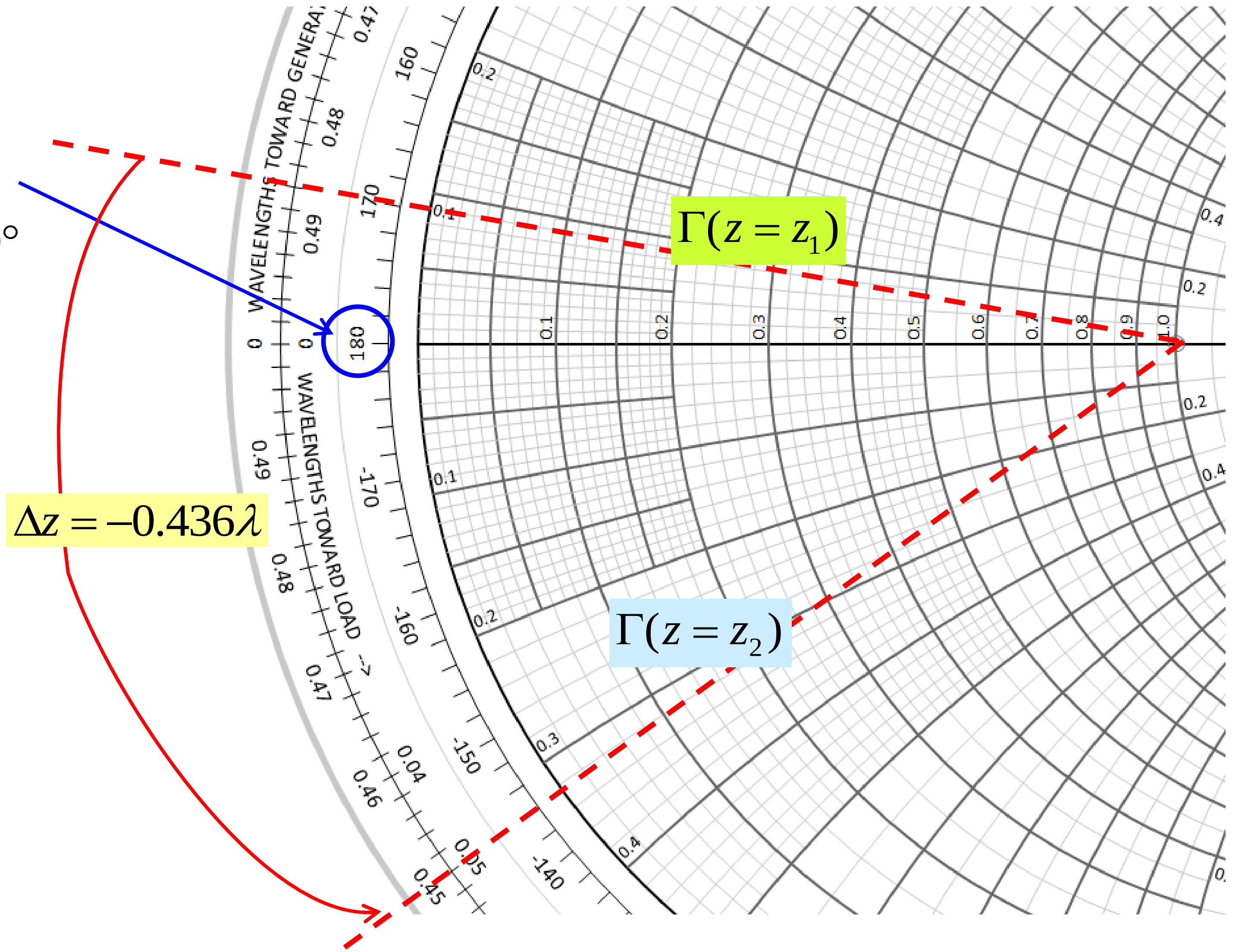
$$\Delta z = 0.050\lambda - 0.486\lambda = -0.436\lambda$$

A large **negative** value! This says that I moved nearly a half wavelength **away** from the load, but I know that I moved just a short distance **toward** the load! **What happened?**

The Smith Chart – Outer Scale (contd.)

The electrical length scales on the Smith chart begin and end where $\theta_\Gamma = \pm 180°$

In your example, when rotating counter-clockwise (i.e., moving toward the load) you **passed by this transition**. This makes the calculation of **Δz** a bit more problematic.



The Smith Chart – Outer Scale (contd.)

- As you rotate counter-clockwise around the Smith Chart, the "wavelengths toward load" scale increases in value, until it reaches a **maximum** value of 0.5λ (at $\theta_{\Gamma} = \pm \pi$)
- At that point, the scale "resets" to its **minimum** value of **zero**
- Thus, in such a situation, we must divide the problem into **two steps**:
- **Step 1:** Determine the electrical length from the **initial** point to the **"end"** of the scale at 0.5λ
- **Step 2:** Determine the electrical distance from the **"beginning"** of the scale (i.e., 0) and the **second location** on the transmission line
- **Add** the results of steps 1 and 2, and you have your answer!

For example, let's look at the case that originally gave us the erroneous result. The distance from the initial location to the **end of the scale** is:

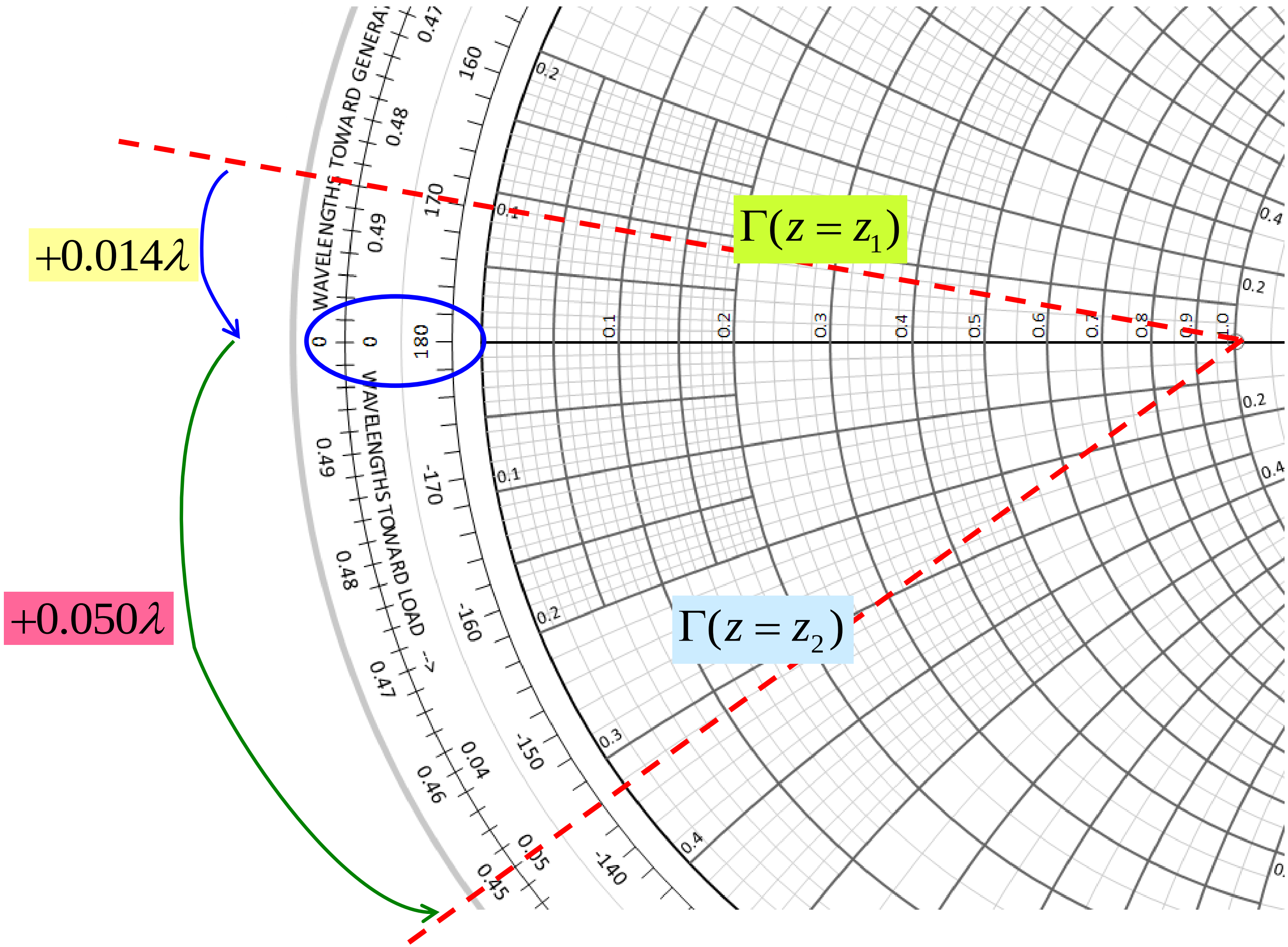
$$0.500\lambda - 0.486\lambda = +0.014\lambda$$

And the distance from the **beginning of the scale** to the second point is:

$$0.050\lambda - 0.000\lambda = +0.050\lambda$$

Thus the distance between the two points is: $+0.014\lambda + 0.050\lambda = +0.064\lambda$

The Smith Chart – Outer Scale (contd.)



The Smith Chart – Outer Scale (contd.)

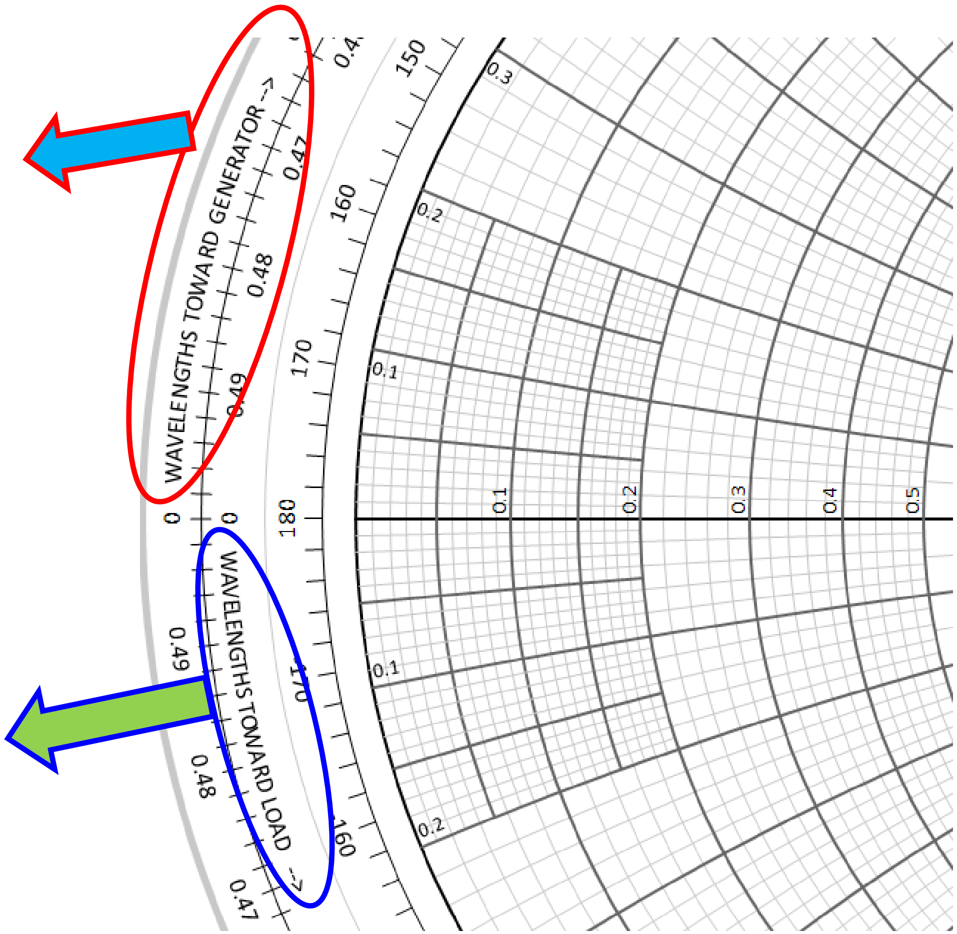
- The Δz towards generator could also be mentioned as a +ve term if we consider the upper metric in the “Outer Scale”

Clockwise Rotation

- **gives +ve distance when moving towards generator**
- **gives –ve distance when moving towards load**

Counter-clockwise Rotation

- **gives -ve distance when moving towards generator**
- **gives +ve distance when moving towards load**



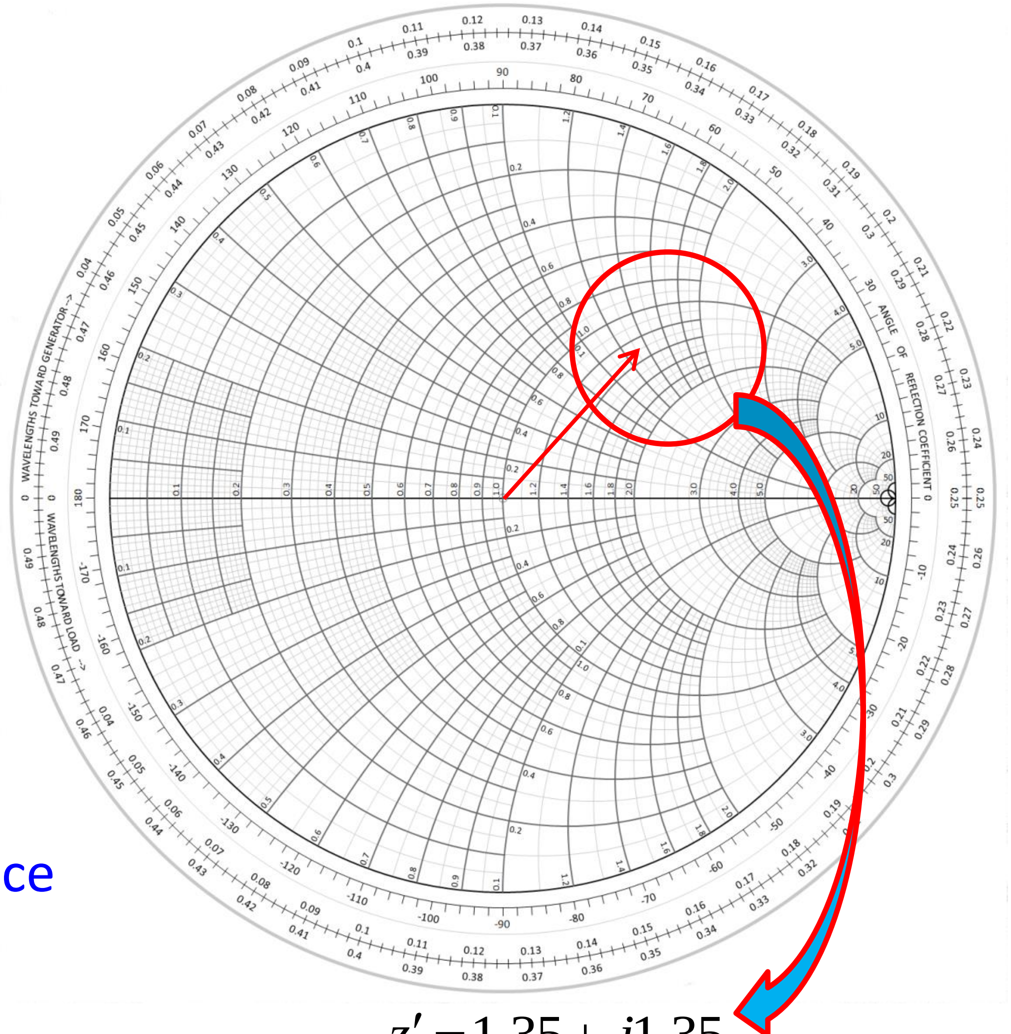
Example-1

Given:

$\Gamma_0 = 0.5\angle 45°$

$Z_0 = 50\Omega$

What is load impedance, Z_L?



$z' = 1.35 + j1.35$

Steps:

- Locate Γ_0 on the smith chart
- Read the normalized impedance
- Then multiply the identified normalized impedance by Z_0

$$\therefore Z_L = 50\Omega * (1.35 + j1.35) = 67.5\Omega + j67.5\Omega$$

Example-2

Given:

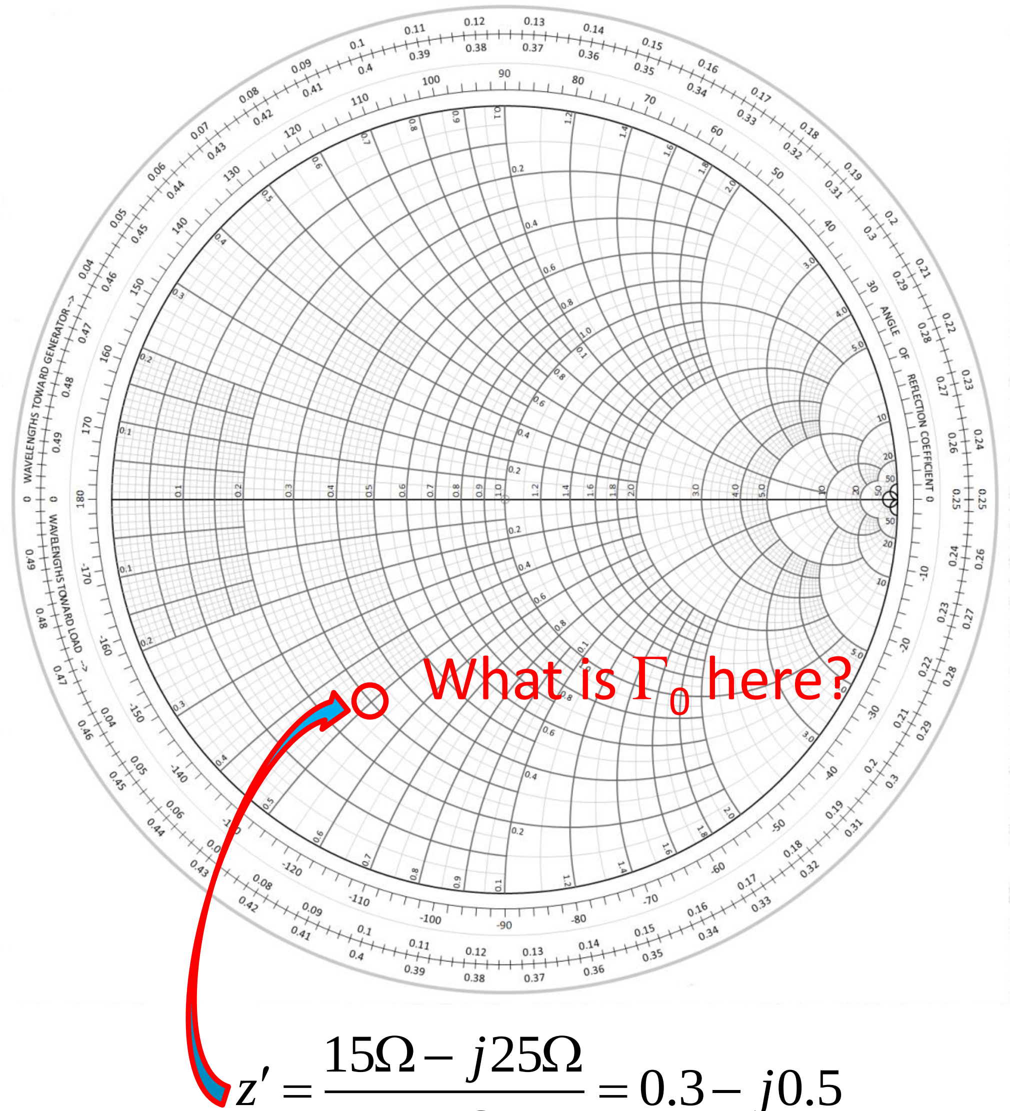
$Z_L = (15 - j25)\Omega$

$Z_0 = 50\Omega$

What is load impedance, Γ_0?

Steps:

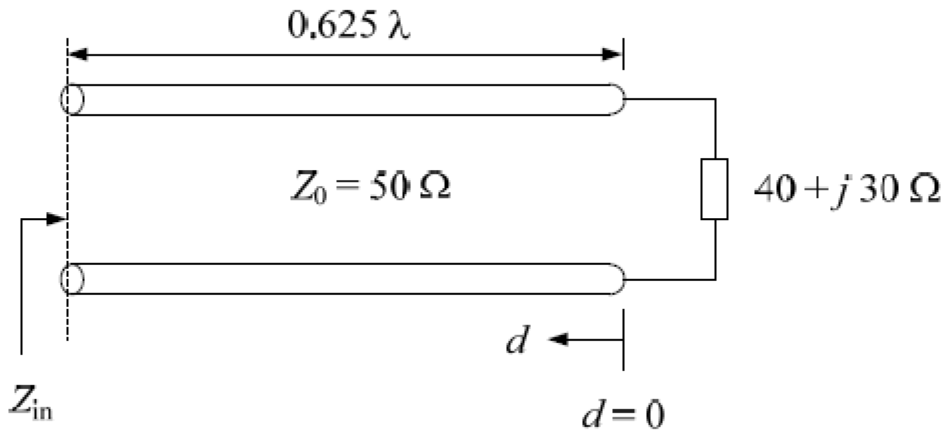
- Normalize the given Z_L
- Mark the normalized impedance Smith chart
- Read the value of Γ_0 from Smith chart



$$z' = \frac{15\Omega - j25\Omega}{50\Omega} = 0.3 - j0.5$$

Example-3

- Using Smith chart, determine the voltage reflection coefficient at the load and the input impedance of the following TL



1. $z_L^{'}\left(d=0\right)=\dfrac{Z(d=0)}{Z_0}=\dfrac{Z_L}{Z_0}=0.8+j0.6$ ⟸ Mark this on Smith chart

2. What is Γ_0? Read this directly from Smith chart.

$$\left|\Gamma_0\right|=0.33 \qquad \angle\Gamma_0=90^o$$

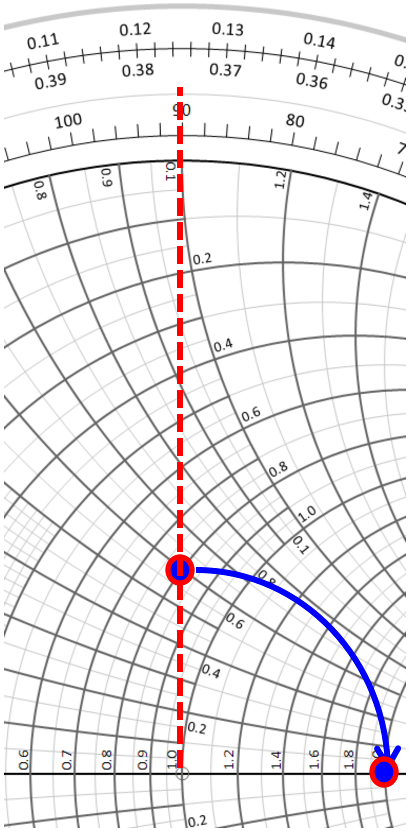
Example-3 (contd.)

3. For Z_{in}, rotate the load reflection coefficient point clockwise (towards generator) by d = 0.625λ (it is full rotation and then additional rotation of 0.125λ) → Then read normalized input impedance from Smith chart

$$z_{in}^{'} = 2 + j0$$

Therefore the input impedance of the TL is:

$$Z_{in} = 50 * z_{in}^{'} = 100\Omega$$



Example – 4

- A load impedance Z_L = (30 + j60)Ω is connected to a 50Ω TL of 2cm length and operated at 2 GHz. Use the reflection coefficient concept and find the input impedance Z_{in} under the assumption that the phase velocity is 50% of the speed of light

First Approach

- We first determine the load reflection coefficient:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{30 + j60 - 50}{30 + j60 + 50} = 0.2 + j0.6 = \sqrt{.40}e^{j71.56^o}$$

- Next we compute Γ (l = 2cm) based on the fact that:

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{v_p} = \frac{2\pi f}{0.5c} = 83.77m^{-1}$$

$$\Rightarrow 2\beta l = 192^o$$

How?

Example – 4 (contd.)

- Therefore, reflection coefficient at the other end of the TL is:

$$\Gamma = \Gamma_0 e^{-j2\beta l} = \sqrt{.40} e^{-120.4^o} = -0.32 - j0.55$$

- The corresponding input impedance is:

$$Z_{in} = Z_0 \frac{1+\Gamma}{1-\Gamma} = R + jX = (14.7 - j26.7)\Omega$$

Second Approach

Using Smith chart

Example – 4 (contd.)

Using Smith Chart

1. The normalized load impedance is:

$$z_L^{'} = (30 + j60)\Omega / 50\Omega = 0.6 + j1.2$$

2. This point on the Smith chart can be identified as the intersection of the circle of constant resistance *r = 0.6* with the circle of constant reactance $x = 1.2$

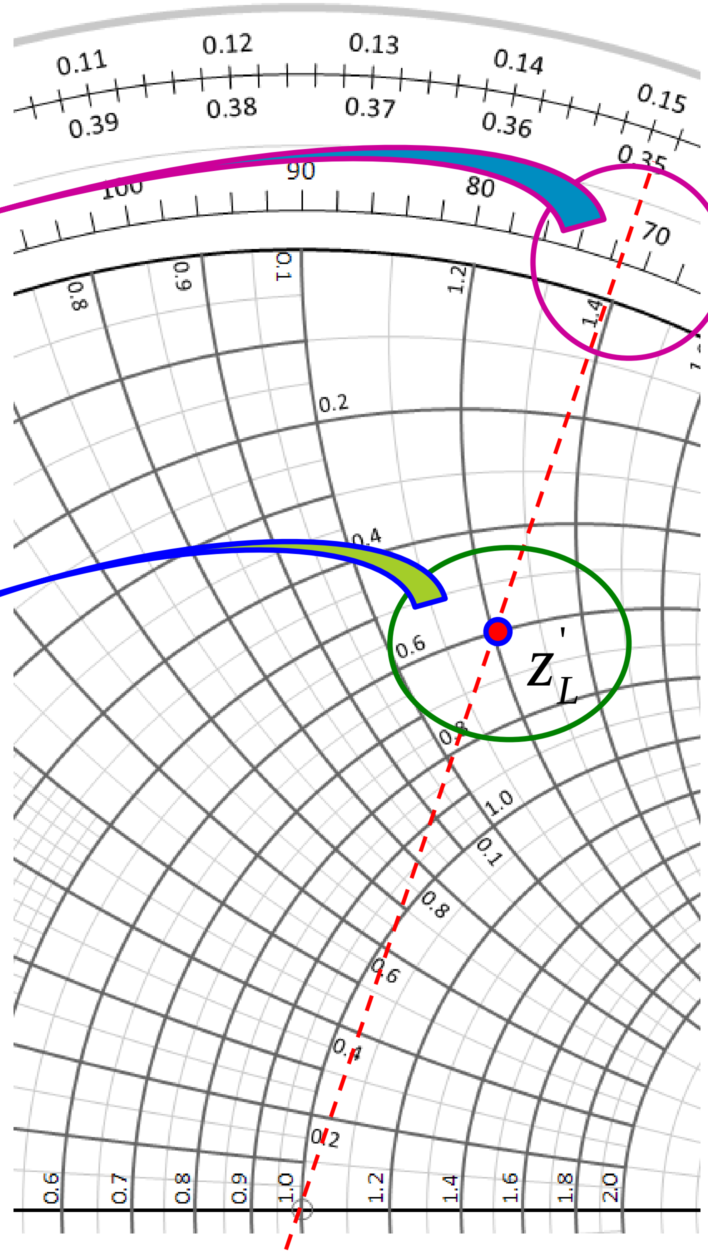
3. The straight line connecting the origin to *normalized load impedance* determines the load reflection coefficient Γ_0. The associated angle is recorded with respect to the positive real axis. From Smith chart we can find that $|\Gamma_0| = 0.6325$ and phase of $\Gamma_0 = 71.56^o$.

4. Rotate this by $2\beta l = 192^o$ to obtain Γ_{in}

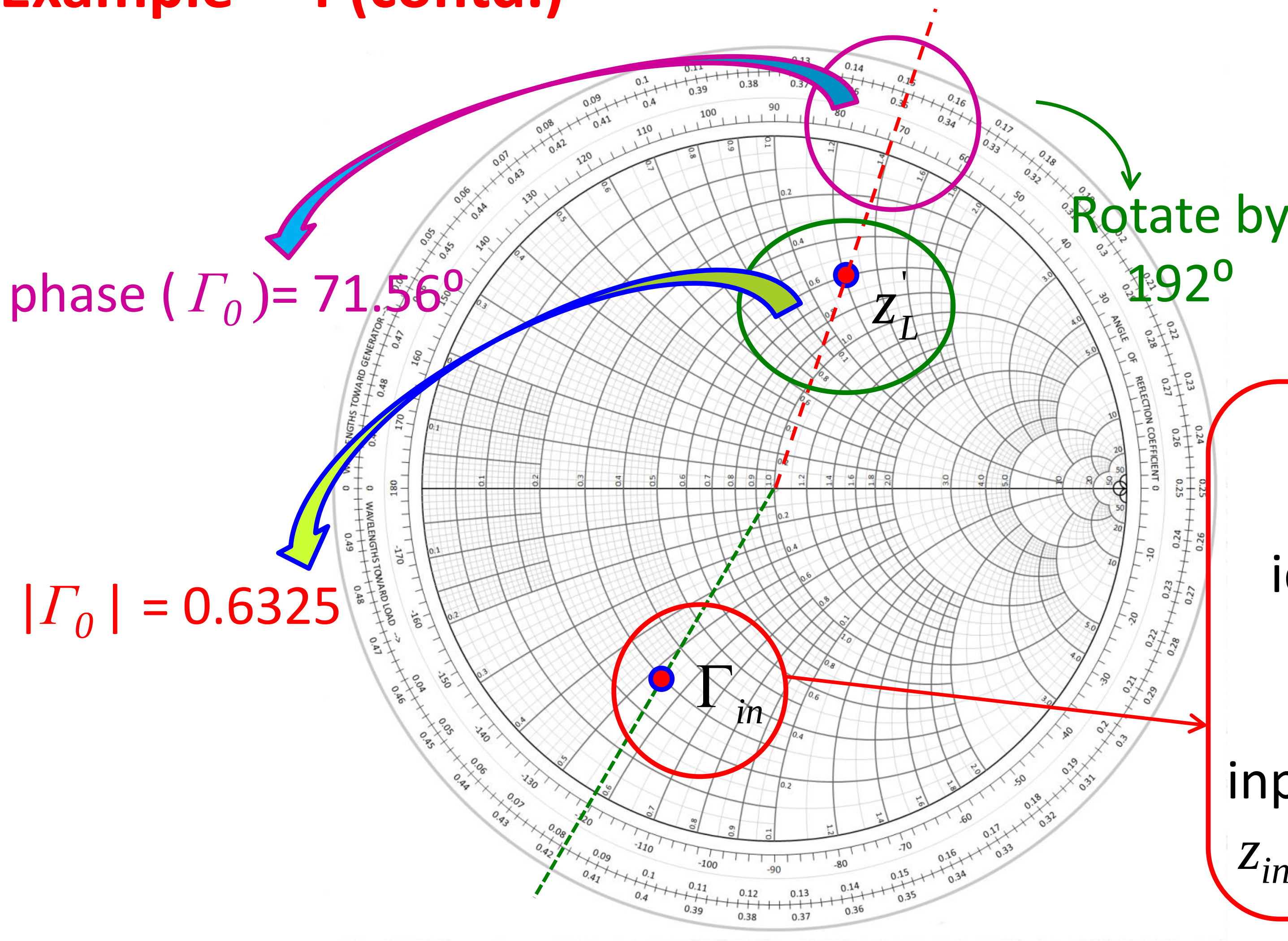
Example – 4 (contd.)

phase (Γ_0)= 71.56°

$|\Gamma_0|$ = 0.6325



Example – 4 (contd.)



This point uniquely identifies the associated normalized input impedance $z_{in}' = 0.3 - j0.53$

Example – 4 (contd.)

5. The Γ_{in} uniquely identifies the associated normalized input impedance $z_{in}' = 0.3 - j0.53$
6. The preceding normalized impedance can be converted back to actual input impedance values by multiplying it by Z_0 = 50Ω, resulting in the final solution $Z_{in} = (15 - j26.5)\Omega$

The exact value of Z_{in} computed earlier was (14.7 – j26.7)Ω. The small anomaly is expected considering the approximate processing of graphical data in Smith chart

Special Transformation Conditions in Smith Chart

- The rotation angle of the normalized TL impedance around the Smith chart is regulated by the length of TL or operating frequency
- Thus, both capacitive and inductive impedances can be generated based on the length of TL and the termination conditions at a given frequency
- The open- and short-circuit terminations are very popular in generating inductive and capacitive elements

Open Circuit Transformations

- **For an arbitrary terminated line the input impedance is:**

$$Z_{in}(z) = Z_0 \frac{Z_L + jZ_0 \tan(\beta z)}{Z_0 + jZ_L \tan(\beta z)} \xrightarrow{\text{For an open circuit}} Z_{in}(z) = -jZ_0 \cot(\beta z)$$

- **For a capacitive impedance of $X_C = 1/j\omega C$ we get:**

$$\frac{1}{j\omega C} \cdot \frac{1}{Z_0} \equiv z_{in}^{'} = -j\cot(\beta z_1) \Rightarrow z_1 = \frac{1}{\beta}\left[\cot^{-1}\left(\frac{1}{\omega C Z_0}\right) + n\pi\right]$$

Special Transformation Conditions in Smith Chart (contd.)

Open Circuit Transformations

- **For an inductive impedance of X_L = jωL we get:**

$$j\omega L\frac{1}{Z_0}\equiv z_{in}^{'}=-j\cot(\beta z_2) \Rightarrow z_2=\frac{1}{\beta}\left[\pi-\cot^{-1}\left(\frac{\omega L}{Z_0}\right)+n\pi\right]$$

Short Circuit Transformations

- **For an arbitrary terminated line the input impedance is:**

$$Z_{in}(z)=Z_0\frac{Z_L+jZ_0\tan(\beta z)}{Z_0+jZ_L\tan(\beta z)}$$

For a short circuit

$$Z_{in}(z)=jZ_0\tan(\beta z)$$

- **For a capacitive impedance of X_C = 1/jωC we get:**

$$\frac{1}{j\omega C}.\frac{1}{Z_0}\equiv z_{in}^{'}=j\tan(\beta z_1) \Rightarrow z_1=\frac{1}{\beta}\left[\pi-\tan^{-1}\left(\frac{1}{\omega C Z_0}\right)+n\pi\right]$$

- **For an inductive impedance of X_L = jωL we get:**

$$j\omega L\frac{1}{Z_0}\equiv z_{in}^{'}=j\tan(\beta z_2) \Rightarrow z_2=\frac{1}{\beta}\left[\tan^{-1}\left(\frac{\omega L}{Z_0}\right)+n\pi\right]$$

Example – 5

- For an open-circuited 50Ω TL operated at 3GHz and with a phase velocity of 77% of speed of light, find the line lengths to create a 2pF capacitor and 5.3nH inductor. Use Smith Chart for solving this problem.

➢ For the given phase velocity, the propagation constant is:

$$\beta = \frac{2\pi f}{v_p} = \frac{2\pi f}{0.77c} = 81.6m^{-1}$$

➢ We know that an open-circuit can create a capacitor as per following equation:

$$z_1 = \frac{1}{\beta}\left[\cot^{-1}\left(\frac{1}{\omega C Z_0}\right) + n\pi\right]$$

$\beta = 81.6m^{-1}$, $C = 2pF$, $f = 3GHz$ ⇒ $z_1 = 13.27 + n38.5$

➢ We know that an open-circuit can create an inductor as per following equation:

$$z_2 = \frac{1}{\beta}\left[\pi - \cot^{-1}\left(\frac{\omega L}{Z_0}\right) + n\pi\right]$$

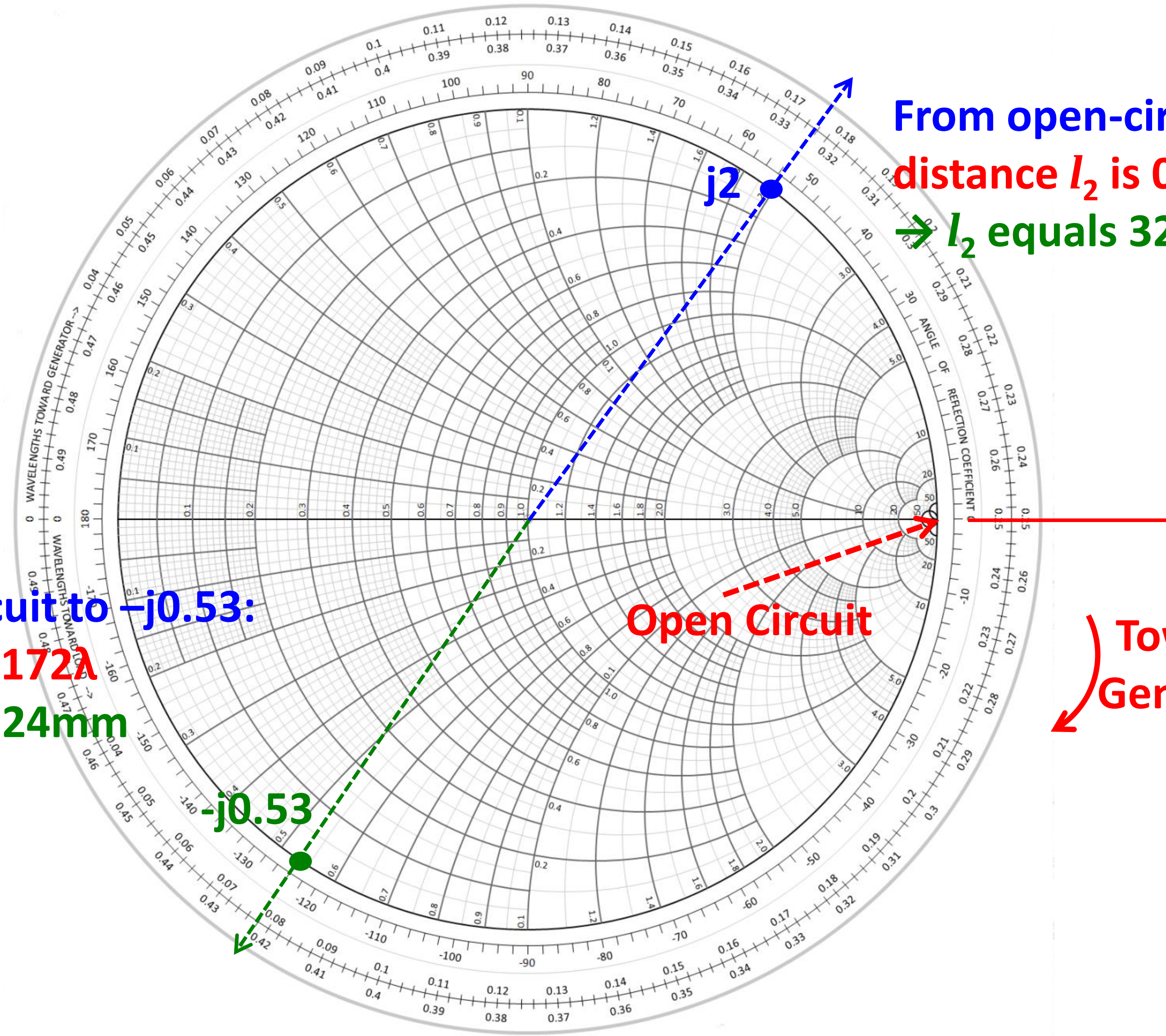
$\beta = 81.6m^{-1}$, $L = 5.3nH$, $f = 3GHz$ ⇒ $z_2 = 32.81 + n38.5$

Example – 5 (contd.)

Using Smith Chart

- At 3GHz, the reactance of a 2pF capacitor is: $X_C = \frac{1}{j\omega C} = -j26.5\Omega$
- Therefore, the normalized capacitive reactance is: $z_c^{'} = \frac{X_C}{Z_0} = -j0.53$
- At 3GHz, the reactance of a 5.3nH inductor is: $X_L = j\omega L = j100\Omega$
- Therefore, the normalized inductive reactance is: $z_L^{'} = \frac{X_L}{Z_0} = j2$
- The wavelength is: $\lambda = \frac{v_p}{f} = 77mm$

Example – 5 (contd.)



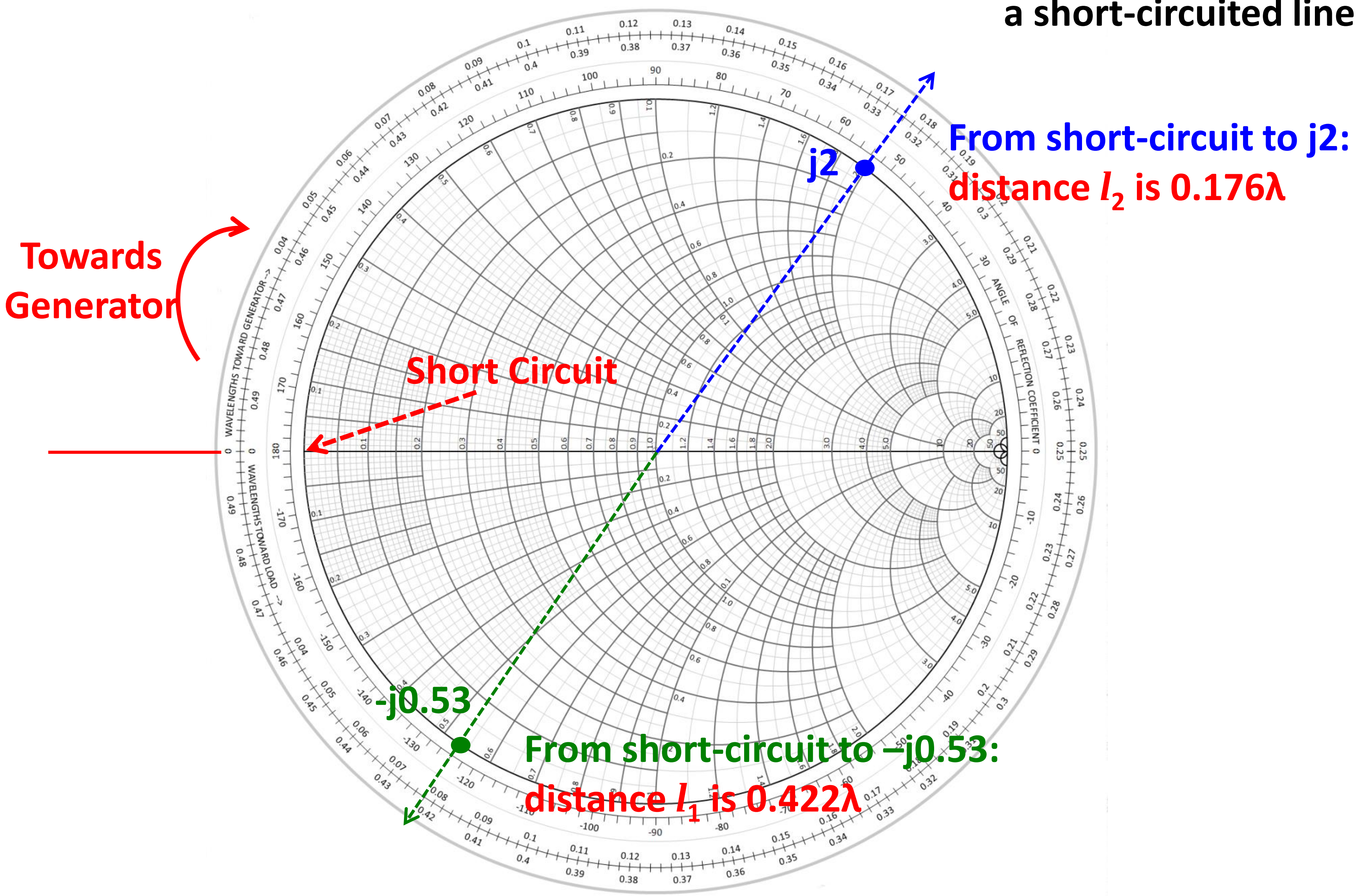
From open-circuit to j2:
distance l_2 is 0.426λ
→ l_2 equals 32.8mm

From open-circuit to –j0.53:
distance l_1 is 0.172λ
→ l_1 equals 13.24mm

Towards Generator

Example – 6

- **Same problem but for a short-circuited line**



Special Transformation Conditions in Smith Chart (contd.)

Summary

- It is apparent that both open-circuit and short-circuit TLs can achieve desired capacitance or inductance. Which configuration is more useful?
- At high frequencies, its difficult to maintain perfect open-circuit conditions → due to changing temperatures, humidity, and other parameters of the medium surrounding the open TL → short-circuit TLs are, therefore, more popular
- However, short-circuit TL is problematic at higher frequencies → through-hole short connections create parasitic inductances (why? → HW # 0)
- Sometimes board size regulates the choice of open or short TL → for example, an open-circuit TL will always require smaller TL segment for realizing any specified capacitance as compared to a short-circuit TL segment

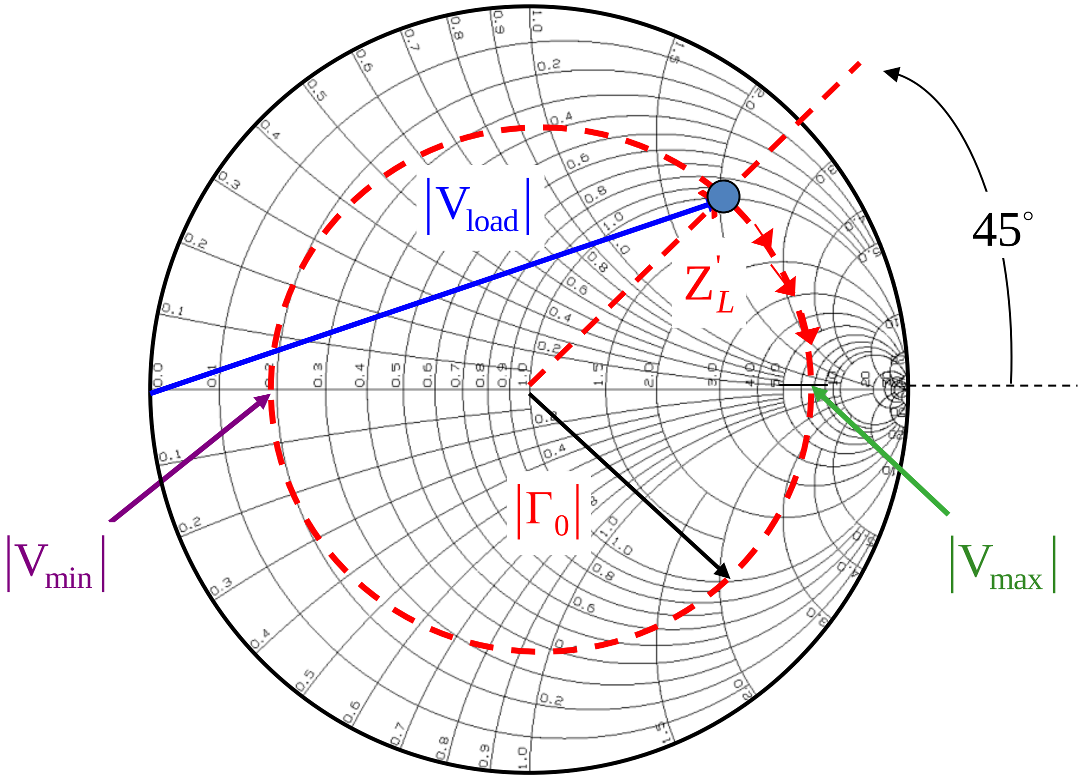
Example – 7

Given: $\Gamma_0 = 0.707\angle 45^\circ$

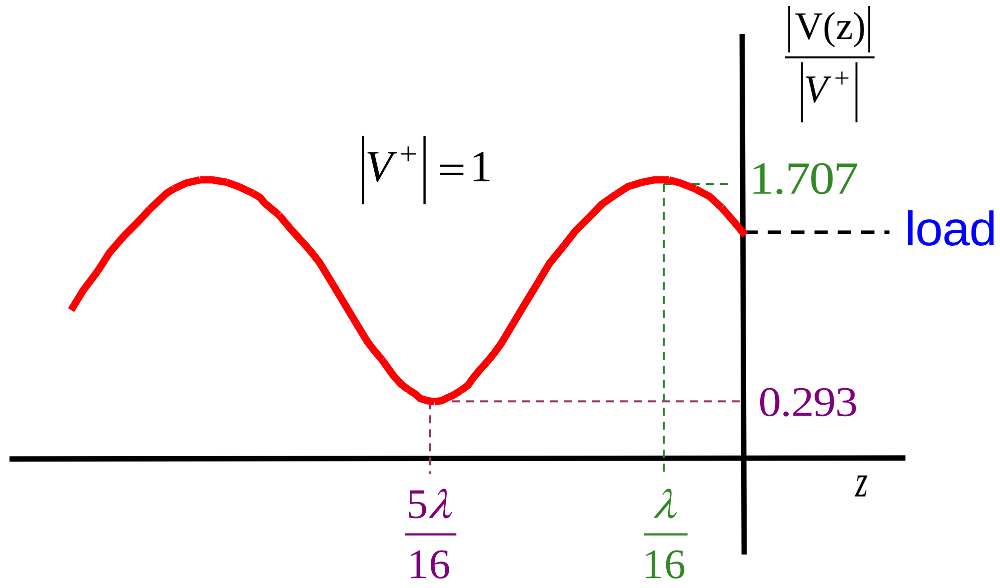
Use the Smith chart to plot the voltage magnitude, find the SWR, and the normalized load admittance

$$Z_L' = 1 + j2$$

$$45^\circ \Leftrightarrow \frac{\lambda}{16}$$

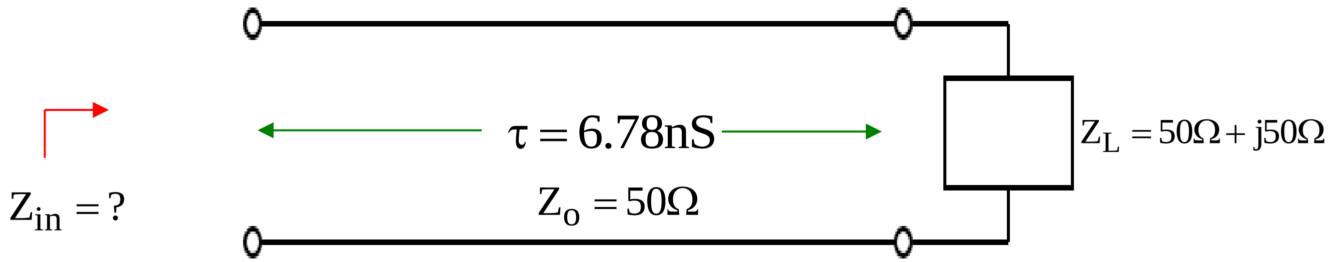


Example – 7 (contd.)



Example – 8

What is Z_{in} at 50 MHZ for the following circuit?



1. Normalized Impedance: $z_L^{'} = \dfrac{50\Omega + j50\Omega}{50\Omega} = 1.0 + j1.0$
2. Mark the normalized impedance on the Smith chart
3. Read reflection coefficient from Smith Chart: $\Rightarrow \Gamma_0 = 0.445\angle 64°$
4. Transform the load reflection coefficient to the input:

$$\Gamma_{in} = \Gamma_0 e^{-j2\beta l} = \Gamma_0 e^{-j2\omega\tau}$$

$$2\omega\tau = 244°$$

Rotate clockwise (towards generator)

$$\Rightarrow \Gamma_{in} = 0.445\angle 180°$$

Read the normalized input impedance in the Smith chart

$$z_{in}^{'} = 0.38 + j0.0$$

Example – 8 (contd.)