

Lecture – 6

Date: 21.08.2014

- Lossy Transmission Line
- Introduction to Smith Chart: The complex Γ – plane
- Transformations on the complex Γ – plane
- Mapping Z to Γ
- Smith Chart – Construction
- Smith Chart – Geography

Lossy Transmission Lines

- Recall that we have been **approximating** low-loss transmission lines as lossless ($R = G = 0$):

$$\alpha = 0$$

$$\beta = \omega\sqrt{LC}$$

- But, **long** low-loss lines require a **better** approximation:

$$\alpha = \frac{1}{2} \left(\frac{R}{Z_0} + GZ_0 \right)$$

$$\beta = \omega\sqrt{LC}$$

- Now, if we have **really long** transmission lines (e.g., long distance communications), we can apply **no** approximations at all:

$$\alpha = \text{Re}\{\gamma\}$$

$$\beta = \text{Im}\{\gamma\}$$

For these **very** long transmission lines, we find that $\beta = \text{Im}\{\gamma\}$ is a **function** of signal **frequency** ω . This results in an extremely serious problem—signal **dispersion**.

Lossy Transmission Lines (contd.)

- Recall that the **phase velocity** v_p (i.e., propagation velocity) of a wave in a transmission line is:

$$v_p = \frac{\omega}{\beta}$$

$$\beta = \text{Im}\{\gamma\} = \text{Im}\left\{\sqrt{(R + j\omega L)(G + j\omega C)}\right\}$$

Thus, for a lossy line, the phase velocity v_p is a function of frequency ω (i.e., $v_p(\omega)$)—this is **bad**!

- Any signal that carries significant **information** must have some non-zero **bandwidth**. In other words, the signal energy (as well as the information it carries) is **spread** across many frequencies.
- If the different frequencies that comprise a signal travel at different velocities, that signal will arrive at the end of a transmission line **distorted**. We call this phenomenon signal **dispersion**.
- Recall for **lossless** lines, however, the phase velocity is **independent** of frequency—**no** dispersion will occur!

Lossy Transmission Lines (contd.)

- For lossless line:

$$v_p = \frac{1}{\sqrt{LC}}$$

however, a perfectly lossless line is impossible, but we find phase velocity is **approximately** constant if the line is low-loss.

Therefore, dispersion distortion on low-loss lines is **most often** not a problem.

Q: You say “**most** often” not a problem—that phrase seems to imply that dispersion sometimes is a problem!



Lossy Transmission Lines (contd.)

A: Even for low-loss transmission lines, dispersion can be a problem **if** the lines are **very** long—just a small difference in phase velocity can result in significant differences in propagation delay **if** the line is very long!

- Modern examples of long transmission lines include phone lines and cable TV. However, the **original** long transmission line problem occurred with the **telegraph**, a device invented and implemented in the 19th century.
- Early telegraph “engineers” discovered that if they made their telegraph lines **too long**, the dots and dashes characterizing Morse code turned into a muddled, indecipherable **mess**. Although they did not realize it, they had fallen victim to the heinous effects of **dispersion**!
- Thus, to send messages over long distances, they were forced to implement a series of intermediate “**repeater**” stations, wherein a human operator received and then **retransmitted** a message on to the next station. This **really** slowed things down!

Lossy Transmission Lines (contd.)



Q: Is there any way to **prevent** dispersion from occurring?

A: You bet! **Oliver Heaviside** figured out how in the **19th** Century!

- Heaviside found that a transmission line would be distortionless (i.e., no dispersion) **if** the line parameters exhibited the following **ratio**:

$$\frac{R}{L} = \frac{G}{C}$$

- Let's see **why** this works. Note the complex propagation constant γ can be expressed as:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{LC(R/L + j\omega)(G/C + j\omega)}$$

Lossy Transmission Lines (contd.)

- Then IF:

$$\frac{R}{L} = \frac{G}{C}$$

- we find:

$$\gamma = \sqrt{LC(R/L + j\omega)(R/L + j\omega)} = (R/L + j\omega)\sqrt{LC} = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$$

- Thus:

$$\alpha = \operatorname{Re}\{\gamma\} = R\sqrt{\frac{C}{L}}$$

$$\beta = \operatorname{Im}\{\gamma\} = \omega\sqrt{LC}$$

- The propagation **velocity** of the wave is thus:

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

The propagation velocity is **independent** of frequency! This lossy transmission line is **not** dispersive!

Lossy Transmission Lines (contd.)



Q: Right. All the transmission lines I use have the property that $R/L > G/C$. I've **never** found a transmission line with this **ideal** property $R/L = G/C$!

A: It is true that typically $R/L > G/C$. But, we can reduce the ratio R/L (until it is equal to G/C) by adding series **inductors** periodically along the transmission line.

This was **Heaviside's** solution—and it worked! **Long** distance transmission lines were made possible.

Q: Why don't we increase G instead?

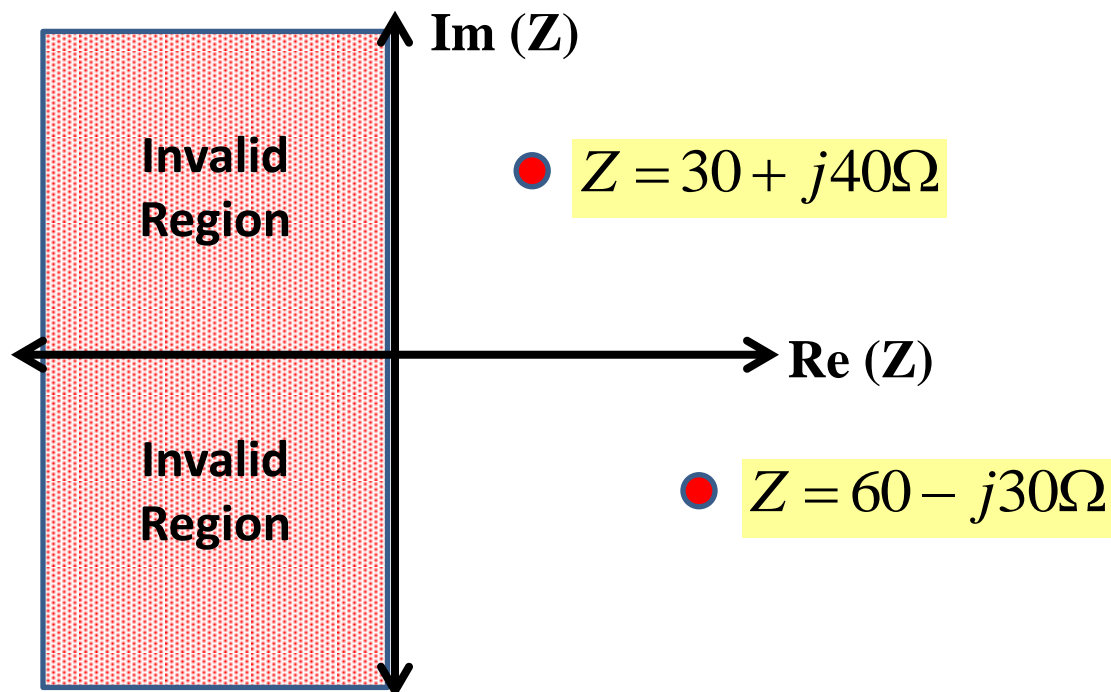
A:

Smith Chart

- Smith chart – what?
- The Smith chart is a very convenient graphical tool for analyzing TLs studying their behavior.
- It is mapping of impedance in standard complex plane into a suitable complex reflection coefficient plane.
- It provides graphical display of reflection coefficients.
- The impedances can be directly determined from the graphical display (ie, from Smith chart)
- Furthermore, Smith charts facilitate the analysis and design of complicated circuit configurations.

The Complex Γ - Plane

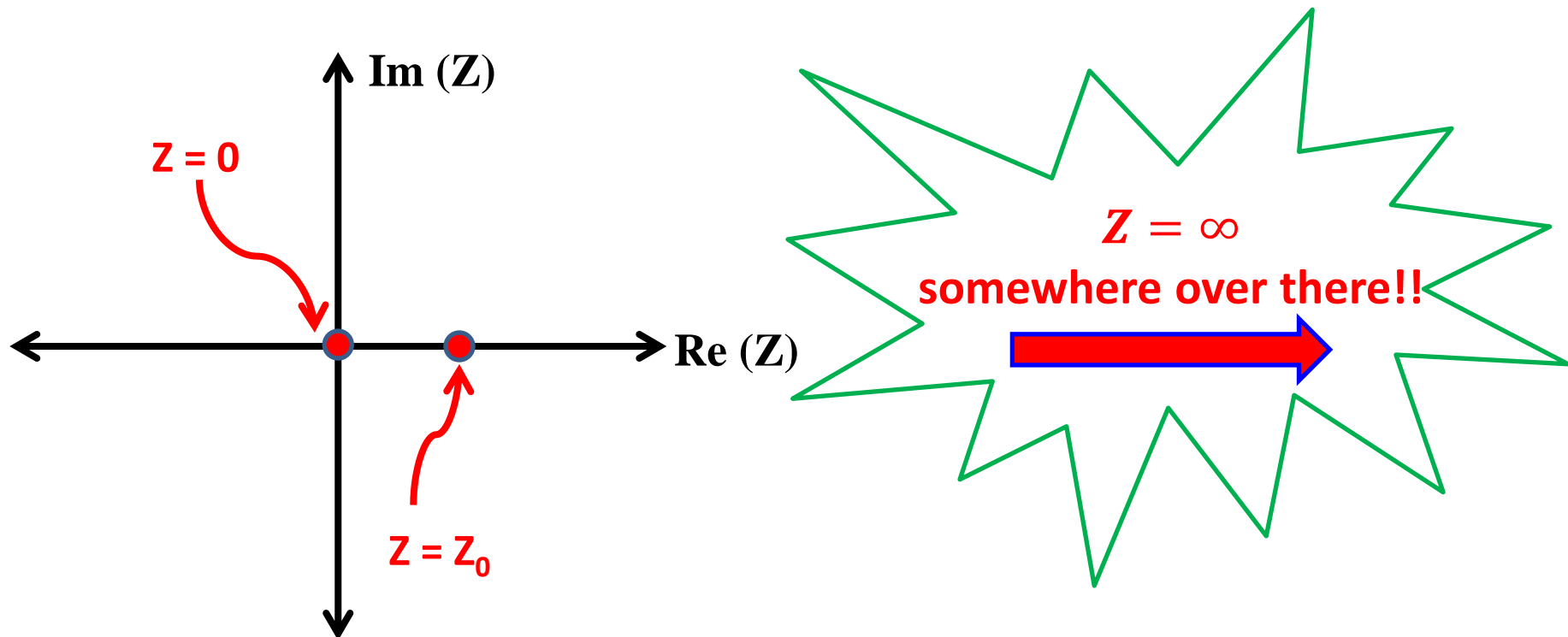
- Let us first display the impedance Z on complex Z -plane



- Note that each dimension is defined by a single real line: the **horizontal line (axis)** indicates the **real component of Z** , and the **vertical line (axis)** indicates the **imaginary component of Z** \rightarrow **Intersection** of these lines indicate the complex impedance

The Complex Γ - Plane (contd.)

- How do we plot an **open circuit** (i.e, $Z = \infty$), **short circuit** (i.e, $Z = 0$), and **matching condition** (i.e, $Z = Z_0 = 50\Omega$) on the complex Z -plane



It is apparent that complex Z - plane is not very useful

The Complex Γ -Plane (contd.)

- The **limitations** of **complex Z-plane** can be **overcome** by **complex Γ -plane**
- We know $\mathbf{Z} \leftrightarrow \mathbf{\Gamma}$ (i.e, if you know **one**, you know the **other**).
- We can therefore define a **complex Γ -plane** in the same manner that we defined a complex Z-plane.
- Let us revisit the reflection coefficient in complex form:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

Where,

$$\theta_0 = \tan^{-1} \left(\frac{\Gamma_{0i}}{\Gamma_{0r}} \right)$$

Real part of Γ_0

Imaginary part of Γ_0

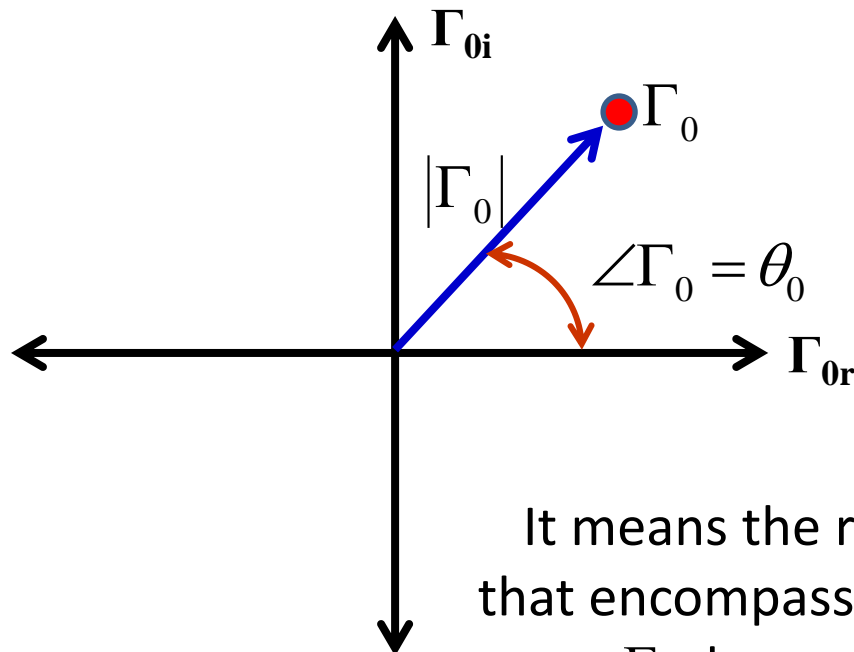
- In the special terminated conditions of **pure short-circuit and pure open-circuit conditions** the corresponding Γ_0 are **-1 and +1** located on the real axis in the complex Γ -plane.

The Complex Γ -Plane (contd.)

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$



Representation of reflection
coefficient in polar form



Observations:

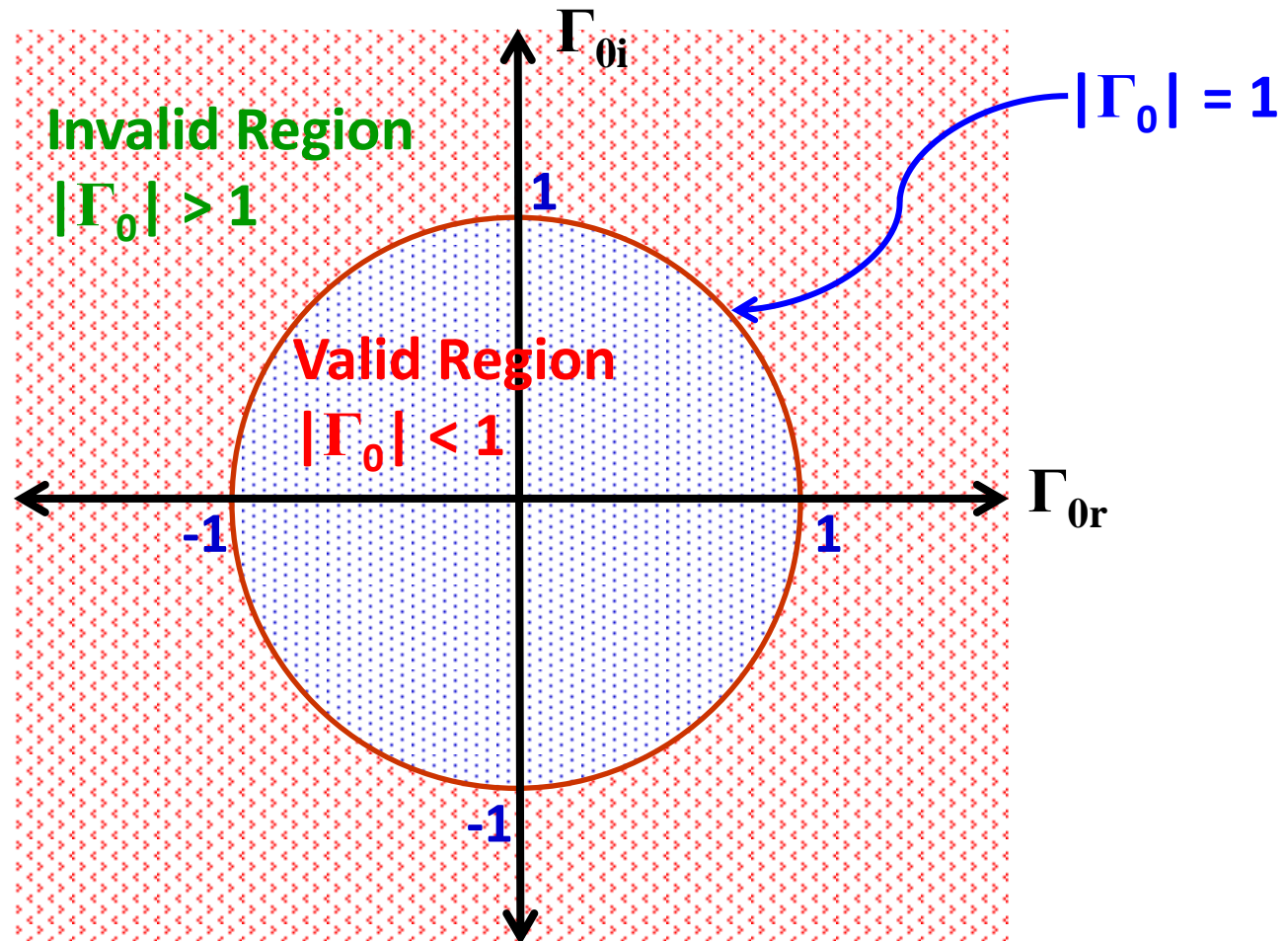
- A radial line is formed by the locus of all points whose phase is θ_0
- A circle is formed by the locus of all points whose magnitude is $|\Gamma_0|$

It means the reflection coefficient has a valid region that encompasses all the four quadrants in the complex Γ -plane within the -1 to $+1$ bounded region

In complex Z -plane the valid region was unbounded on the right half of the plane \rightarrow as a result many important impedances could **not** be plotted

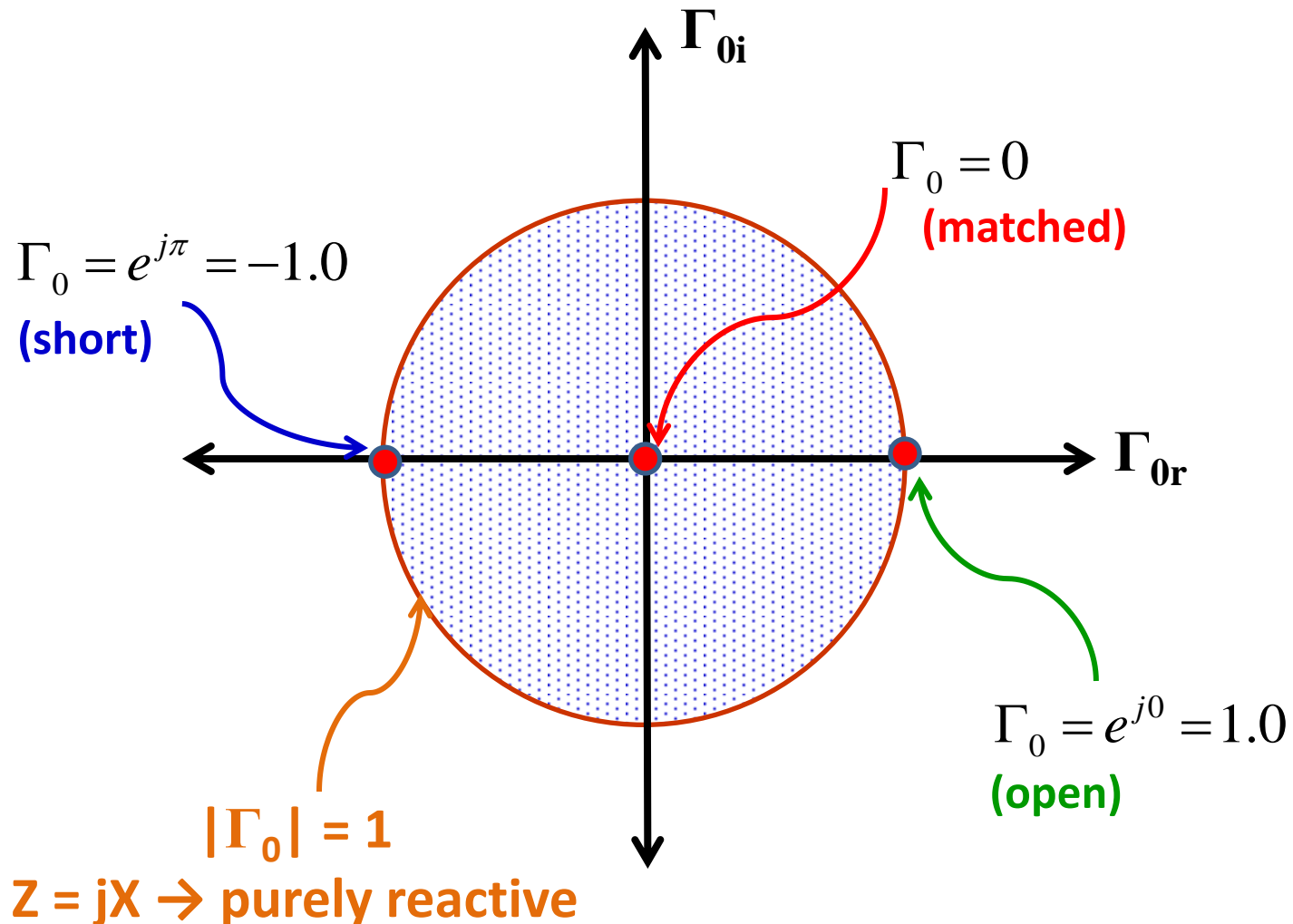
The Complex Γ -Plane (contd.)

- Validity Region



The Complex Γ -Plane (contd.)

- We can plot all the valid impedances (i.e $R > 0$) within this bounded region.



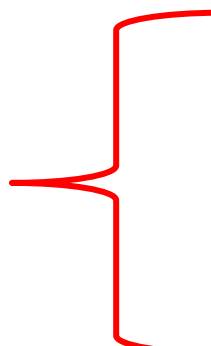
Example – 1

- A TL with a characteristic impedance of $Z_0 = 50\Omega$ is terminated into following load impedances:
 - (a) $Z_L = 0$ (Short Circuit)
 - (b) $Z_L \rightarrow \infty$ (Open Circuit)
 - (c) $Z_L = 50\Omega$
 - (d) $Z_L = (16.67 - j16.67)\Omega$
 - (e) $Z_L = (50 + j50)\Omega$

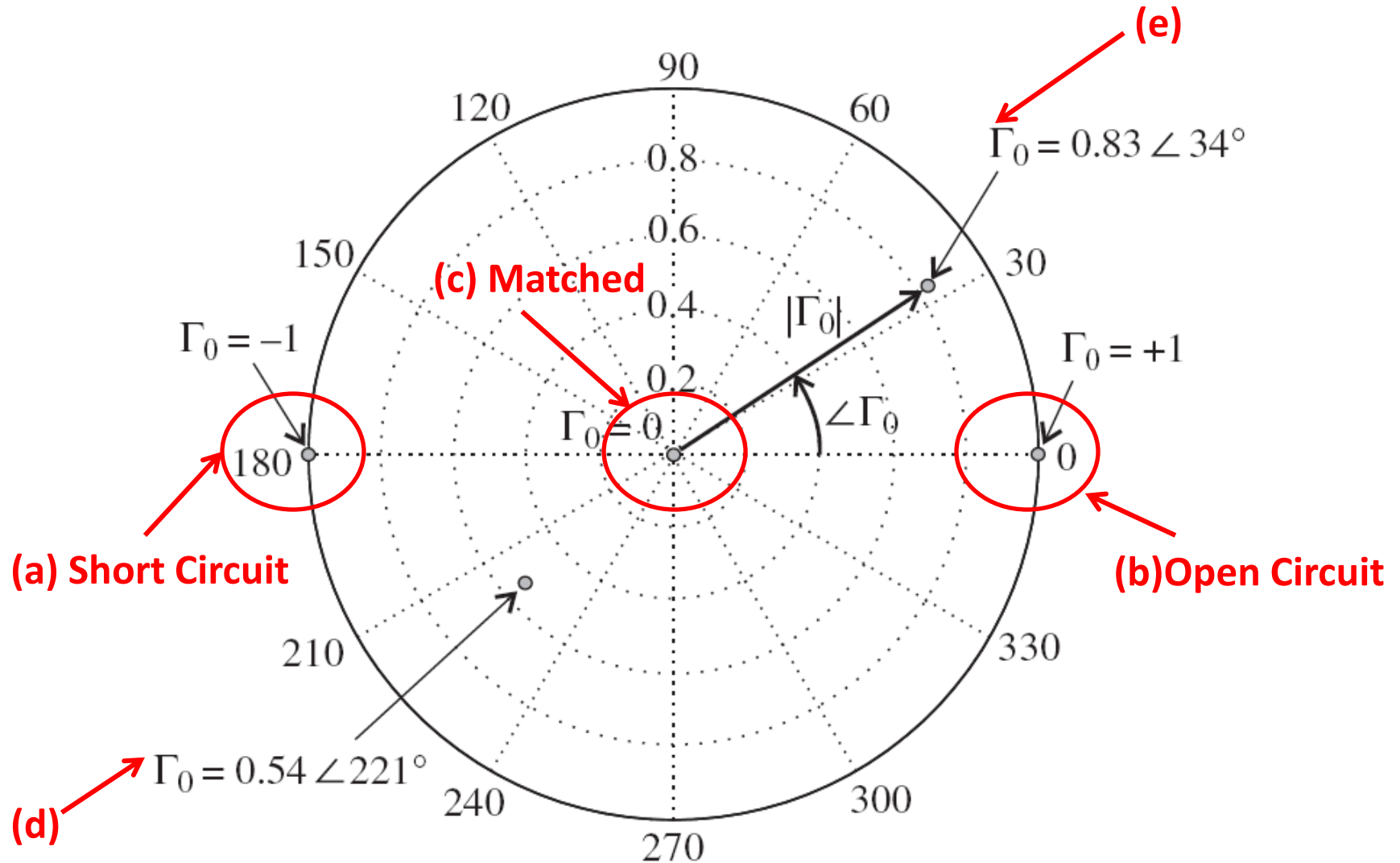
Display the respective reflection coefficients in complex Γ -plane

- Solution:** We know the relationship between Z and Γ :

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

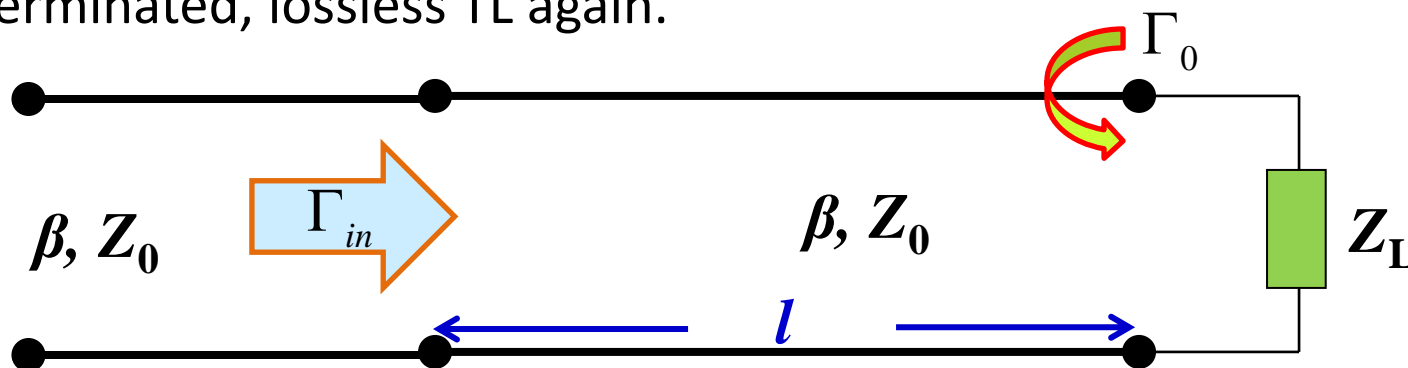
- 
- (a) $\Gamma_0 = -1$ (Short Circuit)
 - (b) $\Gamma_0 = 1$ (Open Circuit)
 - (c) $\Gamma_0 = 0$ (Matched)
 - (d) $\Gamma_0 = 0.54 \angle 221^\circ$
 - (e) $\Gamma_0 = 0.83 \angle 34^\circ$

Example – 1 (contd.)



Transformations on the Complex Γ -Plane

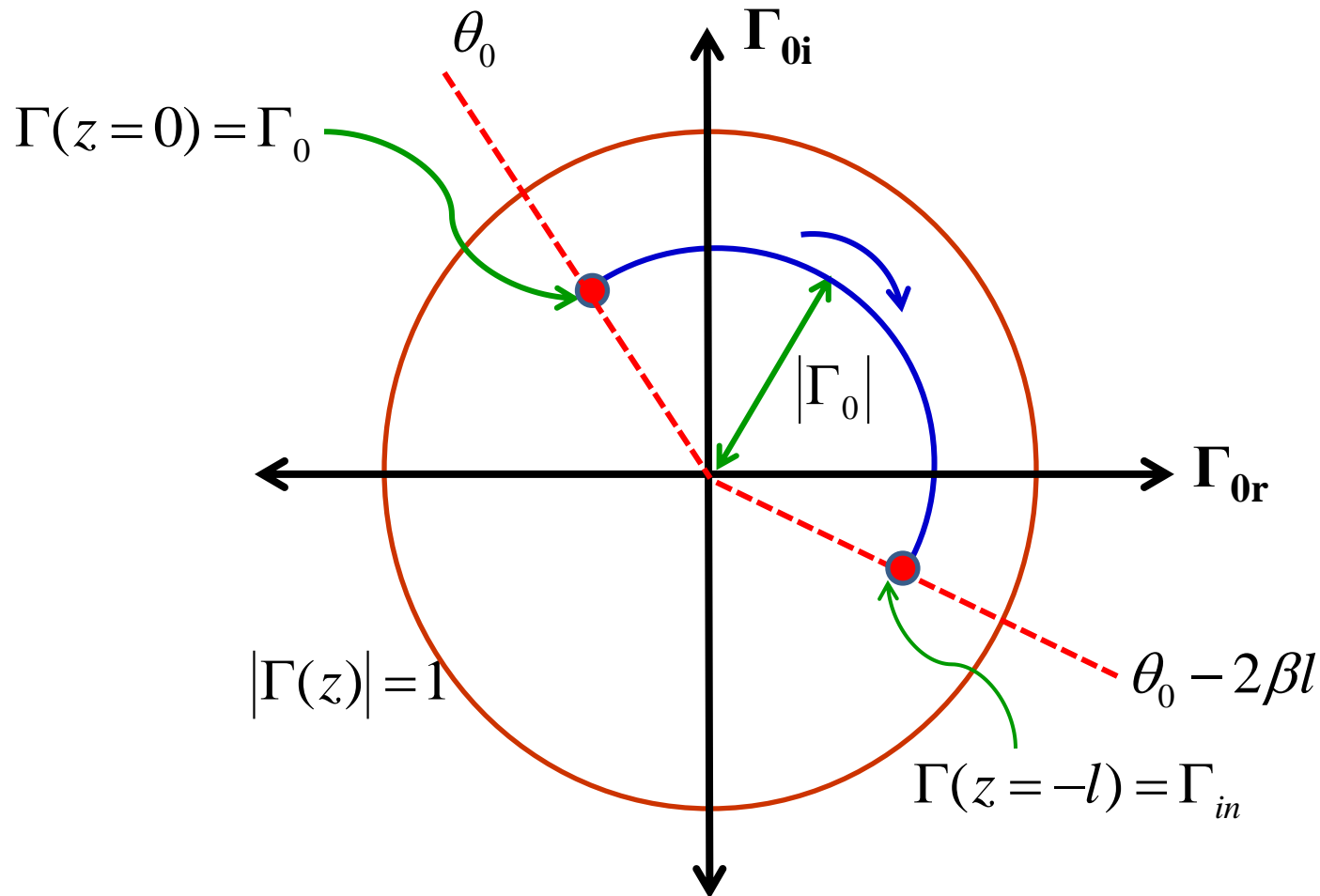
- The usefulness of the complex Γ -plane will be evident when we consider the terminated, lossless TL again.



- At $z=0$, the reflection coefficient is called load reflection coefficient (Γ_0) \rightarrow this actually describes the **mismatch** between the load impedance (Z_L) and the characteristic impedance (Z_0) of the TL.
- The **move away from the load** (or towards the input/source) in the negative z -direction (clockwise rotation) **requires multiplication** of Γ_0 by a factor **$\exp(+j2\beta z)$** in order to explicitly define the mismatch at location ' z ' known as $\Gamma(z)$.
- This **transformation** of Γ_0 to $\Gamma(z)$ is the key ingredient in **Smith chart** as a graphical design/display tool.

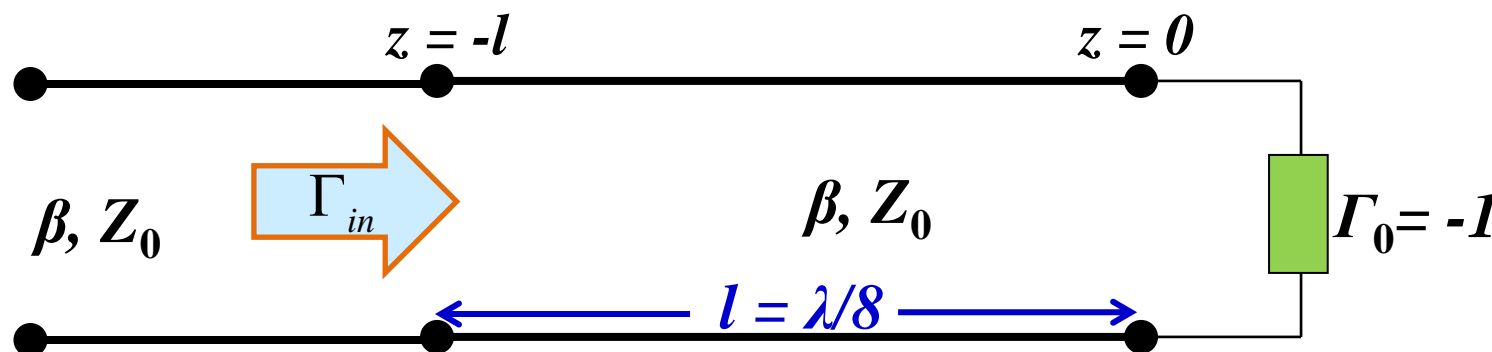
Transformations on the Complex Γ -Plane (contd.)

- Graphical interpretation of $\Gamma(z) = \Gamma_0 e^{+2j\beta z}$



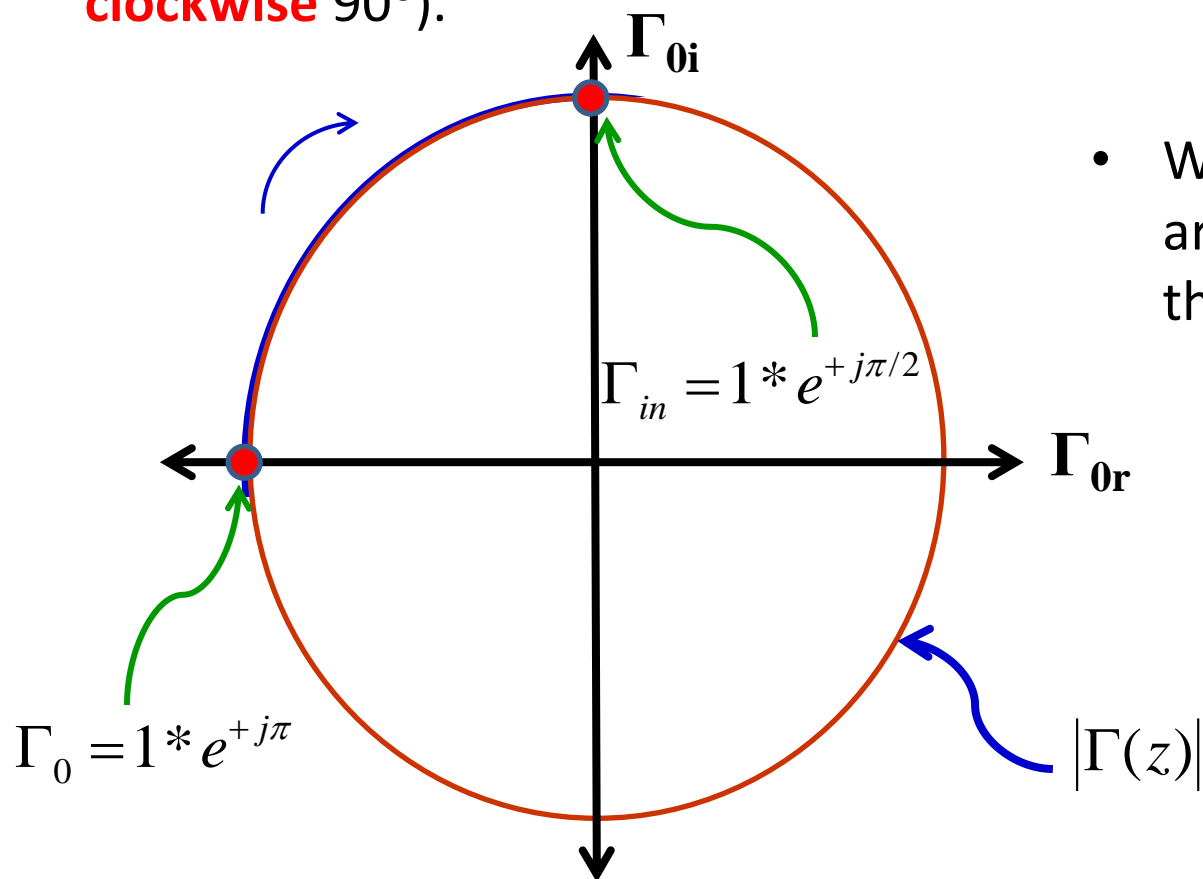
Transformations on the Complex Γ -Plane (contd.)

- It is clear from the graphical display that addition of a length of TL to a load Γ_0 **modifies** the **phase** θ_0 but **not** the **magnitude** Γ_0 , we trace a **circular arc** as we parametrically plot $\Gamma(z)$! This arc has a **radius** Γ_0 and an **arc angle** $2\beta l$ radians.
- We can therefore **easily** solve many interesting TL problems **graphically**—using the complex Γ -plane! For **example**, say we wish to determine Γ_{in} for a transmission line length $l = \lambda/8$ and terminated with a **short** circuit.



Transformations on the Complex Γ -Plane (contd.)

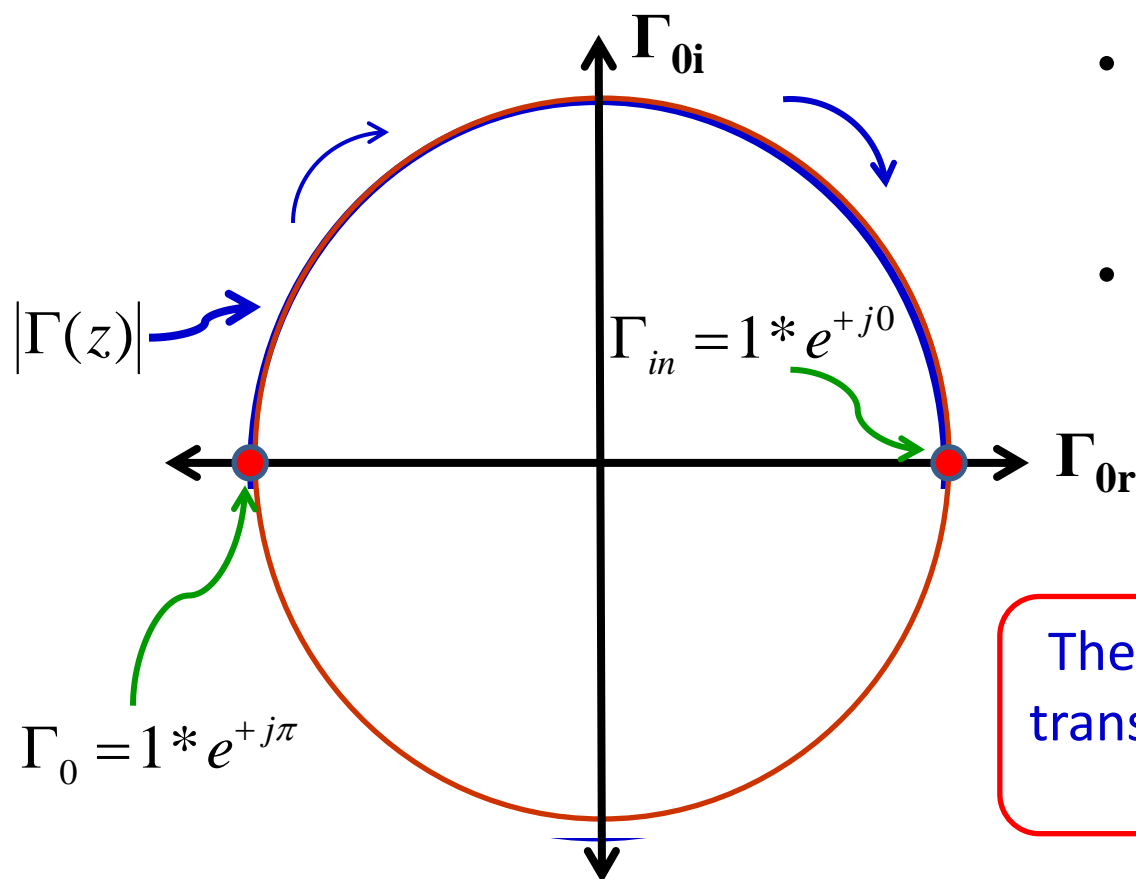
- The reflection coefficient of a **short** circuit is $\Gamma_0 = -1 = 1 * e(j\pi)$, and therefore we **begin** at the leftmost point on the complex Γ -plane. We then move along a **circular arc** $-2\beta l = -2(\pi/4) = -\pi/2$ radians (i.e., rotate **clockwise** 90°).



- When we stop, we find we are at the point for Γ_{in} ; in this case $\Gamma_{in} = 1 * e(j\pi/2)$

Transformations on the Complex Γ -Plane (contd.)

- Now let us consider the same problem, only with a new transmission line length $l = \lambda/4$.
- Now we rotate clockwise $2\beta l = \pi$ radians.

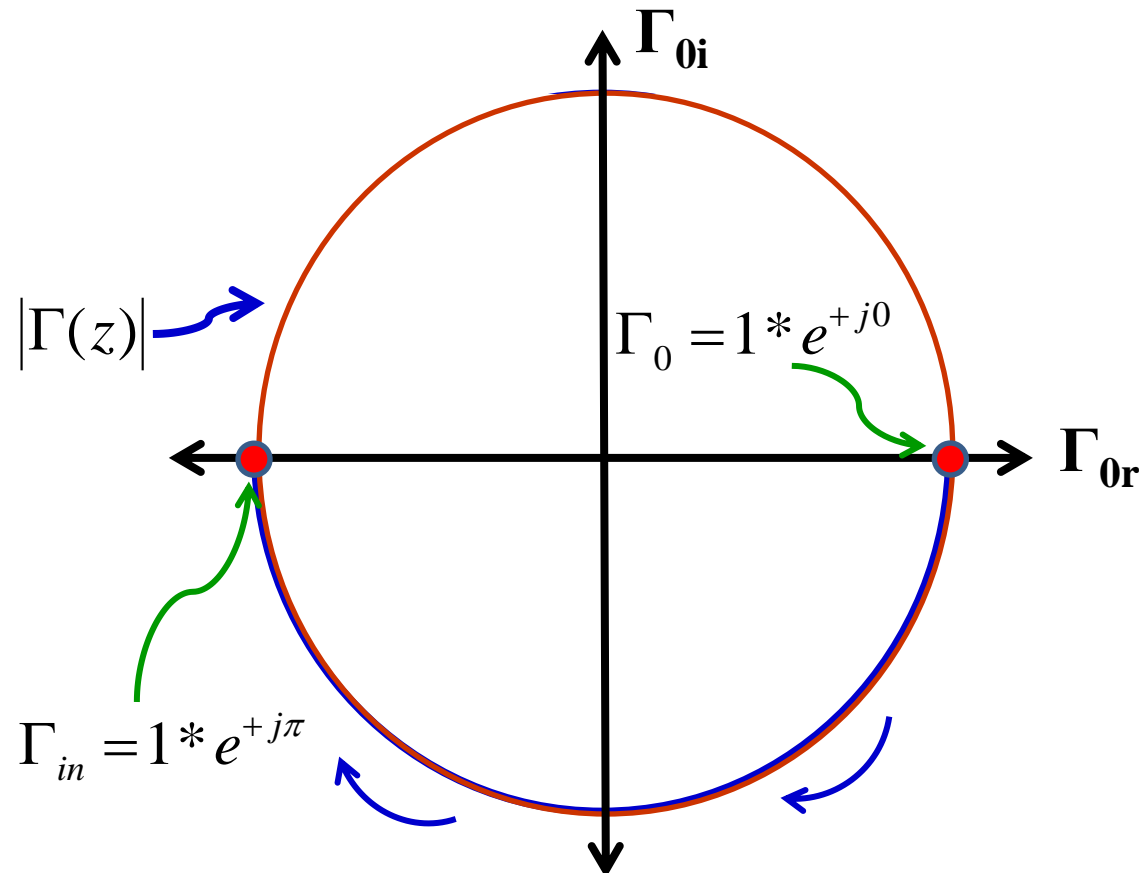


- In this case the input reflection coefficient is $\Gamma_{in} = 1 * e^{+j0} = 1$
- The reflection coefficient of an open circuit

The short circuit load has been transformed into an open circuit with a quarter-wave TL

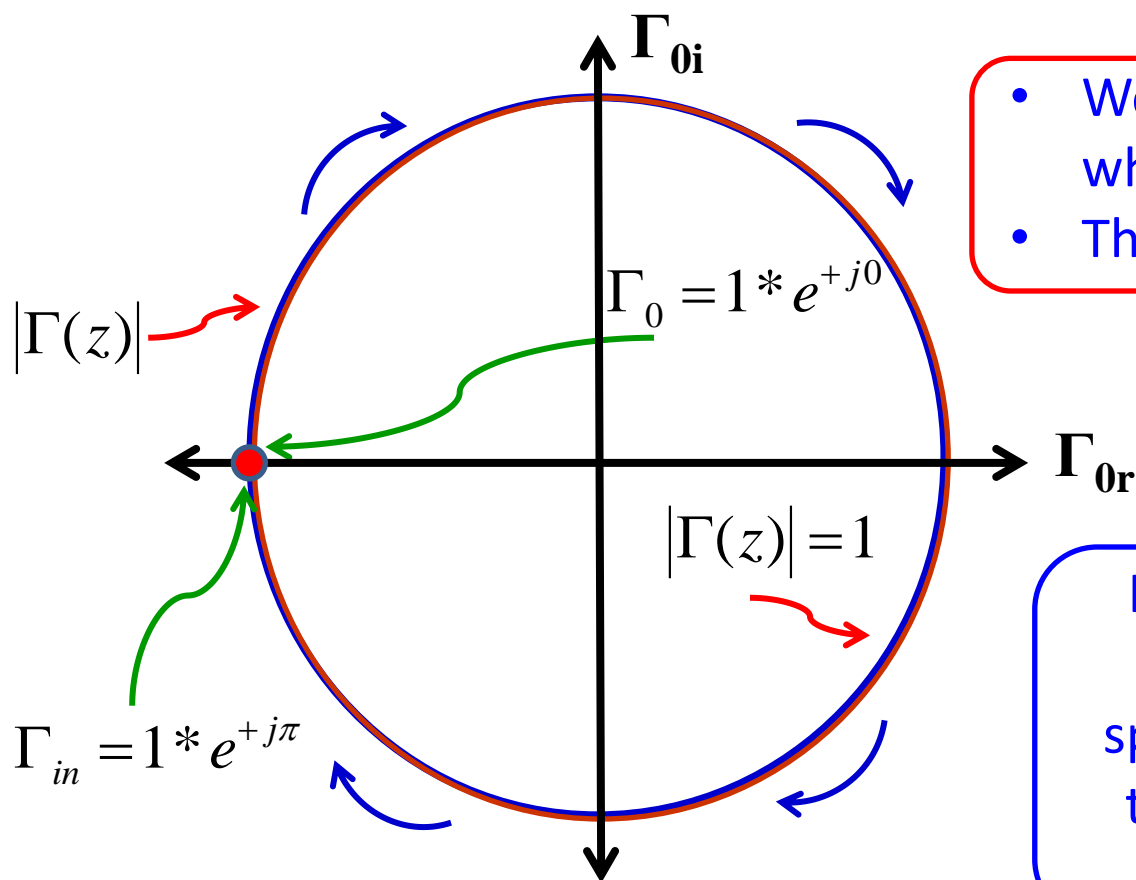
Transformations on the Complex Γ -Plane (contd.)

- We also know that a quarter-wave TL transforms an open-circuit into short-circuit \rightarrow graphically it can be shown as:



Transformations on the Complex Γ -Plane (contd.)

- Now let us consider the same problem again, only with a new transmission line length $l = \lambda/2$.
- Now we rotate clockwise $2\beta l = 2\pi$ radians (360°)

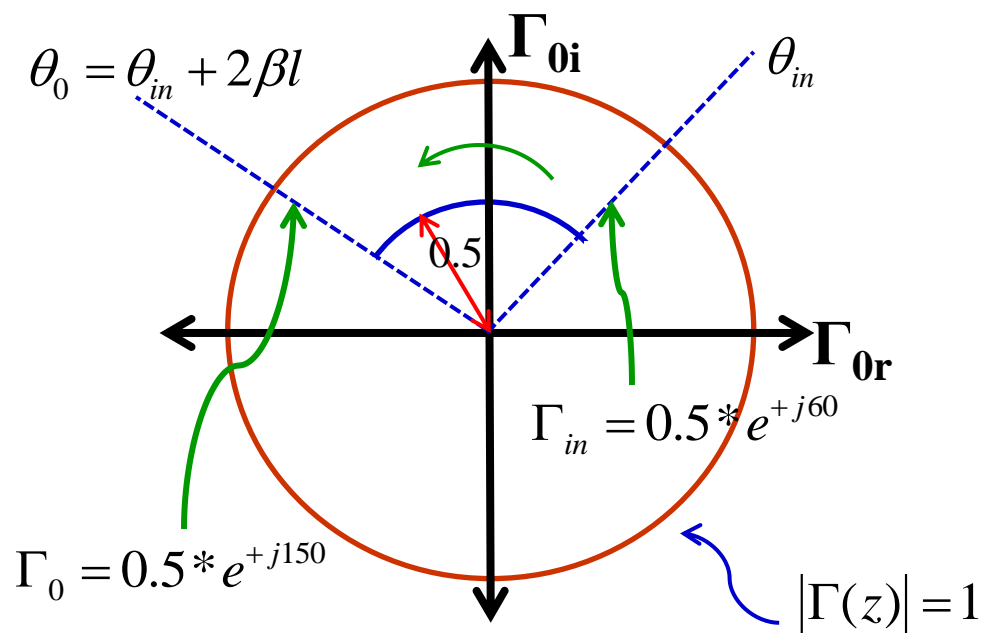


- We came clear around to where we **started!**
- Thus we conclude that $\Gamma_{in} = \Gamma_0$

It comes from the fact that **half-wavelength** TL is a special case, where we know that $\mathbf{Z}_{in} = \mathbf{Z}_L \rightarrow$ eventually it leads to $\Gamma_{in} = \Gamma_0$

Transformations on the Complex Γ -Plane (contd.)

- Now let us consider the **opposite** problem. Say we know that the **input** reflection coefficient at the **beginning** of a TL with length $l = \lambda/8$ is:
 $\Gamma_{in} = 0.5e(j60^\circ)$.
- What is the reflection coefficient at the **load**?
- In this case we rotate **counter-clockwise** along a circular arc (radius = 0.5) by an amount $2\beta l = \pi/2$ radians (90°).
- In essence, we are **removing the phase** associated with the TL.



The reflection coefficient at the load is:

$$\Gamma_0 = 0.5 * e^{+j150}$$

Mapping Z to Γ

- We know that the line impedance and reflection coefficient are **equivalent** – either one can be expressed in terms of the other.

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \longleftrightarrow Z(z) = Z_0 \left(\frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)$$

- The above expressions depend on the characteristic impedance Z_0 of the TL. In order to generalize the relationship, we first define a **normalized** impedance value z' as:

$$z'(z) = \frac{Z(z)}{Z_0} = \frac{R(z)}{Z_0} + j \frac{X(z)}{Z_0} = r(z) + jx(z)$$

therefore

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{(Z(z)/Z_0) - 1}{(Z(z)/Z_0) + 1} = \frac{z'(z) - 1}{z'(z) + 1}$$

$$z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

Mapping Z to Γ (contd.)

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{(Z(z)/Z_0) - 1}{(Z(z)/Z_0) + 1} = \frac{z'(z) - 1}{z'(z) + 1}$$

$$z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

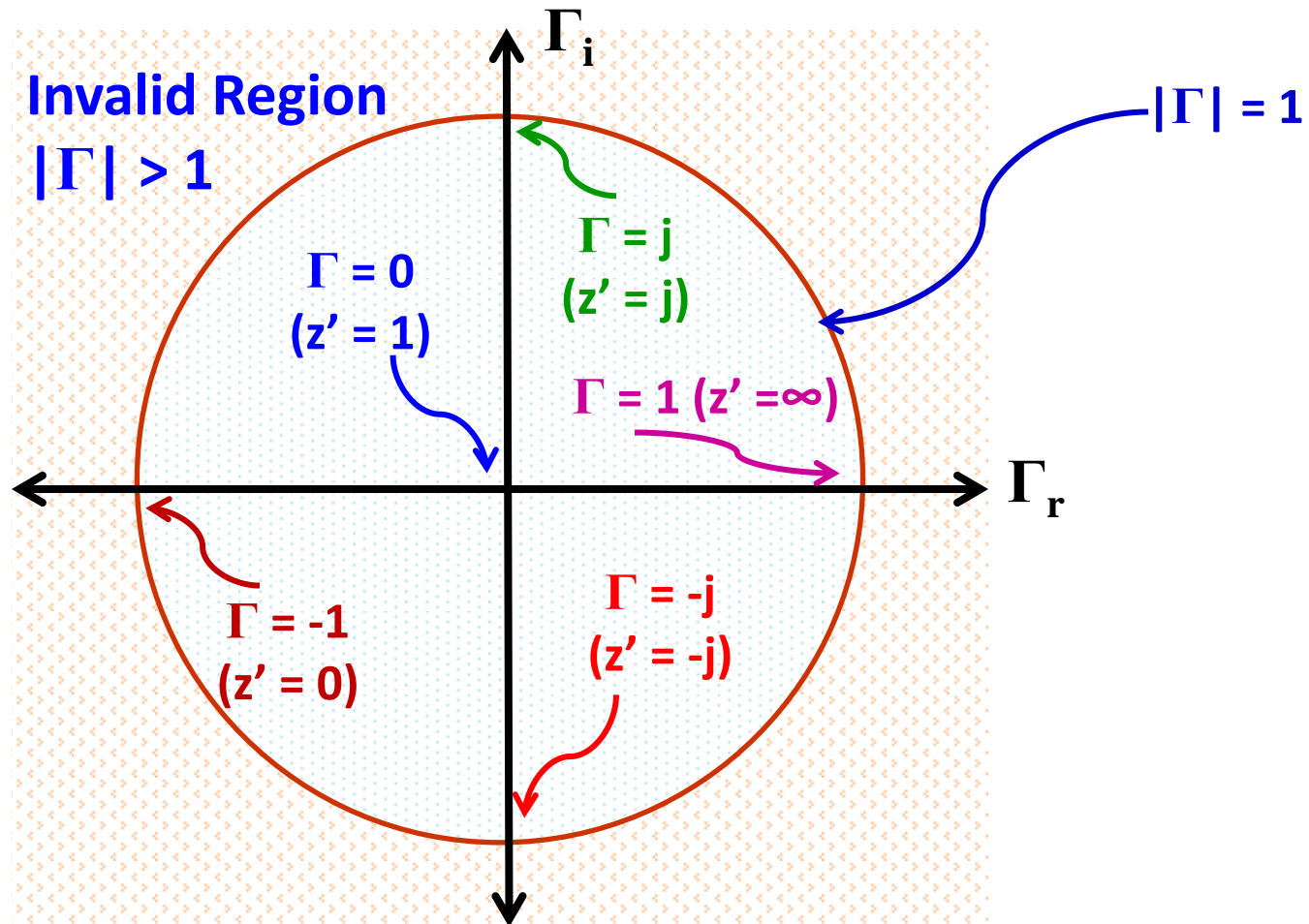
These equations describe a **mapping** between z' and Γ . That means that each and **every normalized impedance** value likewise corresponds to **one specific point** on the complex Γ -plane

- For example, we wish to indicate the values of some common normalized impedances (shown below) on the complex Γ -plane and vice-versa.

Case	Z	z'	Γ
1	∞	∞	1
2	0	0	-1
3	Z_0	1	0
4	jZ_0	j	j
5	$-jZ_0$	$-j$	$-j$

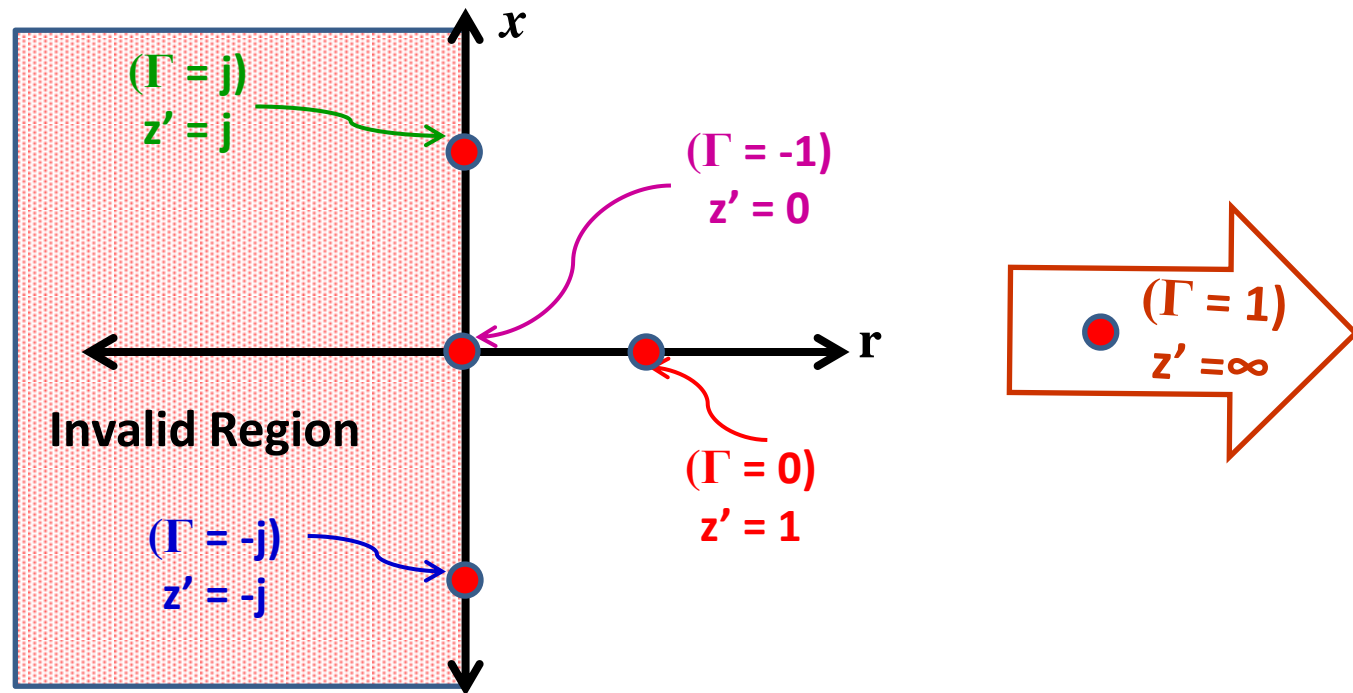
Mapping Z to Γ (contd.)

- The five normalized impedances map five specific points on the complex Γ -plane.



Mapping Z to Γ (contd.)

- The five complex- Γ map onto five points on the normalized Z -plane



- It is apparent that the normalized impedances can be mapped on complex Γ -plane and vice versa
- It gives us a clue that whole impedance contours (i.e, set of points) can be mapped to complex Γ -plane

Mapping Z to Γ (contd.)

Case-I: $Z = R \rightarrow$ impedance is purely real

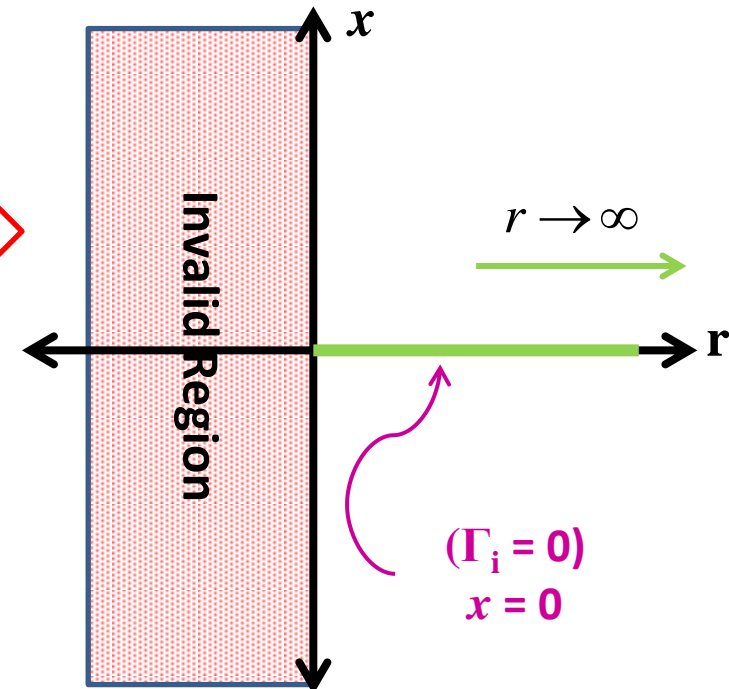
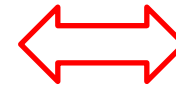
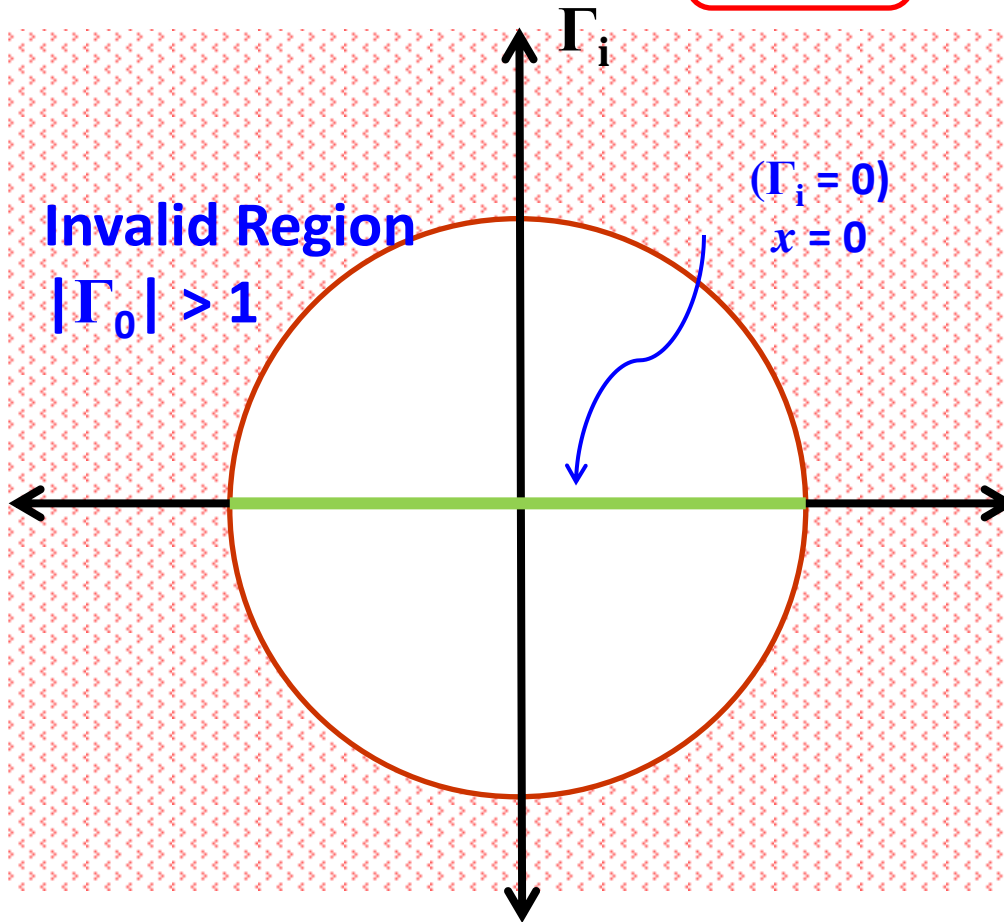
$$z' = r + j0$$



$$\Gamma = \frac{r-1}{r+1}$$



$$\Gamma_r = \frac{r-1}{r+1} \quad \Gamma_i = 0$$



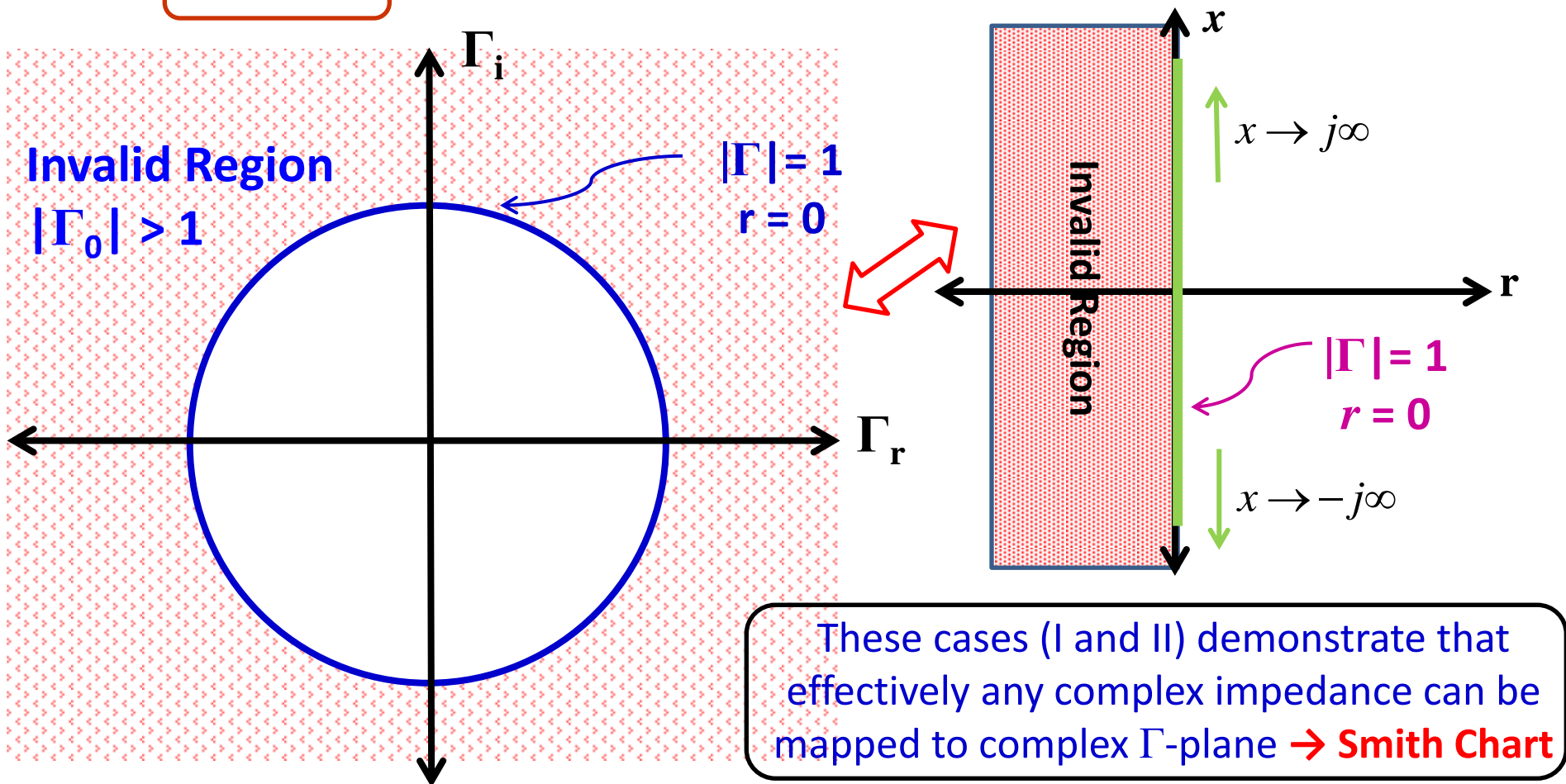
Mapping Z to Γ (contd.)

Case-II: $Z = jX \rightarrow$ impedance is purely imaginary

$$z' = 0 + jx$$

Purely reactive impedance results in a reflection coefficient with unity magnitude

$$|\Gamma| = 1$$



The Smith Chart

In summary

- A vertical line $r = 0$ on complex Z-plane maps to a circle $|\Gamma| = 1$ on the complex Γ -plane
- A horizontal line $x = 0$ on complex Z-plane maps to the line $\Gamma_i = 0$ on the complex Γ -plane



Very fascinating in an academic sense, but are not relevant considering that actual values of impedance generally have both a real and imaginary component

Mappings of more general impedance contours (e.g, $r = 0.5$ and $x = -1.5$ corresponding to normalized impedance $0.5 - j1.5$) can also be mapped

Smith Chart

The Smith Chart (contd.)

- Let us revisit the generalized reflection coefficient formulation:

$$\Gamma(z) = |\Gamma_0| e^{j\theta_0} e^{j2\beta z} = \Gamma_r + j\Gamma_i$$

- Therefore, the normalized impedance can be formulated as:

$$z'(z) = r + jx = \frac{Z(z)}{Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i}$$

$$\Rightarrow ((1 - \Gamma_r) - j\Gamma_i)(r + jx) = (1 + \Gamma_r) + j\Gamma_i$$

- The separation of real and imaginary part results in:

$$r(1 - \Gamma_r) + x\Gamma_i = (1 + \Gamma_r) \quad \leftarrow \text{Real}$$

$$x(1 - \Gamma_r) - r\Gamma_i = \Gamma_i \quad \leftarrow \text{Imaginary}$$

The Smith Chart (contd.)

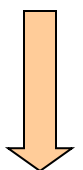
- Simplification and then elimination of **reactance (x)** from these two give:

$$(1 - \Gamma_r)r + \Gamma_i \left[\left(\frac{\Gamma_i}{1 - \Gamma_r} \right) (1 + r) \right] = 1 + \Gamma_r$$

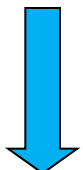


Multiplying through by $1 - \Gamma_r$

$$(1 - \Gamma_r)^2 r + \Gamma_i^2 (1 + r) = (1 + \Gamma_r)(1 - \Gamma_r)$$

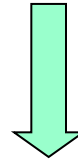


$$(1 - \Gamma_r)^2 r + \Gamma_i^2 (1 + r) = 1 - \Gamma_r^2$$

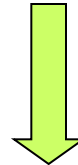


$$\Gamma_r^2 (1 + r) - 2\Gamma_r r + (r - 1) + \Gamma_i^2 (1 + r) = 0$$

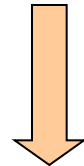
The Smith Chart (contd.)



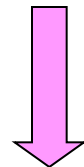
$$\Gamma_r^2(1+r) - 2\Gamma_r r + \Gamma_i^2(1+r) = 1-r$$



$$\Gamma_r^2 - 2\Gamma_r \left(\frac{r}{1+r} \right) + \Gamma_i^2 = \frac{1-r}{1+r}$$




$$\left(\Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{1-r}{1+r} + \left(\frac{r}{1+r} \right)^2$$



$$\left(\Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{(1+r)(1-r) + (r)^2}{(1+r)^2}$$

The Smith Chart (contd.)


$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}$$

Similar equation to circle of radius l ,
centered at (p, q) :

$$(\Gamma_r - p)^2 + (\Gamma_i - q)^2 = l^2$$

This is equation of a circle

center: $(p, q) = \left(\frac{r}{1+r}, 0\right)$ and radius: $l = \frac{1}{1+r}$

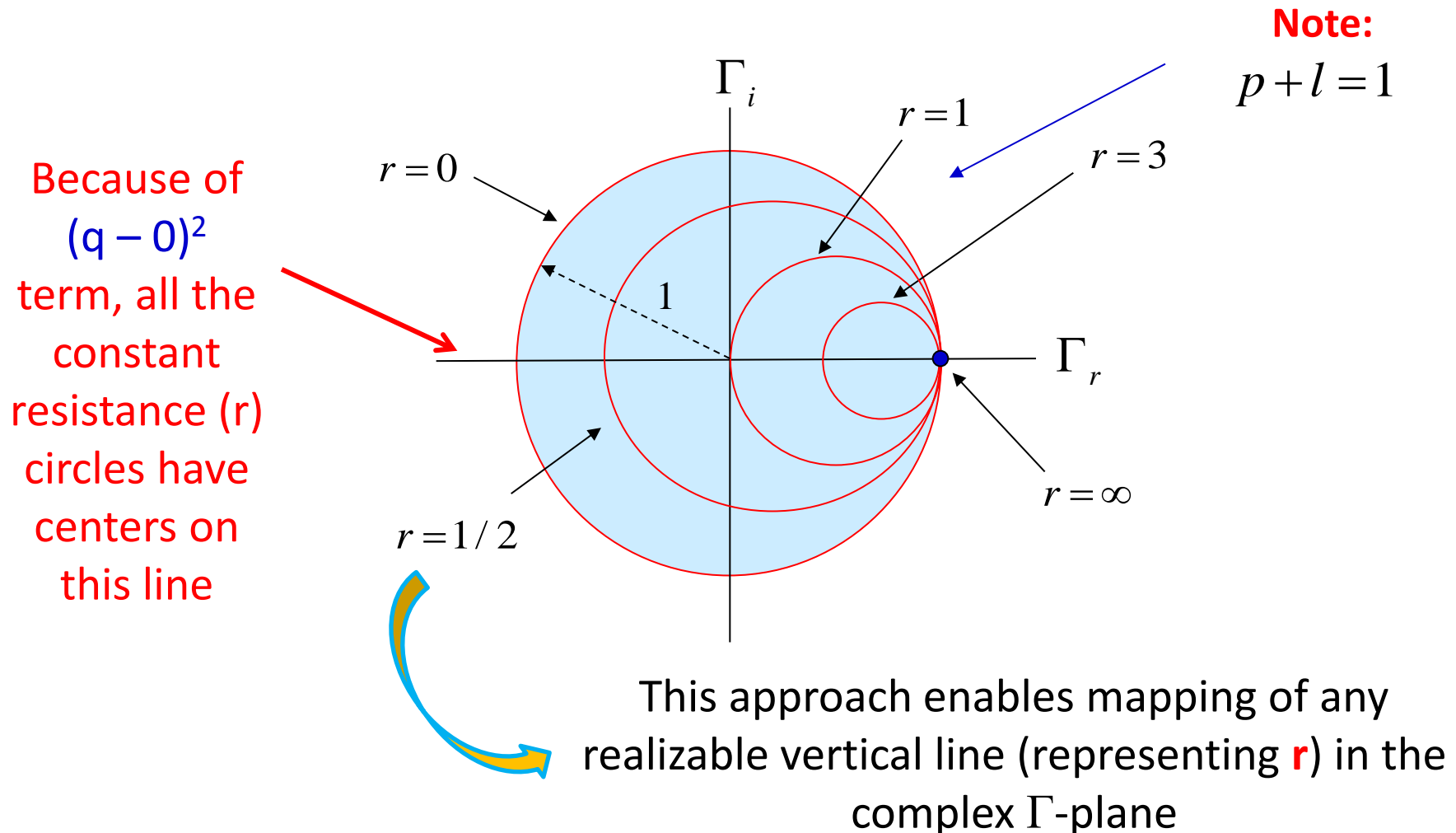
Observations:

- For $r=0$: $p^2 + q^2 = 1$; $(p, q) = (0, 0)$ and $l = 1$
- For $r=1/2$: $(p - 1/3)^2 + q^2 = (2/3)^2$; $(p, q) = (1/3, 0)$ and $l = 2/3$
- For $r=1$: $(p - 1/2)^2 + q^2 = (1/2)^2$; $(p, q) = (1/2, 0)$ and $l = 1/2$
- For $r=3$: $(p - 3/4)^2 + q^2 = (1/4)^2$; $(p, q) = (3/4, 0)$ and $l = 1/4$

Circles of
distinct
centre and
radii

The Smith Chart (contd.)

Therefore the resistance circles on the complex Γ -plane are:



The Smith Chart (contd.)

- For the mapping of horizontal lines of the normalized impedance plane to Γ -plane, let us simplify and eliminate **resistance (r)** from the following:

$$r(1 - \Gamma_r) + x\Gamma_i = (1 + \Gamma_r)$$

← **Real**

$$x(1 - \Gamma_r) - r\Gamma_i = \Gamma_i$$

← **Imaginary**



$$(1 - \Gamma_r) \left[\frac{(1 - \Gamma_r)x - \Gamma_i}{\Gamma_i} \right] + x\Gamma_i = 1 + \Gamma_r$$



$$(1 - \Gamma_r)^2 x - \Gamma_i(1 - \Gamma_r) + x\Gamma_i^2 - \Gamma_i(1 + \Gamma_r) = 0$$



$$(1 - \Gamma_r)^2 x - 2\Gamma_i + x\Gamma_i^2 = 0$$

The Smith Chart (contd.)

$$(\Gamma_r - 1)^2 - \left(\frac{2}{x}\right)\Gamma_i + \Gamma_i^2 = 0$$

center: $(p, q) = (1, 1/x)$

radius: $l = \frac{1}{|x|}$

Note:

$q = \pm l$

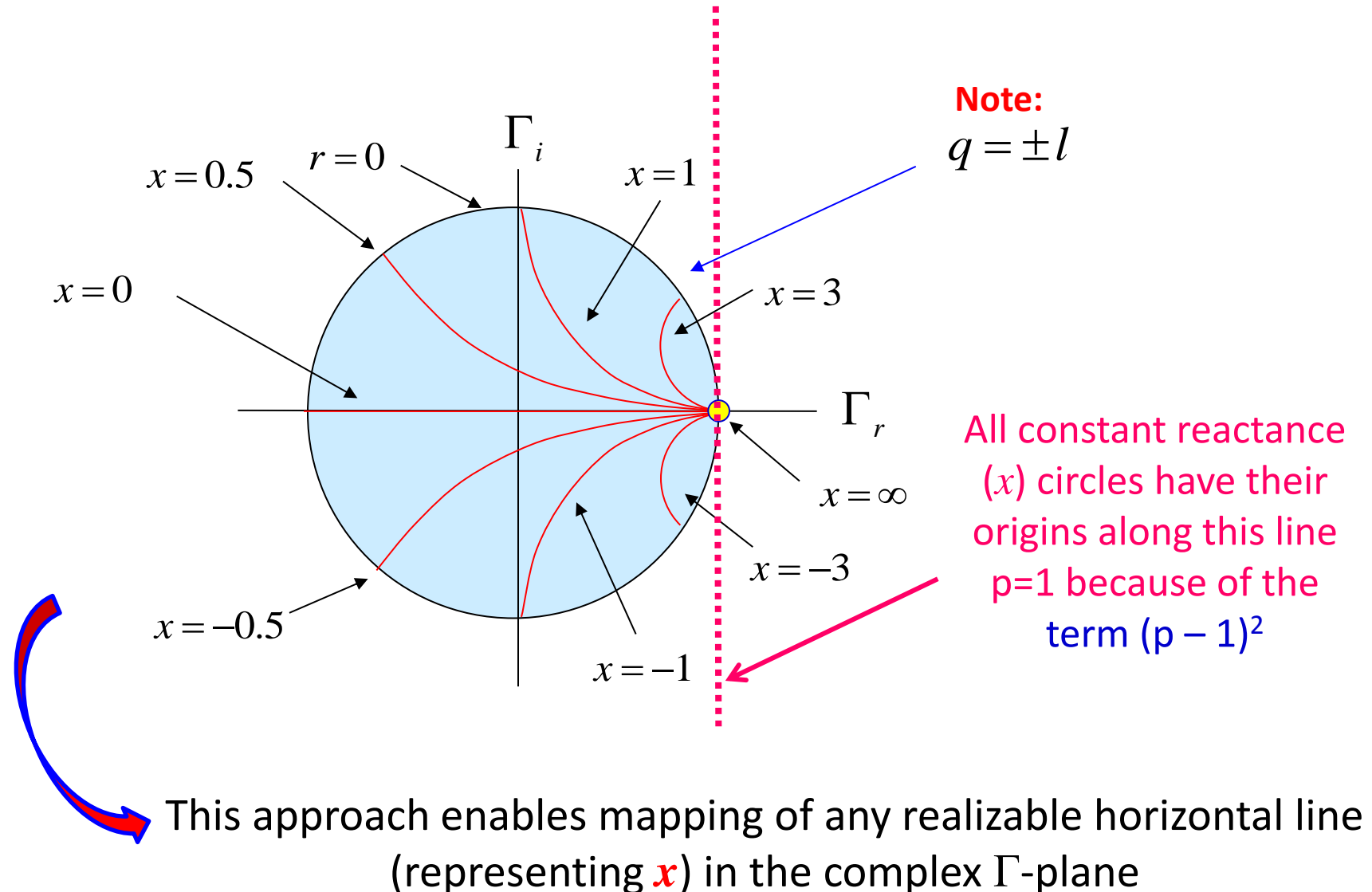
$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

Observations:

- **For $x = 1$:** $(p - 1)^2 + (q - 1)^2 = (1)^2$; $(p, q) = (1, 1)$ and $l = 1$
- **For $x = -1$:** $(p - 1)^2 + (q + 1)^2 = (1)^2$; $(p, q) = (1, -1)$ and $l = 1$
- **For $x = 1/2$:** $(p - 1)^2 + (q - 2)^2 = (2)^2$; $(p, q) = (1, 2)$ and $l = 2$
- **For $x = -1/2$:** $(p - 1)^2 + (q + 2)^2 = (2)^2$; $(p, q) = (1, -2)$ and $l = 2$

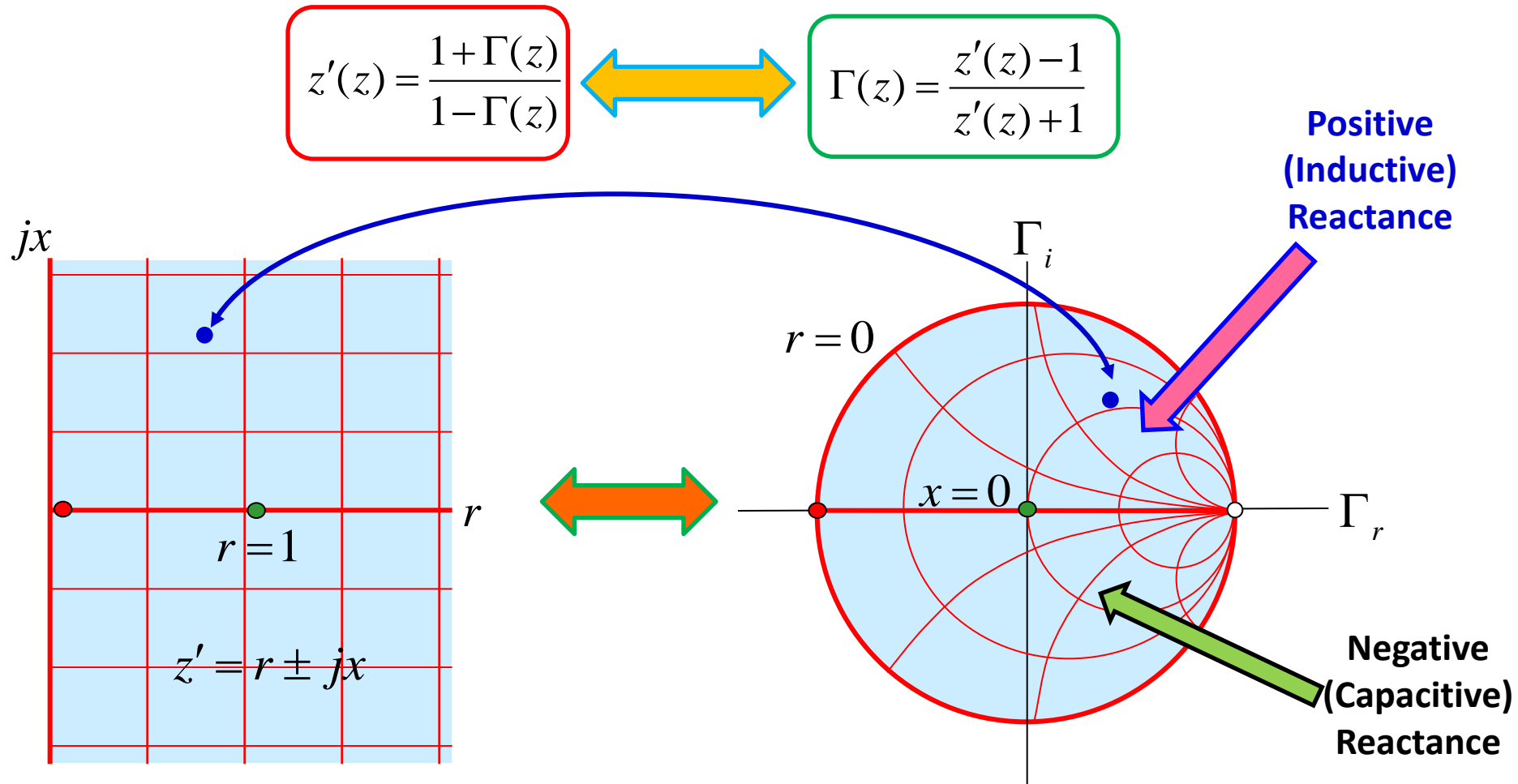
**Circles of
distinct
centre and
radii**

The Smith Chart (contd.)



The Smith Chart (contd.)

- Combination of these **constant resistance** and **reactance circles** define the mappings from **normalized impedance (z') plane** to Γ -plane and is called as Smith chart.



The Smith Chart (contd.) – Important Features

1. By definition:

$$\Gamma(z) = \frac{z'(z) - 1}{z'(z) + 1} = \frac{r + jx - 1}{r + jx + 1}$$



$$|\Gamma(z)| = \frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}$$

- It is apparent: for $r \geq 0$, we get $|\Gamma(z)| \leq 1$. This condition is easily met for passive networks (i.e, no amplifiers) and lossless TLs (real Z_0)
- Consequently, the standard Smith chart only shows only the inside of the unit circle in the Γ -plane. That is, $|\Gamma(z)| \leq 1$ which is bounded by the $r = 0$ circle described by:

$$\Gamma_r^2 + \Gamma_i^2 = 1$$

The Smith Chart (contd.) – Important Features

2. Notice that in the upper semi-circle of the Smith chart, $x \geq 0$ which is an **inductive reactance**. Consequently, the generalized reflection coefficients $\Gamma(z) \equiv \Gamma_r + j\Gamma_i$ in the upper semi-circle are associated with normalized TL impedances $z'(z) \equiv r + jx$ that are inductively reactive.

Conversely, the lower semi-circle of the Smith chart represent capacitive reactive impedances

3. If $z'(z)$ is purely real (ie, $x = 0$) then the reactance term:

$$(1 - \Gamma_r)^2 x - 2\Gamma_i + x\Gamma_i^2 = 0 \quad \xrightarrow{\text{suggests}} \quad \Gamma_i = 0 \text{ except possibly at } \Gamma_r = 1$$

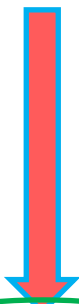
Consequently, purely real $z'(z)$ values are mapped to $\Gamma(z)$ values on the Γ_r axis.

The Smith Chart (contd.) – Important Features

4. If $z'(z)$ is purely imaginary (ie, $r = 0$) then the impedance term:

$$\left(\Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{(1+r)(1-r) + (r)^2}{(1+r)^2}$$

suggests


$$\Gamma_r^2 + \Gamma_i^2 = 1$$

Unit Circle on
 Γ -plane

Consequently, purely imaginary $z'(z)$ values are mapped to $\Gamma(z)$ values on the unit circle in Γ -plane.