

## Lecture – 6

Date: 21.08.2014

- Lossy Transmission Line
- Introduction to Smith Chart: The complex  $\Gamma$ - plane
- Transformations on the complex  $\Gamma$ - plane
- Mapping  $Z$  to  $\Gamma$
- Smith Chart – Construction
- Smith Chart – Geography

## Lossy Transmission Lines

- Recall that we have been **approximating** low-loss transmission lines as lossless ( $R = G = 0$ ):

$$\alpha = 0$$

$$\beta = \omega\sqrt{LC}$$

- But, **long** low-loss lines require a **better** approximation:

$$\alpha = \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right)$$

$$\beta = \omega\sqrt{LC}$$

- Now, if we have **really long** transmission lines (e.g., long distance communications), we can apply **no** approximations at all:

$$\alpha = \text{Re}\{\gamma\}$$

$$\beta = \text{Im}\{\gamma\}$$

For these **very** long transmission lines, we find that  $\beta = \text{Im}\{\gamma\}$  is a **function** of signal **frequency**  $\omega$ . This results in an extremely serious problem—signal **dispersion**.

## Lossy Transmission Lines (contd.)

- Recall that the **phase velocity**  $v_p$  (i.e., propagation velocity) of a wave in a transmission line is:

$$v_p = \frac{\omega}{\beta}$$

$$\beta = \text{Im}\{\gamma\} = \text{Im}\left\{\sqrt{(R + j\omega L)(G + j\omega C)}\right\}$$

Thus, for a lossy line, the phase velocity  $v_p$  is a function of frequency  $\omega$  (i.e.,  $v_p(\omega)$ )—this is **bad!**

- Any signal that carries significant **information** must have some non-zero **bandwidth**. In other words, the signal energy (as well as the information it carries) is **spread** across many frequencies.
- If the different frequencies that comprise a signal travel at different velocities, that signal will arrive at the end of a transmission line **distorted**. We call this phenomenon signal **dispersion**.
- Recall for **lossless** lines, however, the phase velocity is **independent** of frequency—**no** dispersion will occur!

## Lossy Transmission Lines (contd.)

- For lossless line:

$$v_p = \frac{1}{\sqrt{LC}}$$

however, a perfectly lossless line is impossible, but we find phase velocity is **approximately** constant if the line is low-loss.

Therefore, dispersion distortion on low-loss lines is **most often** not a problem.

**Q:** You say “**most often**” not a problem—that phrase seems to imply that dispersion sometimes is a problem!



## Lossy Transmission Lines (contd.)

**A:** Even for low-loss transmission lines, dispersion can be a problem **if** the lines are **very** long—just a small difference in phase velocity can result in significant differences in propagation delay **if** the line is very long!

- Modern examples of long transmission lines include phone lines and cable TV. However, the **original** long transmission line problem occurred with the **telegraph**, a device invented and implemented in the 19<sup>th</sup> century.
- Early telegraph “engineers” discovered that if they made their telegraph lines **too long**, the dots and dashes characterizing Morse code turned into a muddled, indecipherable **mess**. Although they did not realize it, they had fallen victim to the heinous effects of **dispersion**!
- Thus, to send messages over long distances, they were forced to implement a series of intermediate “**repeater**” stations, wherein a human operator received and then **retransmitted** a message on to the next station. This **really** slowed things down!

## Lossy Transmission Lines (contd.)



**Q:** Is there any way to **prevent** dispersion from occurring?

**A:** You bet! **Oliver Heaviside** figured out how in the **19<sup>th</sup>** Century!

- Heaviside found that a transmission line would be distortionless (i.e., no dispersion) **if** the line parameters exhibited the following **ratio**:

$$\frac{R}{L} = \frac{G}{C}$$

- Let's see **why** this works. Note the complex propagation constant  $\gamma$  can be expressed as:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{LC(R/L + j\omega)(G/C + j\omega)}$$

## Lossy Transmission Lines (contd.)

- Then IF:

$$\frac{R}{L} = \frac{G}{C}$$

- we find:

$$\gamma = \sqrt{LC(R/L + j\omega)(R/L + j\omega)} = (R/L + j\omega)\sqrt{LC} = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$$

- Thus:

$$\alpha = \text{Re}\{\gamma\} = R\sqrt{\frac{C}{L}}$$

$$\beta = \text{Im}\{\gamma\} = \omega\sqrt{LC}$$

- The propagation **velocity** of the wave is thus:

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

The propagation velocity is **independent** of frequency! This lossy transmission line is **not** dispersive!

## Lossy Transmission Lines (contd.)



**Q:** Right. All the transmission lines I use have the property that  $R/L > G/C$ . I've **never** found a transmission line with this **ideal** property  $R/L = G/C$ !

**A:** It is true that typically  $R/L > G/C$ . But, we can reduce the ratio  $R/L$  (until it is equal to  $G/C$ ) by adding series **inductors** periodically along the transmission line.

This was **Heaviside's** solution—and it worked! **Long** distance transmission lines were made possible.

**Q:** Why don't we increase  $G$  instead?

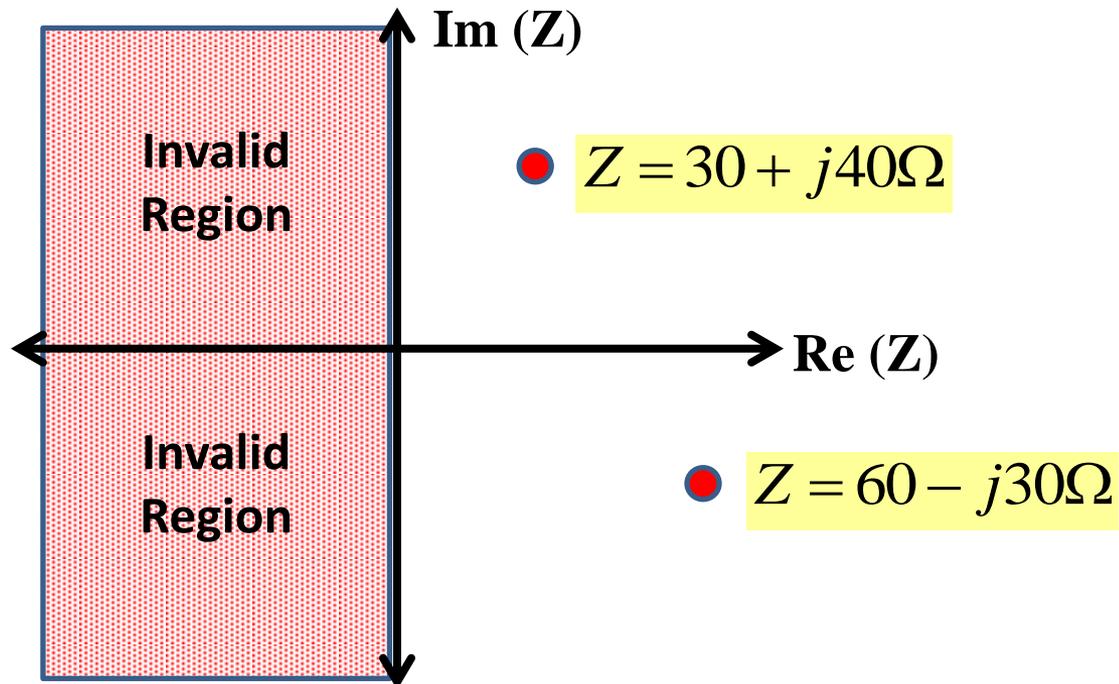
**A:**

## Smith Chart

- Smith chart – what?
- The **Smith chart** is a very convenient graphical tool for analyzing Tl's studying their behavior.
- **It is** mapping of impedance in standard complex plane into a suitable complex reflection coefficient plane.
- It provides graphical display of reflection coefficients.
- The impedances can be directly determined from the graphical display (ie, from Smith chart)
- Furthermore, Smith charts facilitate the analysis and design of complicated circuit configurations.

## The Complex $\Gamma$ - Plane

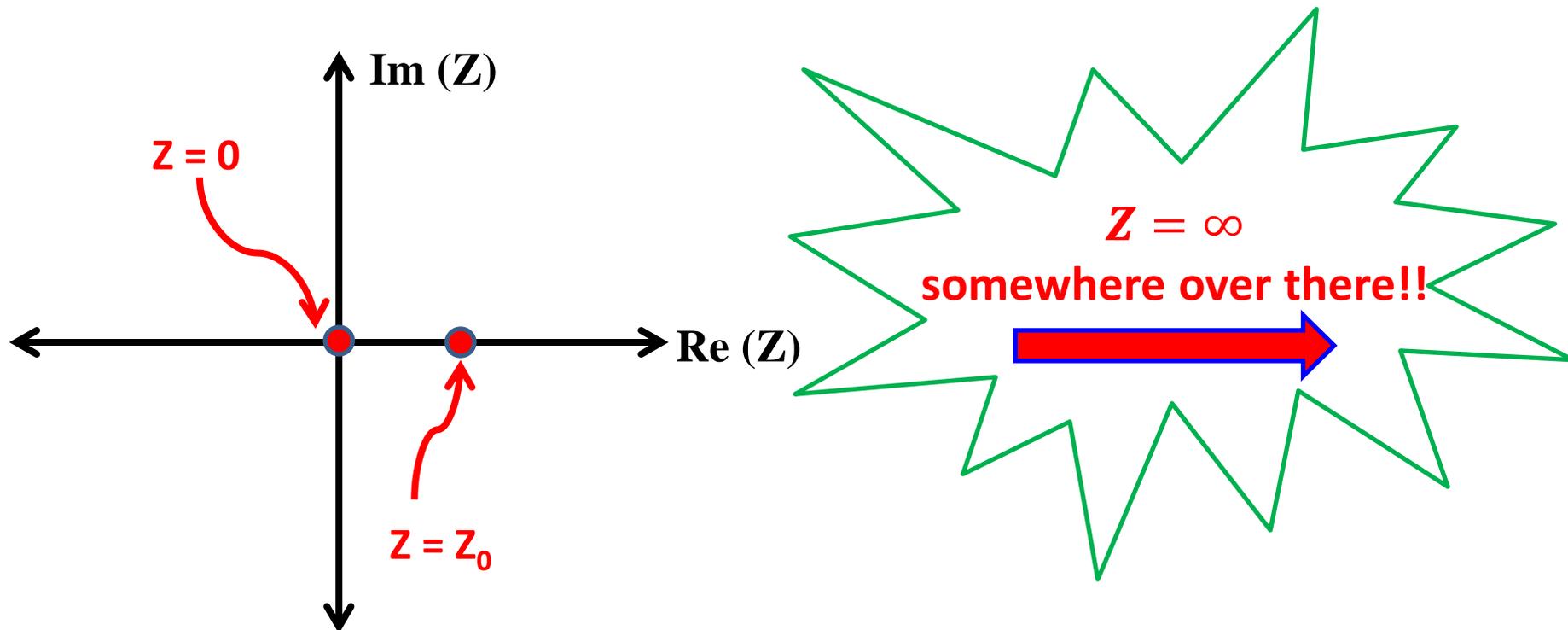
- Let us first display the impedance  $Z$  on complex  $Z$ -plane



- Note that each dimension is defined by a single real line: the **horizontal line (axis)** indicates the **real component of  $Z$** , and the **vertical line (axis)** indicates the **imaginary component of  $Z$**   $\rightarrow$  **Intersection** of these lines indicate the complex impedance

## The Complex $Z$ -Plane (contd.)

- How do we plot an **open circuit** (i.e,  $Z = \infty$ ), **short circuit** (i.e,  $Z = 0$ ), and **matching condition** (i.e,  $Z = Z_0 = 50\Omega$ ) on the complex  $Z$ -plane



It is apparent that complex  $Z$  - plane is not very useful

## The Complex $\Gamma$ -Plane (contd.)

- The **limitations** of **complex Z-plane** can be **overcome** by **complex  $\Gamma$ -plane**
- We know  $\mathbf{Z} \leftrightarrow \mathbf{\Gamma}$  (i.e, if you know **one**, you know the **other**).
- We can therefore define a **complex  $\Gamma$ -plane** in the same manner that we defined a complex Z-plane.
- Let us revisit the reflection coefficient in complex form:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

Where,

$$\theta_0 = \tan^{-1} \left( \frac{\Gamma_{0i}}{\Gamma_{0r}} \right)$$

Real part of  $\Gamma_0$

Imaginary part of  $\Gamma_0$

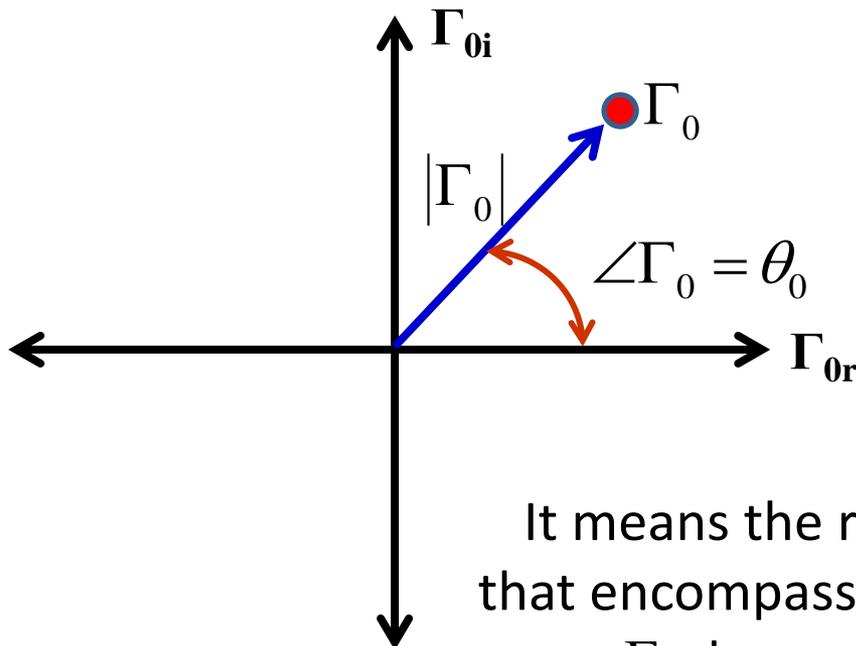
- In the special terminated conditions of **pure short-circuit and pure open-circuit conditions** the corresponding  $\Gamma_0$  are **-1 and +1** located on the real axis in the complex  $\Gamma$ -plane.

## The Complex $\Gamma$ -Plane (contd.)

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$



Representation of reflection coefficient in polar form



### Observations:

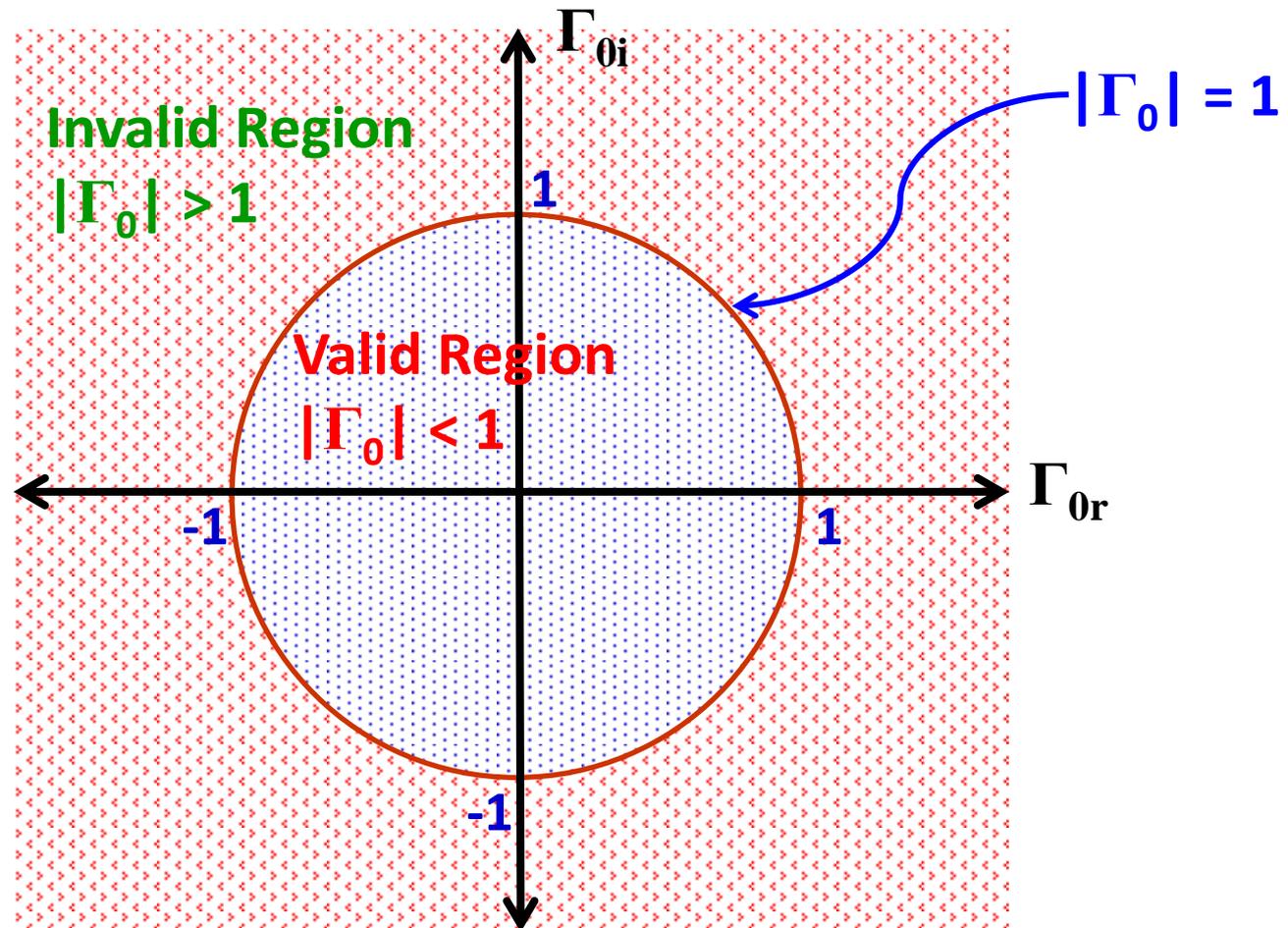
- A radial line is formed by the locus of all points whose phase is  $\theta_0$
- A circle is formed by the locus of all points whose magnitude is  $|\Gamma_0|$

It means the reflection coefficient has a valid region that encompasses all the four quadrants in the complex  $\Gamma$ -plane within the  $-1$  to  $+1$  bounded region

In complex  $Z$ -plane the valid region was unbounded on the right half of the plane  $\rightarrow$  as a result many important impedances could not be plotted

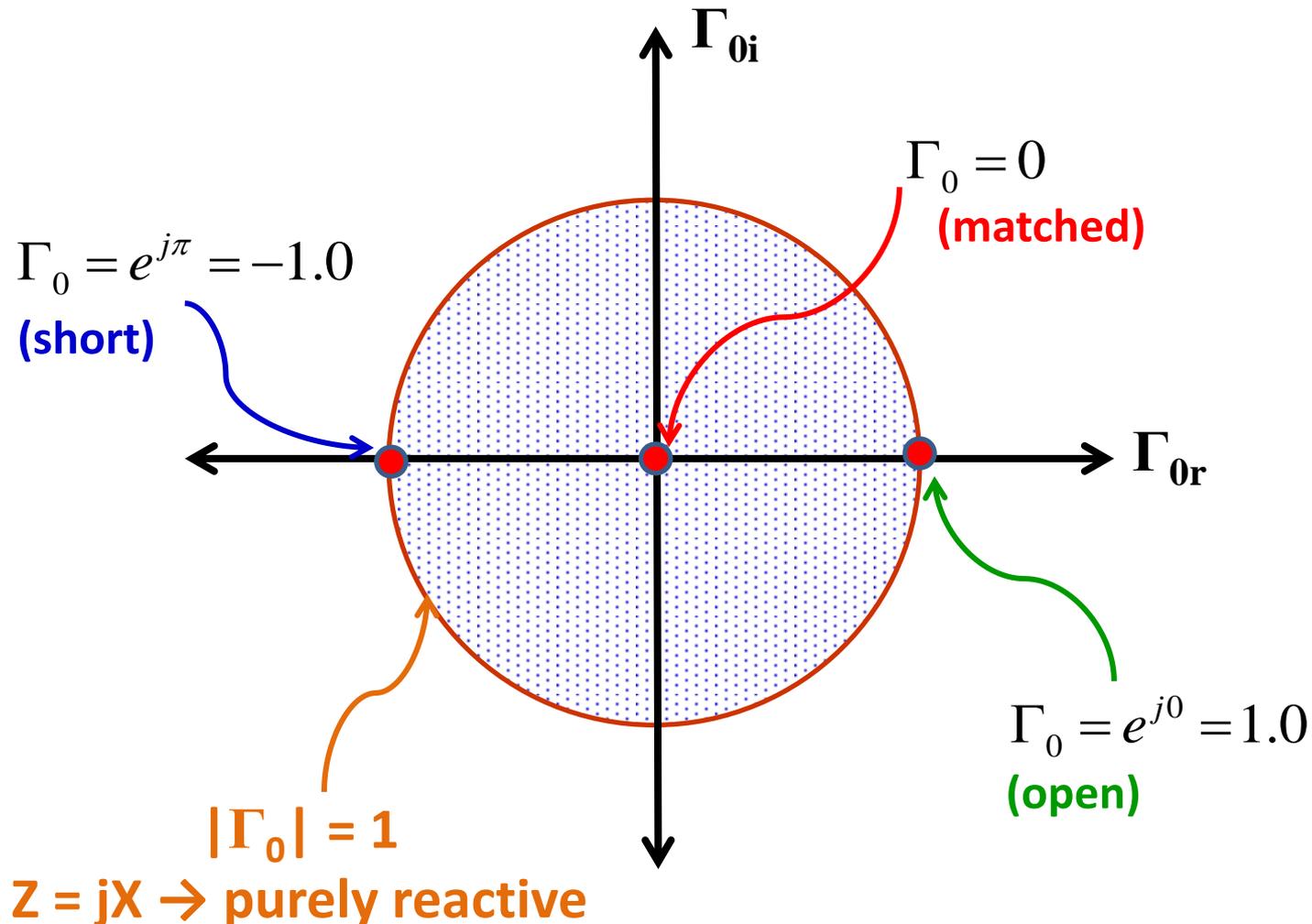
## The Complex $\Gamma$ -Plane (contd.)

- Validity Region



## The Complex $\Gamma$ -Plane (contd.)

- We can plot all the valid impedances (i.e  $R > 0$ ) within this bounded region.



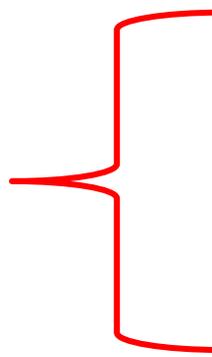
## Example – 1

- A TL with a characteristic impedance of  $Z_0 = 50\Omega$  is terminated into following load impedances:
  - (a)  $Z_L = 0$  (Short Circuit)
  - (b)  $Z_L \rightarrow \infty$  (Open Circuit)
  - (c)  $Z_L = 50\Omega$
  - (d)  $Z_L = (16.67 - j16.67)\Omega$
  - (e)  $Z_L = (50 + j50)\Omega$

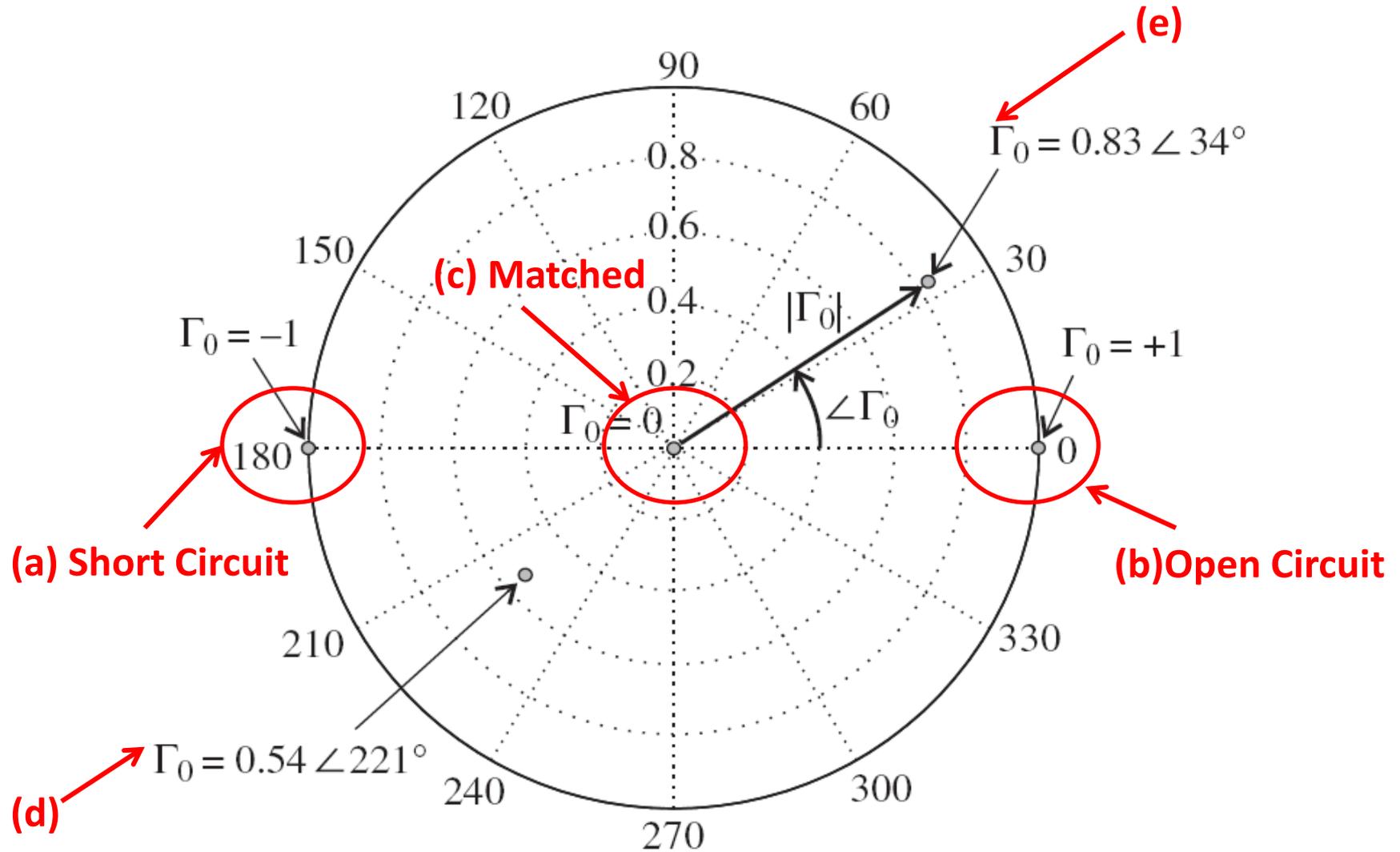
**Display the respective reflection coefficients in complex  $\Gamma$ -plane**

- **Solution:** We know the relationship between  $Z$  and  $\Gamma$ :

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

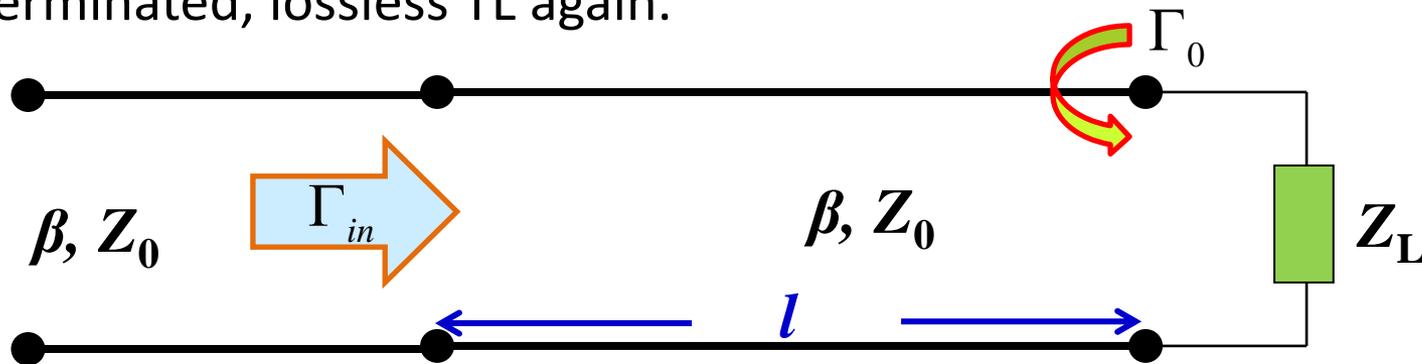
- 
- (a)  $\Gamma_0 = -1$  (Short Circuit)
  - (b)  $\Gamma_0 = 1$  (Open Circuit)
  - (c)  $\Gamma_0 = 0$  (Matched)
  - (d)  $\Gamma_0 = 0.54 \angle 221^\circ$
  - (e)  $\Gamma_0 = 0.83 \angle 34^\circ$

## Example – 1 (contd.)



## Transformations on the Complex $\Gamma$ -Plane

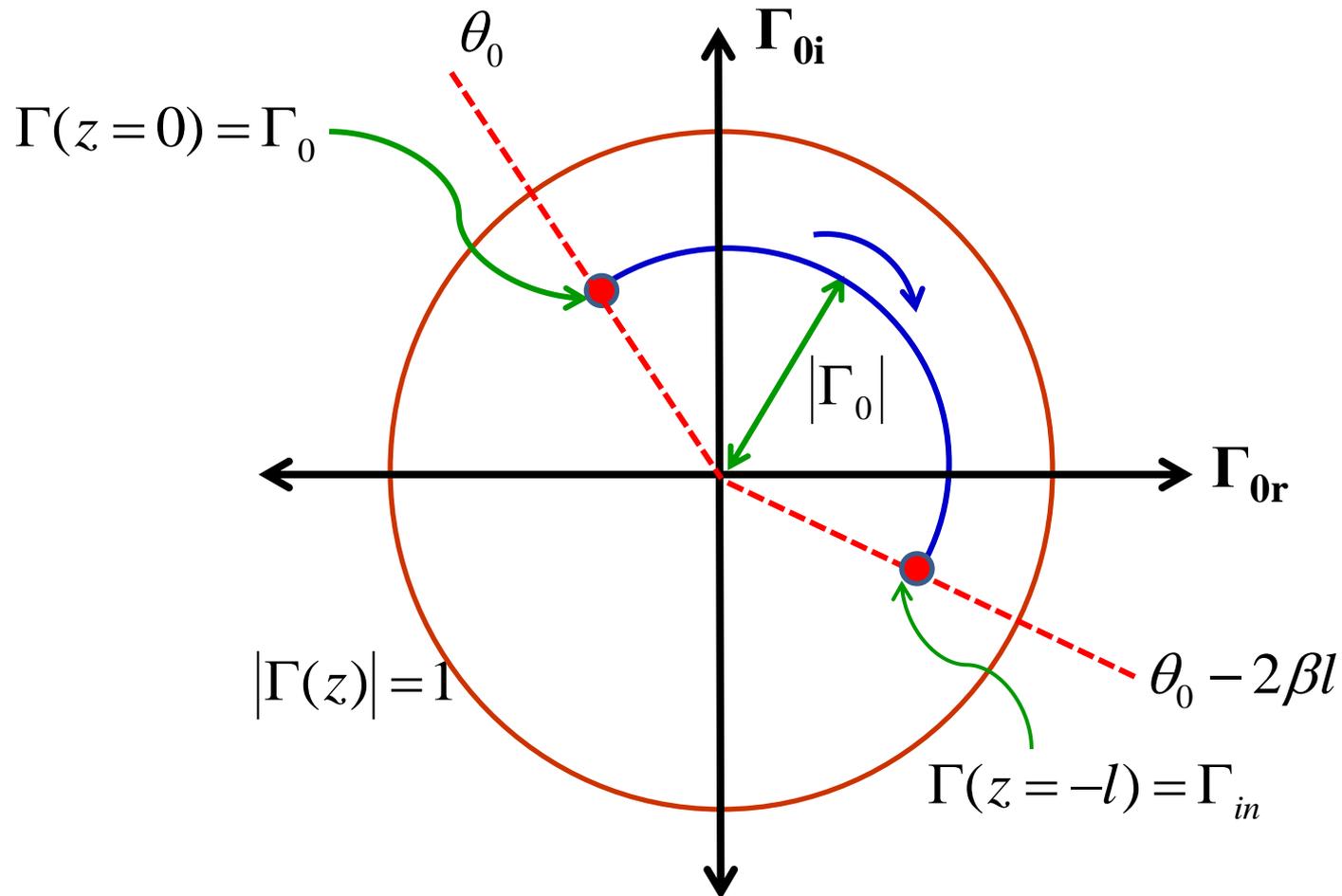
- The usefulness of the complex  $\Gamma$ -plane will be evident when we consider the terminated, lossless TL again.



- At  $z = 0$ , the reflection coefficient is called load reflection coefficient ( $\Gamma_0$ )  $\rightarrow$  this actually describes the **mismatch** between the load impedance ( $Z_L$ ) and the characteristic impedance ( $Z_0$ ) of the TL.
- The **move away from the load** (or towards the input/source) in the negative  $z$ -direction (clockwise rotation) **requires multiplication** of  $\Gamma_0$  by a factor  **$\exp(+j2\beta z)$**  in order to explicitly define the mismatch at location 'z' known as  $\Gamma(z)$ .
- This **transformation** of  $\Gamma_0$  to  $\Gamma(z)$  is the key ingredient in **Smith chart** as a graphical design/display tool.

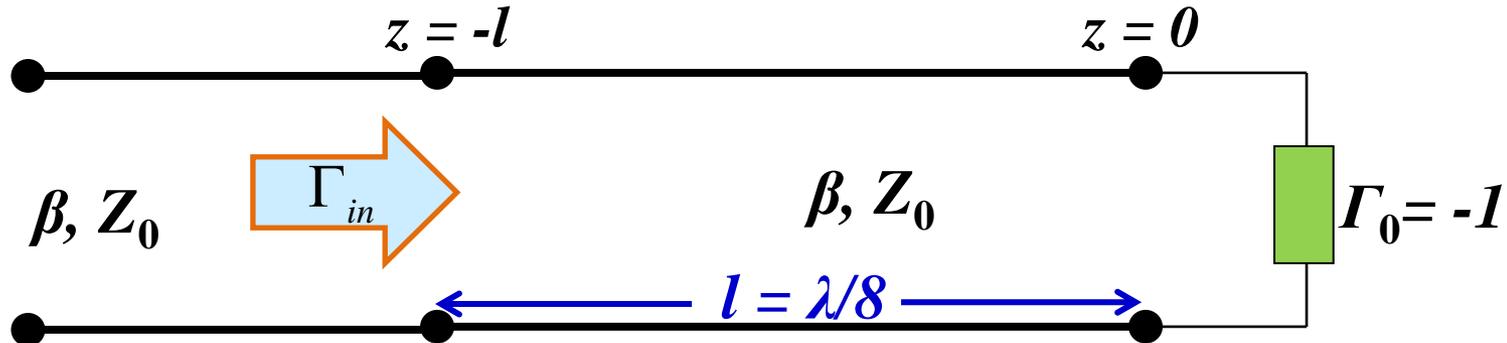
## Transformations on the Complex $\Gamma$ -Plane (contd.)

- Graphical interpretation of  $\Gamma(z) = \Gamma_0 e^{+2j\beta z}$



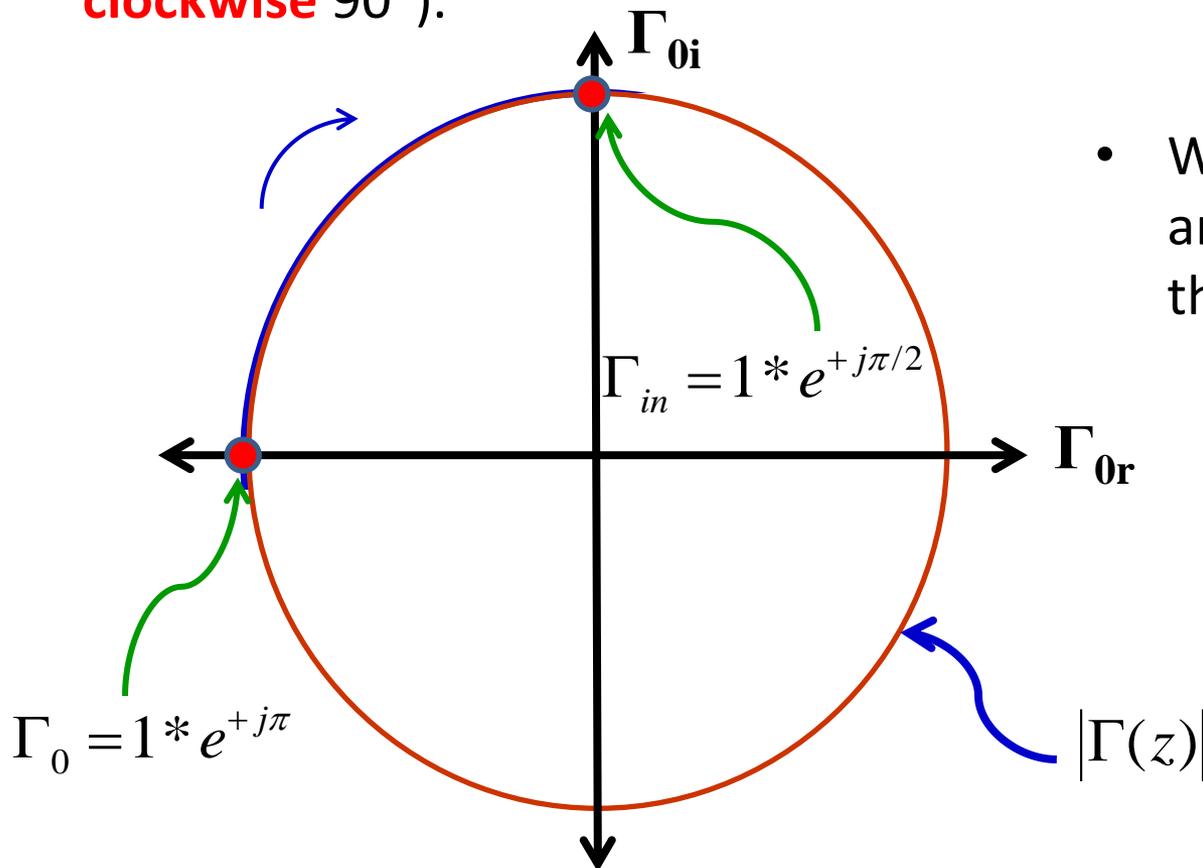
## Transformations on the Complex $\Gamma$ -Plane (contd.)

- It is clear from the graphical display that addition of a length of TL to a load  $\Gamma_0$  **modifies** the **phase**  $\theta_0$  but **not** the **magnitude**  $\Gamma_0$ , we trace a **circular arc** as we parametrically plot  $\Gamma(z)$ ! This arc has a **radius**  $\Gamma_0$  and an **arc angle**  $2\beta l$  radians.
- We can therefore **easily** solve many interesting TL problems **graphically**—using the complex  $\Gamma$ -plane! For **example**, say we wish to determine  $\Gamma_{in}$  for a transmission line length  $l = \lambda/8$  and terminated with a **short** circuit.



## Transformations on the Complex $\Gamma$ -Plane (contd.)

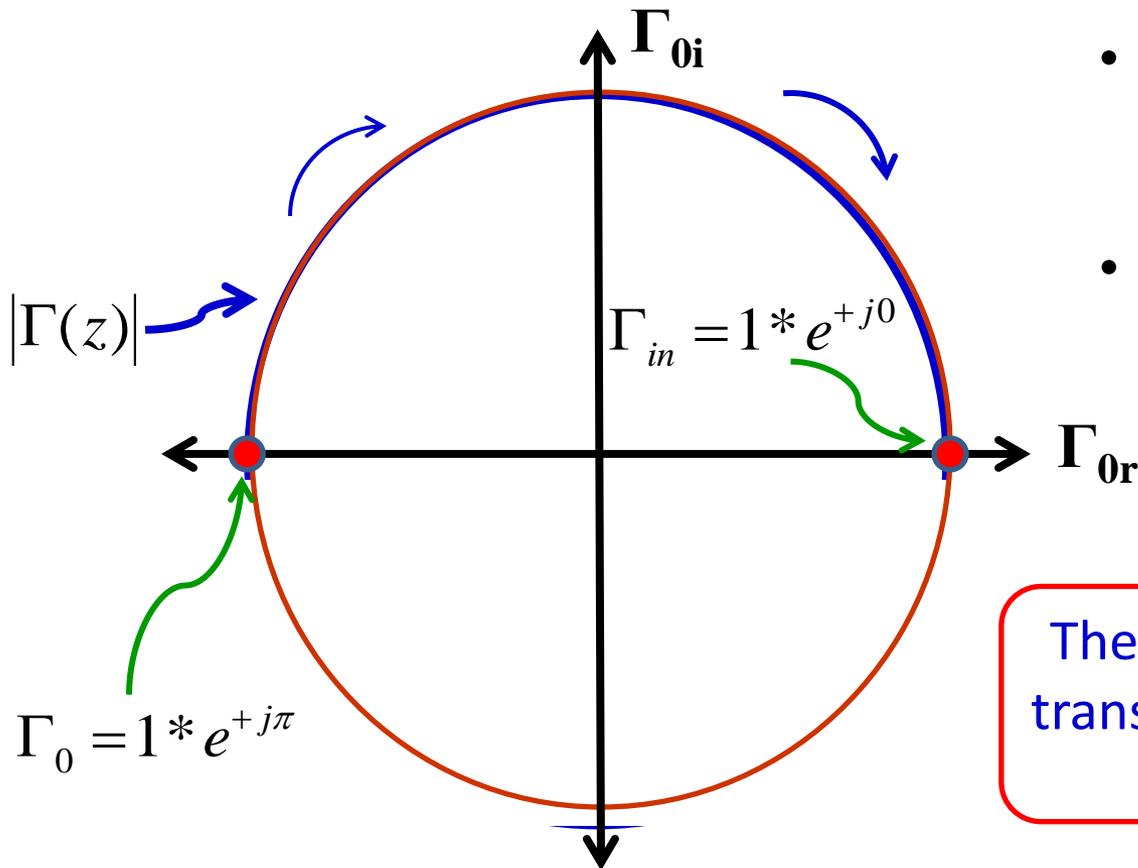
- The reflection coefficient of a **short** circuit is  $\Gamma_0 = -1 = 1 * e(j\pi)$ , and therefore we **begin** at the leftmost point on the complex  $\Gamma$ -plane. We then move along a **circular arc**  $-2\beta l = -2(\pi/4) = -\pi/2$  radians (i.e., rotate **clockwise**  $90^\circ$ ).



- When we stop, we find we are at the point for  $\Gamma_{in}$ ; in this case  $\Gamma_{in} = 1 * e(j\pi/2)$

## Transformations on the Complex $\Gamma$ -Plane (contd.)

- Now let us consider the same problem, only with a new transmission line length  $l = \lambda/4$ .
- Now we rotate clockwise  $2\beta l = \pi$  radians.

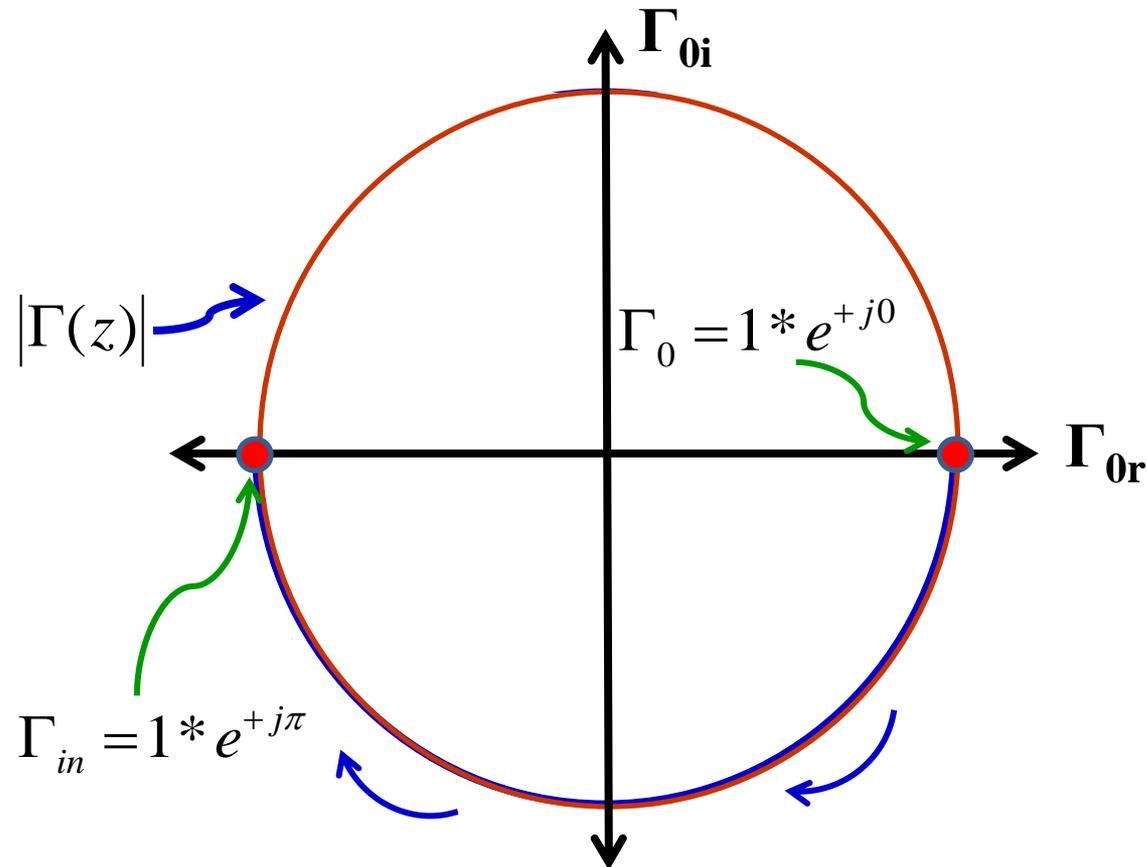


- In this case the input reflection coefficient is  $\Gamma_{in} = 1 * e^{j0} = 1$
- The reflection coefficient of an open circuit

The short circuit load has been transformed into an open circuit with a quarter-wave TL

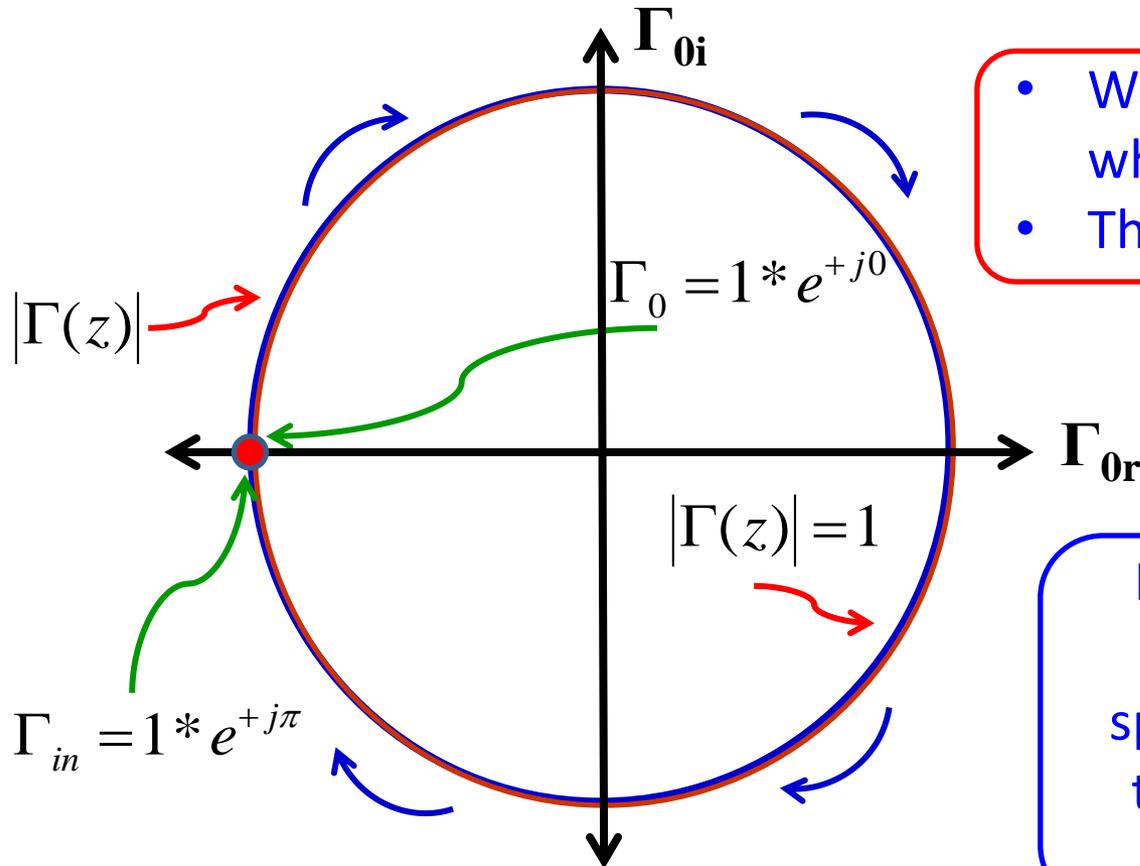
## Transformations on the Complex $\Gamma$ -Plane (contd.)

- We also know that a quarter-wave TL transforms an open-circuit into short-circuit  $\rightarrow$  graphically it can be shown as:



## Transformations on the Complex $\Gamma$ -Plane (contd.)

- Now let us consider the same problem again, only with a new transmission line length  $l = \lambda/2$ .
- Now we rotate clockwise  $2\beta l = 2\pi$  radians ( $360^\circ$ )

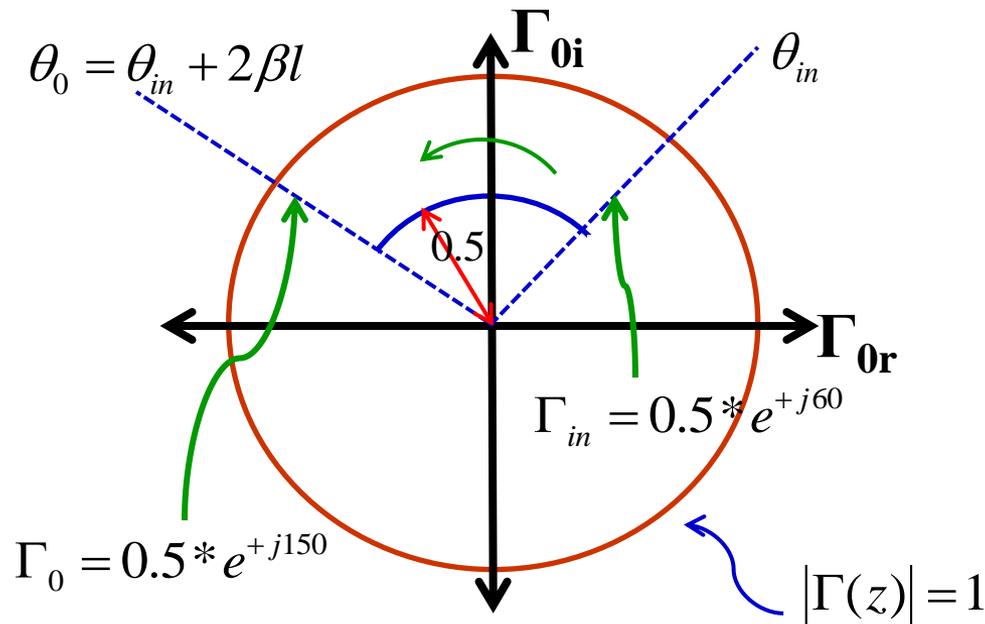


- We came clear around to where we **started!**
- Thus we conclude that  $\Gamma_{in} = \Gamma_0$

It comes from the fact that **half-wavelength** TL is a special case, where we know that  $\mathbf{z}_{in} = \mathbf{z}_L \rightarrow$  eventually it leads to  $\Gamma_{in} = \Gamma_0$

## Transformations on the Complex $\Gamma$ -Plane (contd.)

- Now let us consider the **opposite** problem. Say we know that the **input** reflection coefficient at the **beginning** of a TL with length  $l = \lambda/8$  is:  
 $\Gamma_{in} = 0.5e(j60^\circ)$ .
- What is the reflection coefficient at the **load**?
- In this case we rotate **counter-clockwise** along a circular arc (radius =0.5) by an amount  $2\beta l = \pi/2$  radians ( $90^\circ$ ).
- In essence, we are **removing the phase** associated with the TL.



The reflection coefficient at  
the load is:

$$\Gamma_0 = 0.5 * e^{+j150}$$

## Mapping $Z$ to $\Gamma$

- We know that the line impedance and reflection coefficient are **equivalent** – either one can be expressed in terms of the other.

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \iff Z(z) = Z_0 \left( \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)$$

- The above expressions depend on the characteristic impedance  $Z_0$  of the TL. In order to generalize the relationship, we first define a **normalized** impedance value  $z'$  as:

$$z'(z) = \frac{Z(z)}{Z_0} = \frac{R(z)}{Z_0} + j \frac{X(z)}{Z_0} = r(z) + jx(z)$$

therefore

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{(Z(z)/Z_0) - 1}{(Z(z)/Z_0) + 1} = \frac{z'(z) - 1}{z'(z) + 1}$$

$$z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

## Mapping $Z$ to $\Gamma$ (contd.)

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{(Z(z)/Z_0) - 1}{(Z(z)/Z_0) + 1} = \frac{z'(z) - 1}{z'(z) + 1}$$

$$z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

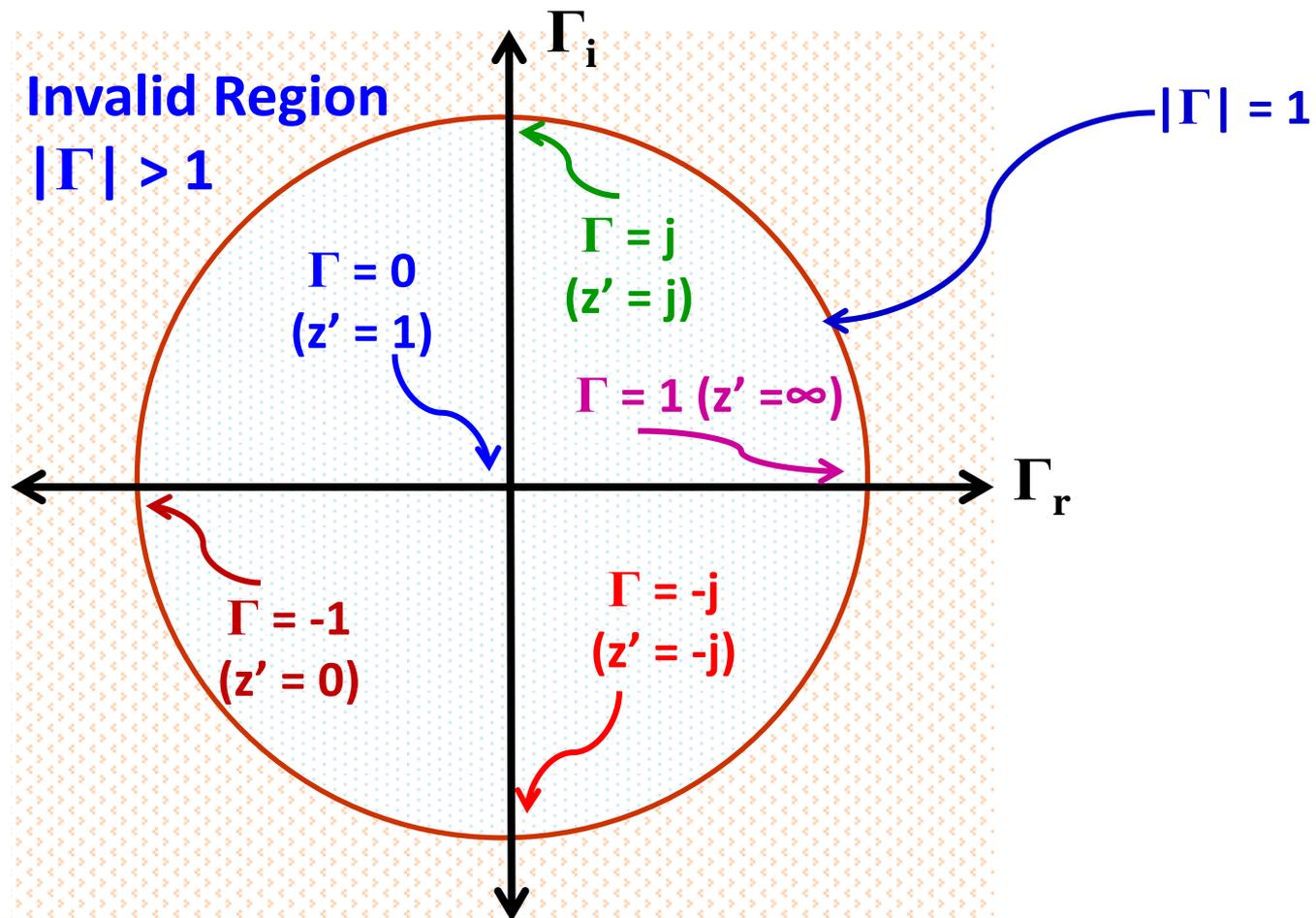
These equations describe a **mapping** between  $z'$  and  $\Gamma$ . That means that each and **every normalized impedance** value likewise corresponds to **one specific point** on the complex  $\Gamma$ -plane

- For example, we wish to indicate the values of some common normalized impedances (shown below) on the complex  $\Gamma$ -plane and vice-versa.

Case	$Z$	$z'$	$\Gamma$
1	$\infty$	$\infty$	1
2	0	0	-1
3	$Z_0$	1	0
4	$jZ_0$	$j$	$j$
5	$-jZ_0$	$-j$	$-j$

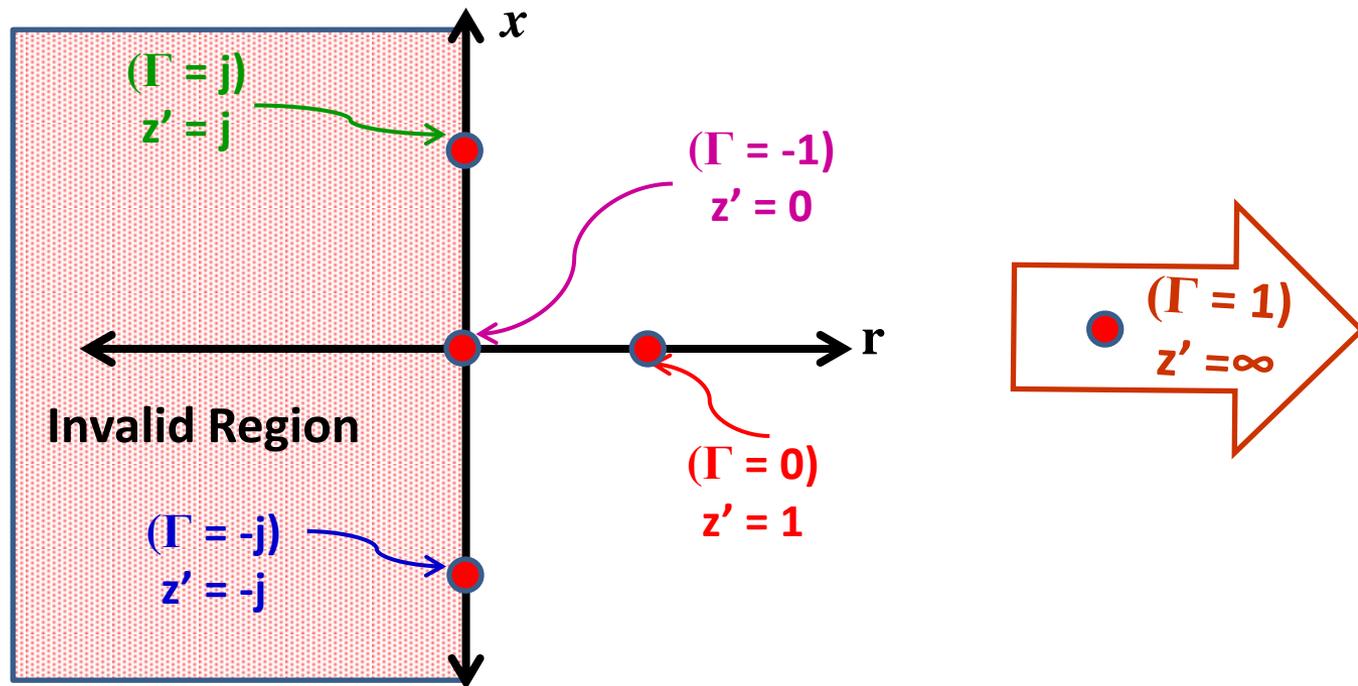
## Mapping $Z$ to $\Gamma$ (contd.)

- The five normalized impedances map five specific points on the complex  $\Gamma$ -plane.



## Mapping $Z$ to $\Gamma$ (contd.)

- The five complex- $\Gamma$  map onto five points on the normalized  $Z$ -plane

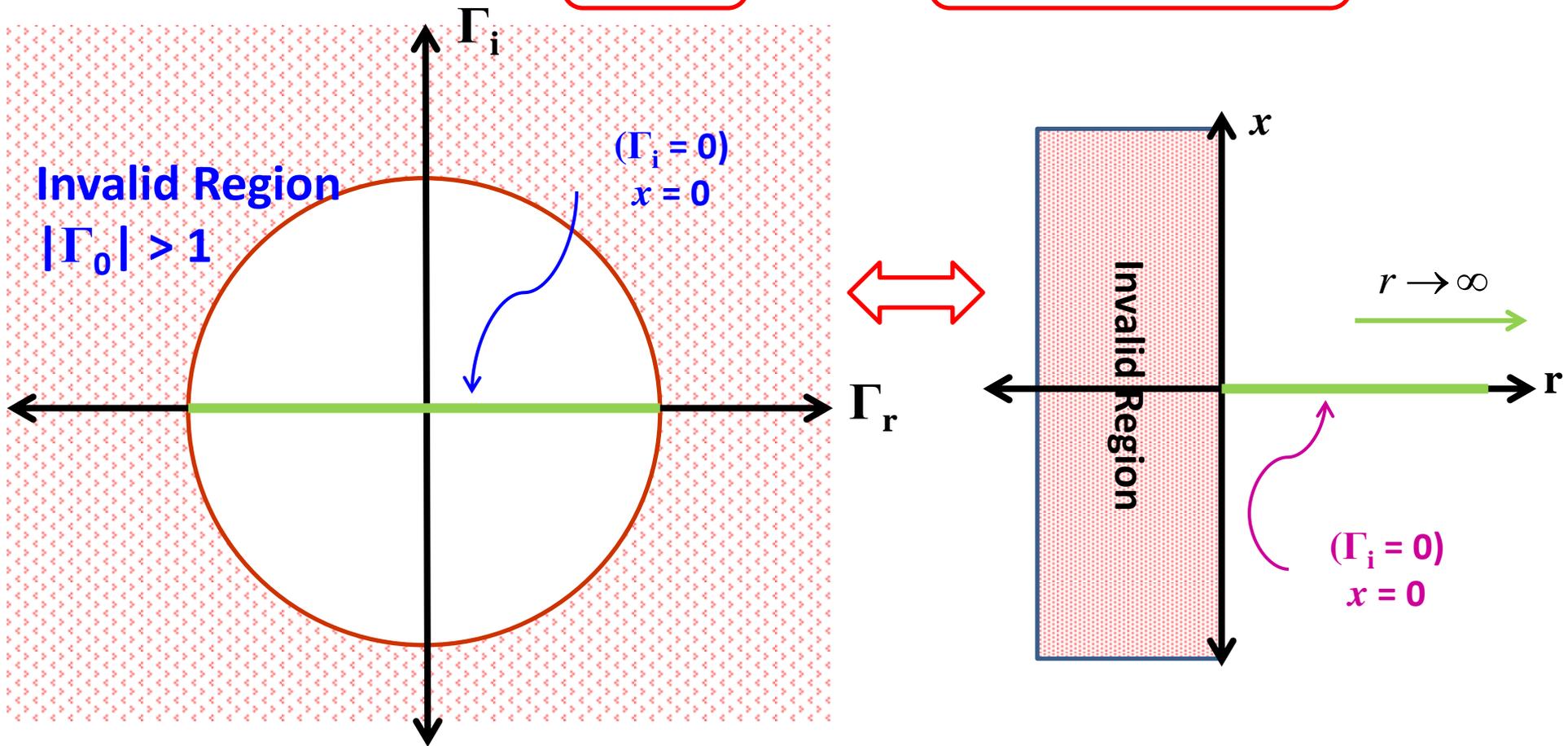


- It is apparent that the normalized impedances can be mapped on complex  $\Gamma$ -plane and vice versa
- It gives us a clue that whole impedance contours (i.e, set of points) can be mapped to complex  $\Gamma$ -plane

## Mapping $Z$ to $\Gamma$ (contd.)

Case-I:  $Z = R \rightarrow$  impedance is purely real

$$z' = r + j0 \quad \rightarrow \quad \Gamma = \frac{r-1}{r+1} \quad \rightarrow \quad \Gamma_r = \frac{r-1}{r+1} \quad \Gamma_i = 0$$



## Mapping $Z$ to $\Gamma$ (contd.)

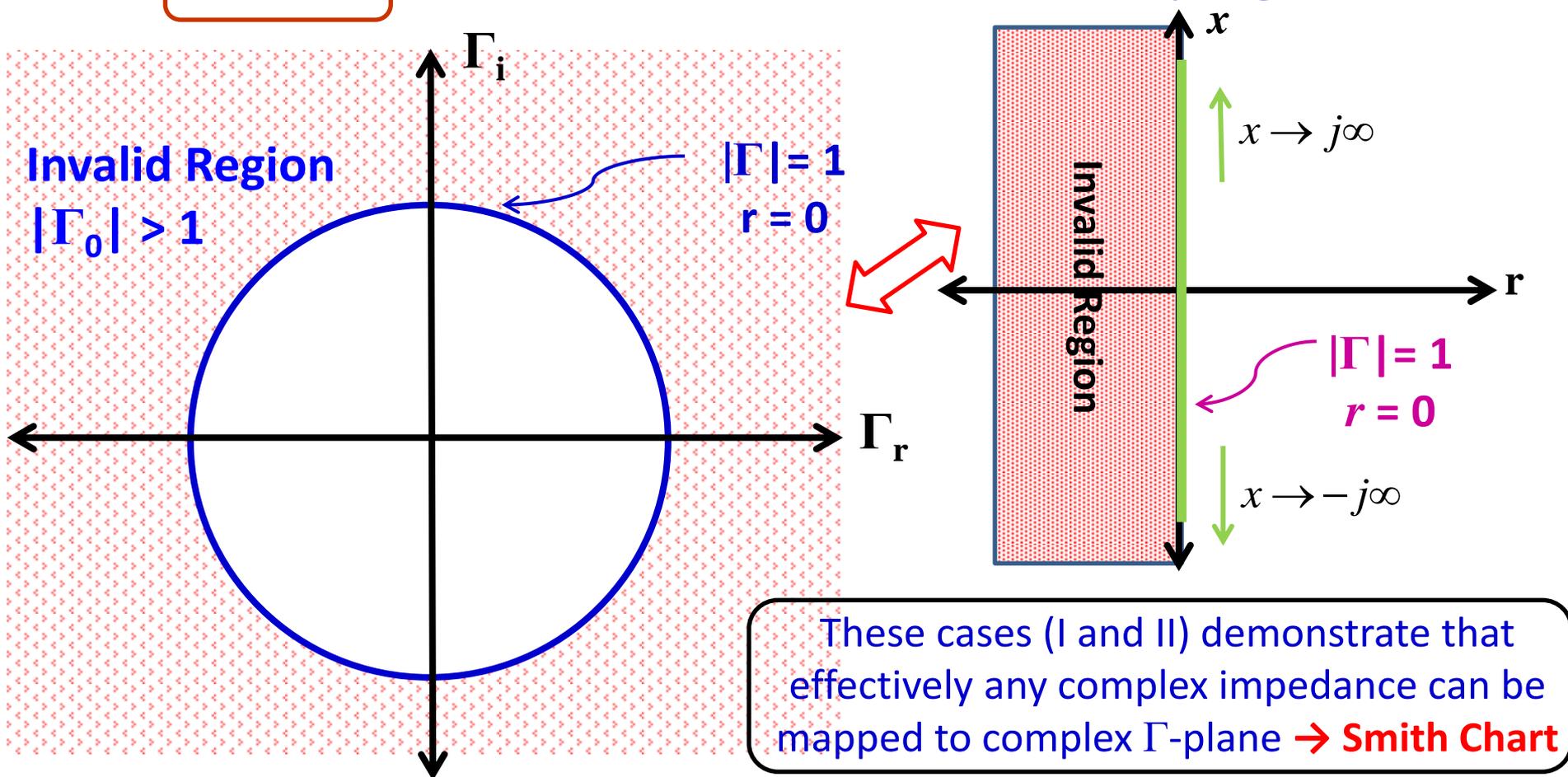
Case-II:  $Z = jX \rightarrow$  impedance is purely imaginary

$$z' = 0 + jx$$



Purely reactive impedance results in a reflection coefficient with unity magnitude

$$|\Gamma| = 1$$



These cases (I and II) demonstrate that effectively any complex impedance can be mapped to complex  $\Gamma$ -plane  $\rightarrow$  **Smith Chart**

# The Smith Chart

## In summary

- A vertical line  $r = 0$  on complex Z-plane maps to a circle  $|\Gamma| = 1$  on the complex  $\Gamma$ -plane
- A horizontal line  $x = 0$  on complex Z-plane maps to the line  $\Gamma_i = 0$  on the complex  $\Gamma$ -plane



Very fascinating in an academic sense, but are not relevant considering that actual values of impedance generally have both a real and imaginary component

Mappings of more general impedance contours (e.g,  $r = 0.5$  and  $x = -1.5$  corresponding to normalized impedance  $0.5 - j1.5$ ) can also be mapped

Smith Chart

## The Smith Chart (contd.)

- Let us revisit the generalized reflection coefficient formulation:

$$\Gamma(z) = |\Gamma_0| e^{j\theta_0} e^{j2\beta z} = \Gamma_r + j\Gamma_i$$

- Therefore, the normalized impedance can be formulated as:

$$z'(z) = r + jx = \frac{Z(z)}{Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i}$$

$$\Rightarrow ((1 - \Gamma_r) - j\Gamma_i)(r + jx) = (1 + \Gamma_r) + j\Gamma_i$$

- The separation of real and imaginary part results in:

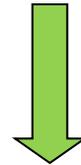
$$r(1 - \Gamma_r) + x\Gamma_i = (1 + \Gamma_r) \quad \leftarrow \text{Real}$$

$$x(1 - \Gamma_r) - r\Gamma_i = \Gamma_i \quad \leftarrow \text{Imaginary}$$

## The Smith Chart (contd.)

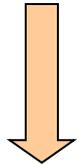
- Simplification and then elimination of **reactance ( $x$ )** from these two give:

$$(1 - \Gamma_r)r + \Gamma_i \left[ \left( \frac{\Gamma_i}{1 - \Gamma_r} \right) (1 + r) \right] = 1 + \Gamma_r$$

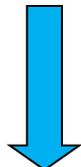


Multiplying through by  $1 - \Gamma_r$

$$(1 - \Gamma_r)^2 r + \Gamma_i^2 (1 + r) = (1 + \Gamma_r)(1 - \Gamma_r)$$

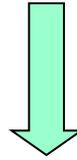


$$(1 - \Gamma_r)^2 r + \Gamma_i^2 (1 + r) = 1 - \Gamma_r^2$$

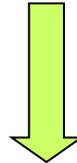


$$\Gamma_r^2 (1 + r) - 2\Gamma_r r + (r - 1) + \Gamma_i^2 (1 + r) = 0$$

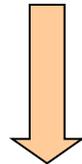
## The Smith Chart (contd.)



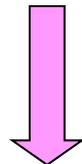
$$\Gamma_r^2(1+r) - 2\Gamma_r r + \Gamma_i^2(1+r) = 1-r$$



$$\Gamma_r^2 - 2\Gamma_r \left( \frac{r}{1+r} \right) + \Gamma_i^2 = \frac{1-r}{1+r}$$



$$\left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{1-r}{1+r} + \left( \frac{r}{1+r} \right)^2$$



$$\left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{(1+r)(1-r) + (r)^2}{(1+r)^2}$$

## The Smith Chart (contd.)

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}$$

Similar equation to circle of radius  $l$ ,  
centered at  $(p, q)$ :

$$(\Gamma_r - p)^2 + (\Gamma_i - q)^2 = l^2$$

This is equation of a **circle**

center:  $(p, q) = \left(\frac{r}{1+r}, 0\right)$  **and** radius:  $l = \frac{1}{1+r}$

### Observations:

- For  $r=0$ :  $p^2 + q^2 = 1$ ;  $(p, q) = (0, 0)$  and  $l = 1$
- For  $r=1/2$ :  $(p - 1/3)^2 + q^2 = (2/3)^2$ ;  $(p, q) = (1/3, 0)$  and  $l = 2/3$
- For  $r=1$ :  $(p - 1/2)^2 + q^2 = (1/2)^2$ ;  $(p, q) = (1/2, 0)$  and  $l = 1/2$
- For  $r=3$ :  $(p - 3/4)^2 + q^2 = (1/4)^2$ ;  $(p, q) = (3/4, 0)$  and  $l = 1/4$

Circles of  
distinct  
centre and  
radii

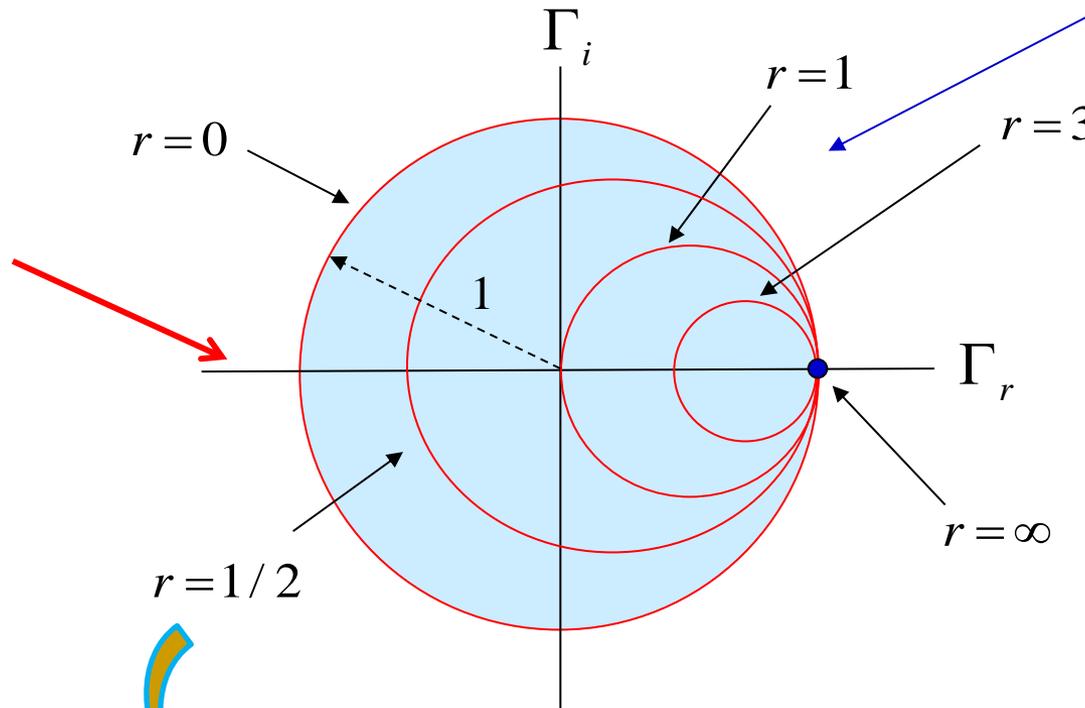
## The Smith Chart (contd.)

Therefore the resistance circles on the complex  $\Gamma$ -plane are:

**Note:**

$$p + l = 1$$

Because of  
 $(q - 0)^2$   
term, all the  
constant  
resistance ( $r$ )  
circles have  
centers on  
this line



This approach enables mapping of any  
realizable vertical line (representing  $r$ ) in the  
complex  $\Gamma$ -plane

## The Smith Chart (contd.)

- For the mapping of horizontal lines of the normalized impedance plane to  $\Gamma$ -plane, let us simplify and eliminate **resistance ( $r$ )** from the following:

$$r(1 - \Gamma_r) + x\Gamma_i = (1 + \Gamma_r) \quad \leftarrow \text{Real}$$

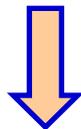
$$x(1 - \Gamma_r) - r\Gamma_i = \Gamma_i \quad \leftarrow \text{Imaginary}$$



$$(1 - \Gamma_r) \left[ \frac{(1 - \Gamma_r)x - \Gamma_i}{\Gamma_i} \right] + x\Gamma_i = 1 + \Gamma_r$$



$$(1 - \Gamma_r)^2 x - \Gamma_i(1 - \Gamma_r) + x\Gamma_i^2 - \Gamma_i(1 + \Gamma_r) = 0$$



$$(1 - \Gamma_r)^2 x - 2\Gamma_i + x\Gamma_i^2 = 0$$

## The Smith Chart (contd.)

$$(\Gamma_r - 1)^2 - \left(\frac{2}{x}\right)\Gamma_i + \Gamma_i^2 = 0$$

center:  $(p, q) = (1, 1/x)$

radius:  $l = \frac{1}{|x|}$

**Note:**

$$q = \pm l$$

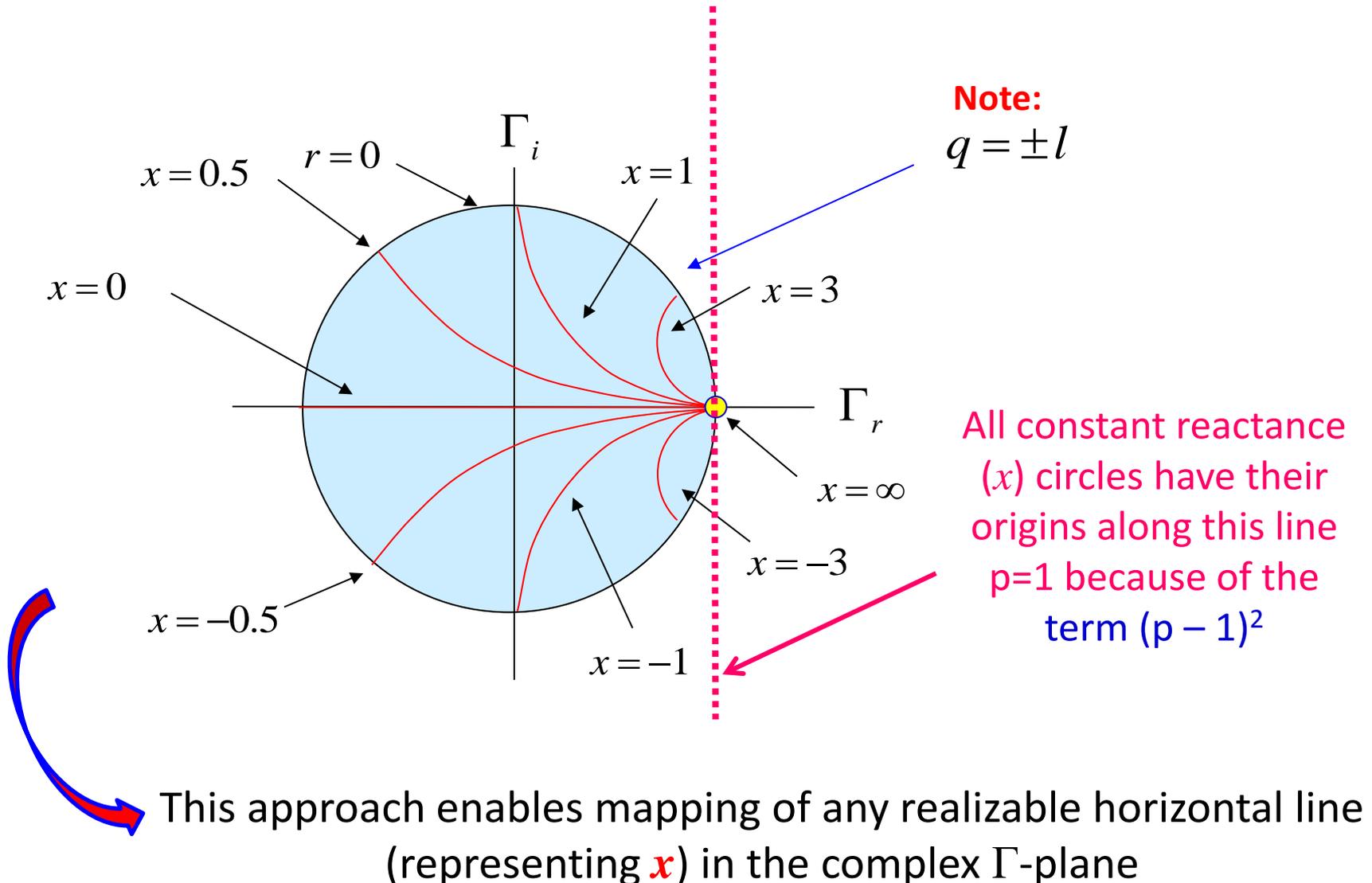
$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

### Observations:

- **For  $x = 1$ :**  $(p - 1)^2 + (q - 1)^2 = (1)^2$ ;  $(p, q) = (1, 1)$  and  $l = 1$
- **For  $x = -1$ :**  $(p - 1)^2 + (q + 1)^2 = (1)^2$ ;  $(p, q) = (1, -1)$  and  $l = 1$
- **For  $x = 1/2$ :**  $(p - 1)^2 + (q - 2)^2 = (2)^2$ ;  $(p, q) = (1, 2)$  and  $l = 2$
- **For  $x = -1/2$ :**  $(p - 1)^2 + (q + 2)^2 = (2)^2$ ;  $(p, q) = (1, -2)$  and  $l = 2$

Circles of  
distinct  
centre and  
radii

## The Smith Chart (contd.)



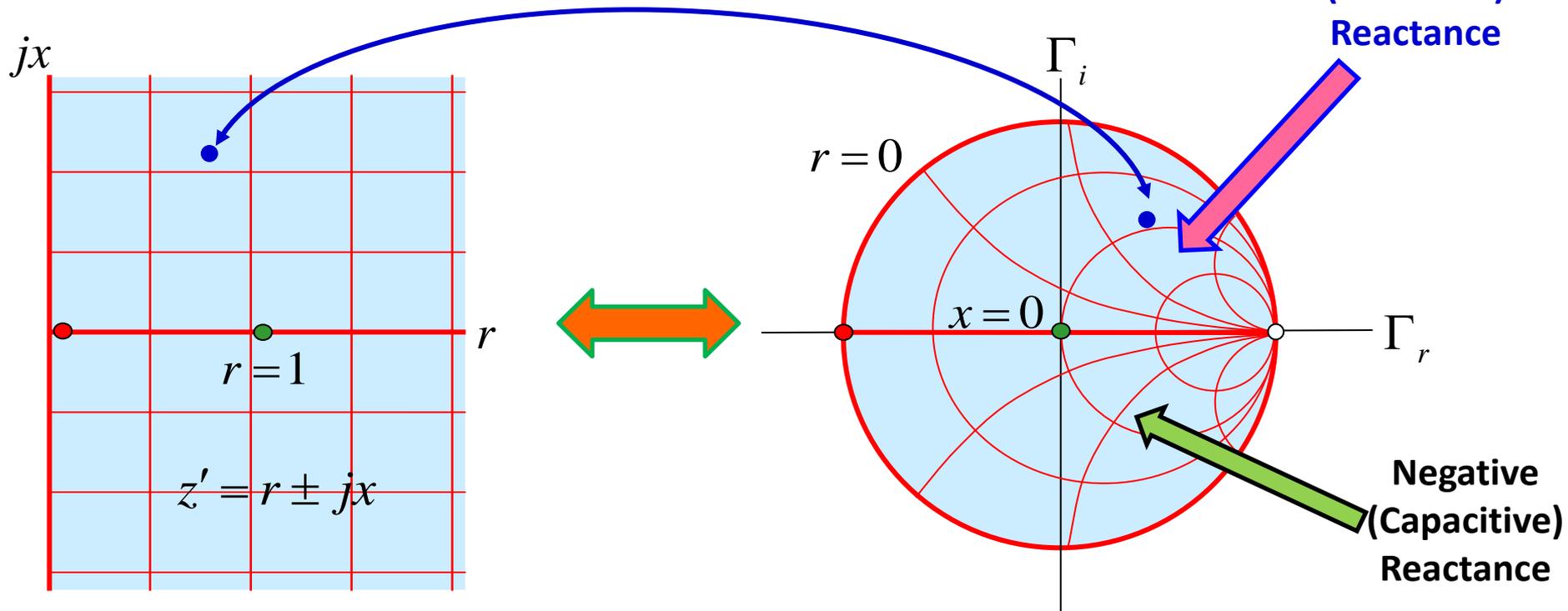
## The Smith Chart (contd.)

- Combination of these **constant resistance** and **reactance circles** define the mappings from **normalized impedance ( $z'$ ) plane** to  $\Gamma$ -plane and is called as Smith chart.

$$z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$



$$\Gamma(z) = \frac{z'(z) - 1}{z'(z) + 1}$$



## The Smith Chart (contd.) – Important Features

1. By definition:

$$\Gamma(z) = \frac{z'(z) - 1}{z'(z) + 1} = \frac{r + jx - 1}{r + jx + 1}$$



$$|\Gamma(z)| = \frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}$$

- It is apparent: for  $r \geq 0$ , we get  $|\Gamma(z)| \leq 1$ . This condition is easily met for passive networks (i.e, no amplifiers) and lossless TLs (real  $Z_0$ )
- Consequently, the standard Smith chart only shows only the inside of the unit circle in the  $\Gamma$ -plane. That is,  $|\Gamma(z)| \leq 1$  which is bounded by the  $r = 0$  circle described by:

$$\Gamma_r^2 + \Gamma_i^2 = 1$$

## The Smith Chart (contd.) – Important Features

2. Notice that in the upper semi-circle of the Smith chart,  $x \geq 0$  which is an **inductive reactance**. Consequently, the generalized reflection coefficients  $\Gamma(z) \equiv \Gamma_r + j\Gamma_i$  in the upper semi-circle are associated with normalized TL impedances  $z'(z) \equiv r + jx$  that are inductively reactive.

**Conversely**, the lower semi-circle of the Smith chart represent capacitive reactive impedances

3. If  $z'(z)$  is purely real (ie,  $x = 0$ ) then the reactance term:

$$(1 - \Gamma_r)^2 x - 2\Gamma_i + x\Gamma_i^2 = 0 \quad \xrightarrow{\text{suggests}} \quad \Gamma_i = 0 \text{ except possibly at } \Gamma_r = 1$$

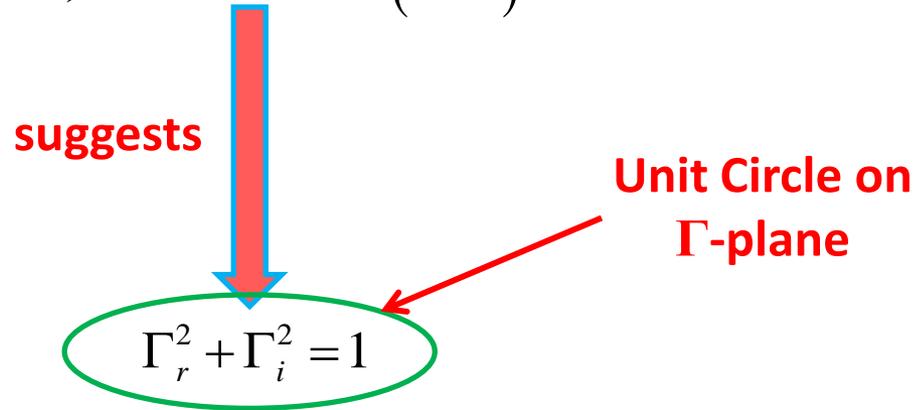
Consequently, purely real  $z'(z)$  values are mapped to  $\Gamma(z)$  values on the  $\Gamma_r$  axis.

## The Smith Chart (contd.) – Important Features

4. If  $z'(z)$  is purely imaginary (ie,  $r = 0$ ) then the impedance term:

$$\left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{(1+r)(1-r) + (r)^2}{(1+r)^2}$$

suggests


$$\Gamma_r^2 + \Gamma_i^2 = 1$$

Unit Circle on  
 $\Gamma$ -plane

Consequently, purely imaginary  $z'(z)$  values are mapped to  $\Gamma(z)$  values on the unit circle in  $\Gamma$ -plane.