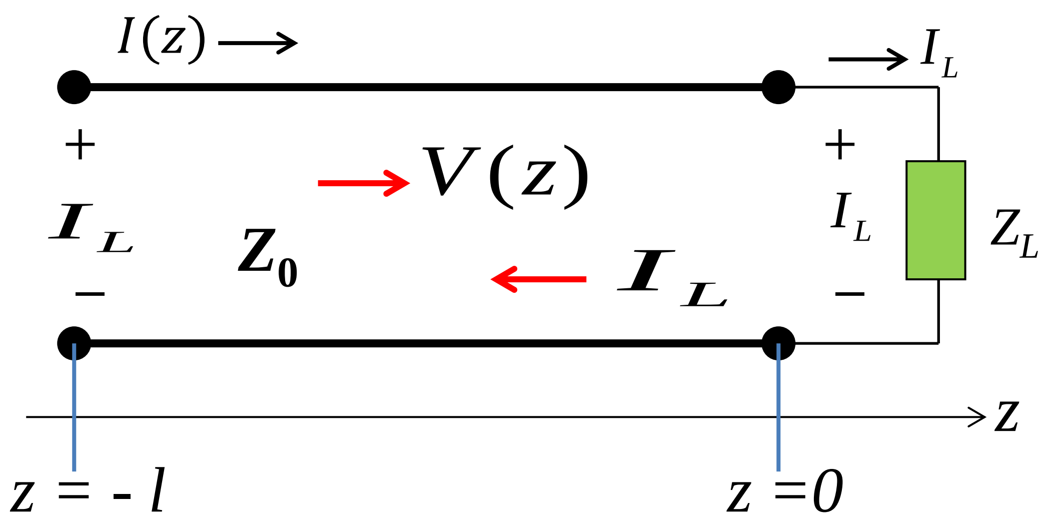
Lecture-4

Date: 14.08.2014

- Review – Lecture 3
- Terminated Lossless Transmission Line
- TL Input Impedance
- Solved Examples

Review – Lecture 3



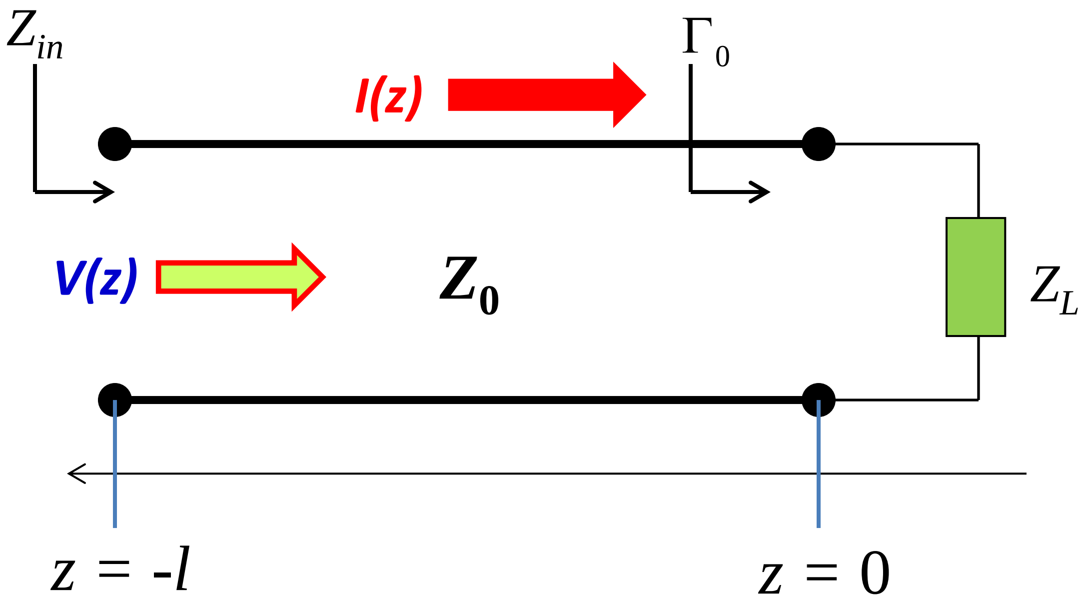
- For a terminated lossless transmission line, the current and voltage along the line is:

$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma_0 e^{+j\beta z}\right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma_0 e^{+j\beta z}\right]$$

Special Termination Conditions

- Let us consider a generic TL terminated in arbitrary impedance Z_L



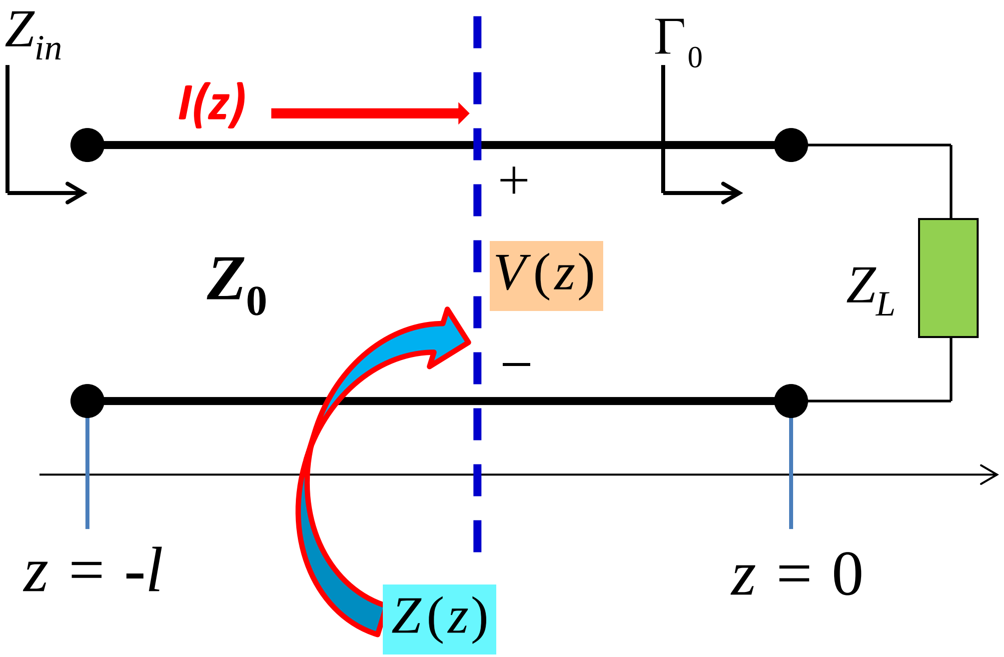
It's interesting to note that the load $\mathbf{Z_L}$ enforces a boundary condition that explicitly determines neither **V(z)** nor **I(z)**—but **completely** specifies **line impedance** Z(z)!

Special Termination Conditions (contd.)

- We define the generalized impedance at any point on the line as:

$$Z(z) = \frac{V(z)}{I(z)}$$

This is the impedance we would measure if we cut the line at *z* and measured its impedance there.



$$Z(z) = \frac{V(z)}{I(z)} = \frac{V^{+}e^{+j\beta z}\left(1+\Gamma_0 e^{-j(2\beta z)}\right)}{\frac{V^{+}e^{+j\beta z}}{Z_0}\left(1-\Gamma_0 e^{-j(2\beta z)}\right)}$$

Special Termination Conditions (contd.)

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_0 e^{+j\beta z}}{e^{-j\beta z} - \Gamma_0 e^{+j\beta z}} = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

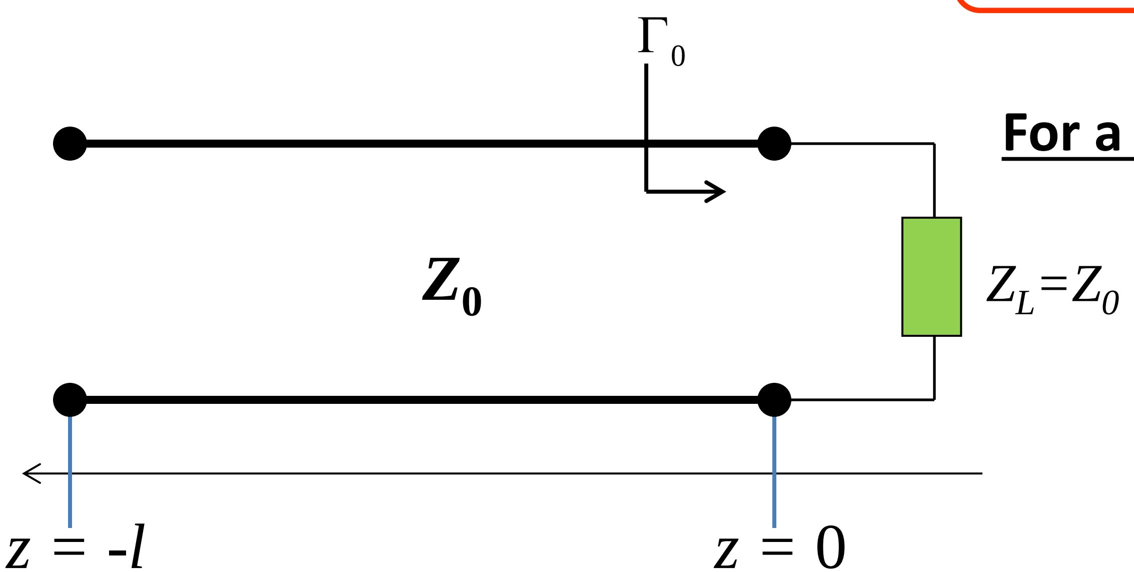
- Likewise, the load boundary condition leaves $V^{+}(z)$ and $V^{-}(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(\boldsymbol{z})$!

$$\Gamma(z) = \frac{V^{-}(z)}{V^{+}(z)} = \Gamma_0 e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L - Z_0} e^{+j2\beta z}$$

Let's look at some **specific** values of load impedance $Z_L = R_L + jX_L$ and see what functions $Z(z)$ and $\Gamma(\boldsymbol{z})$ result!

Special Termination Conditions (contd.)

- $Z_L = Z_0$ ← **Matched Line** → the load impedance equals the characteristic impedance of the TL



For a lossless TL: $R_L = Z_0$, $X_L = 0$ ← Purely Real

The load reflection coefficient:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$

means no reflected wave $V^{-}(z)$

reflection coefficient is zero at all points along the line

The impedance at position z:

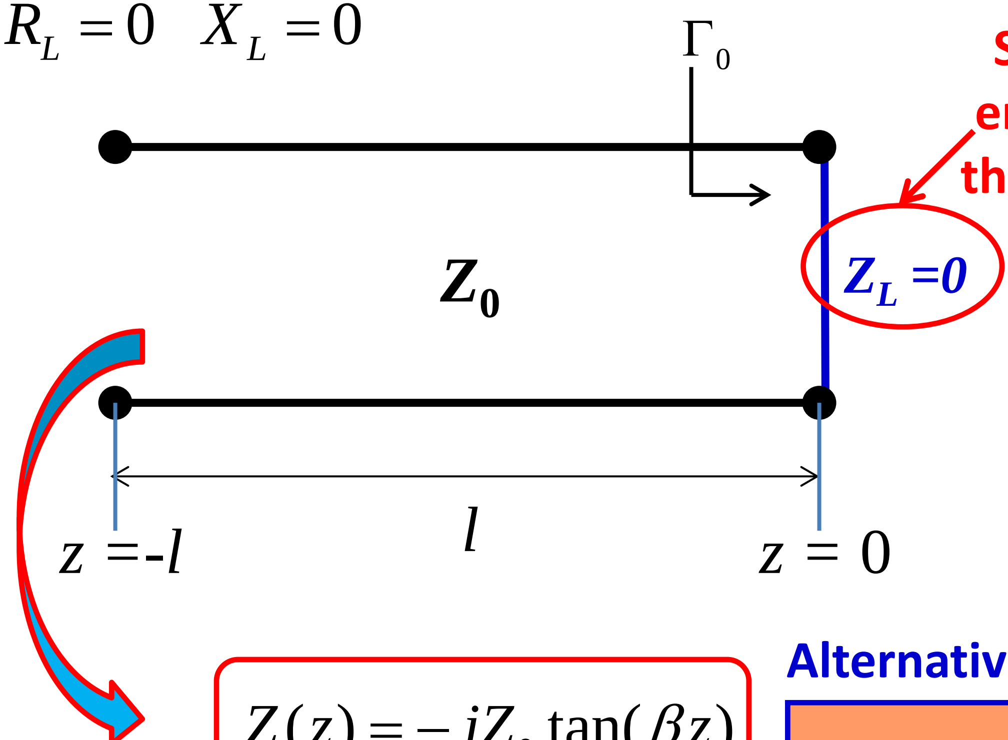
$$Z(z) = Z_0$$

The line impedance equals Z_0 → matched condition

Special Termination Conditions (contd.)

- $Z_L = 0$ ⇐ **Short-Circuited Line** ⇒ A device with no load is called short circuit

$R_L = 0 \quad X_L = 0$



Short-circuit entails setting this impedance to zero

$$\Gamma_0 = \frac{0 - Z_0}{0 + Z_0} = -1$$

$$Z(z) = -jZ_0 \tan(\beta z)$$

Alternatively

$$Z(z) = -jZ_0 \tan\left(\frac{2\pi z}{\lambda}\right)$$

Note that this impedance is **purely reactive**. This means that the current and voltage on the transmission line will be everywhere 90° **out of phase**.

Special Termination Conditions (contd.)

- **Short-Circuited Line**
- The current and voltage along the TL is:

$$V(z) = V_0^+\left[e^{-j\beta z} - e^{+j\beta z}\right] = -j2V_0^+\sin(\beta z)$$

$$I(z) = \frac{2V_0^+}{Z_0}\cos(\beta z)$$

- Finally, the reflection coefficient **function** is:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{-V_0^+e^{+j\beta z}}{V_0^+e^{-j\beta z}} = -e^{j2\beta z}$$

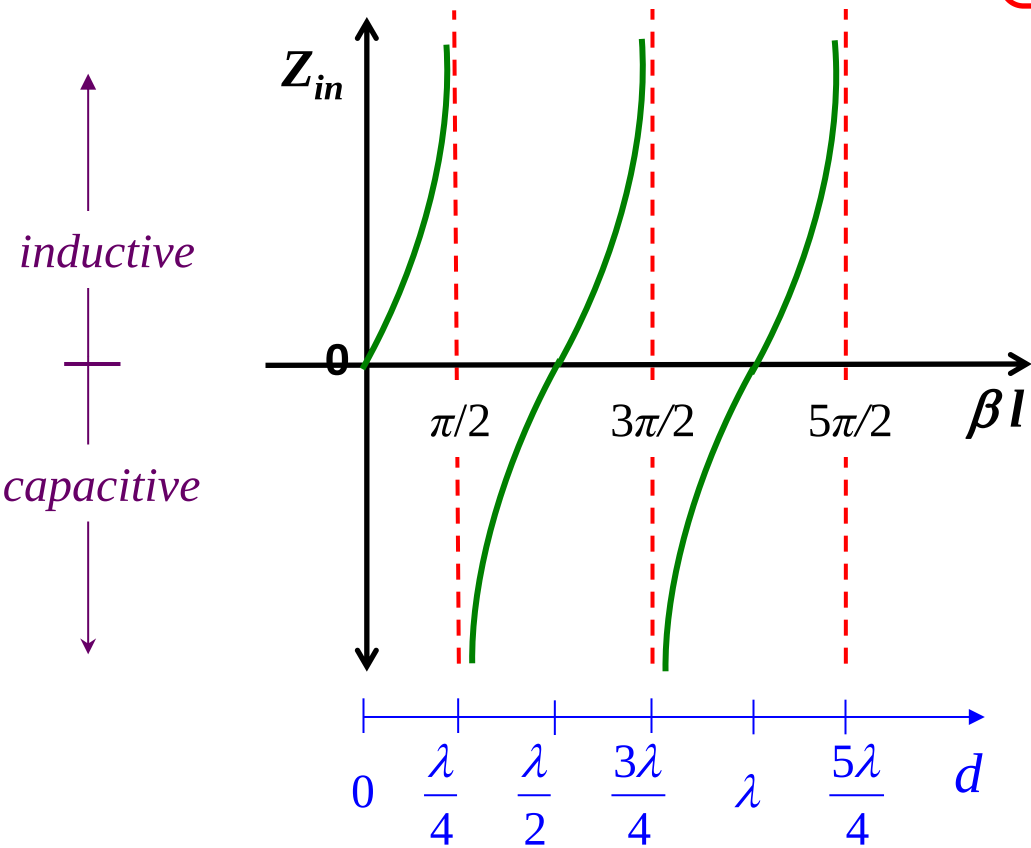
$\Rightarrow$ $|\Gamma(z)| = 1$ $\Rightarrow$ $|V^-(z)| = |V^+(z)|$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

Special Termination Conditions (contd.)

- **Short-Circuited Line**

$$Z(-l) = jZ_0 \tan(\beta l)$$



It can be observed:

- **At -*l*=0,** the impedance is zero (short-circuit condition)
- **Increase in -*l*** leads to inductive behavior
- **At -*l*=λ/4,** the impedance equals infinity (open-circuit condition)
- **Further increase in -*l*** leads to capacitive behavior
- **At -*l*=λ/2,** the impedance becomes zero (short-circuit condition)
- **The entire periodic sequence repeats for -*l*>λ/2 and so on…**

Special Termination Conditions (contd.)

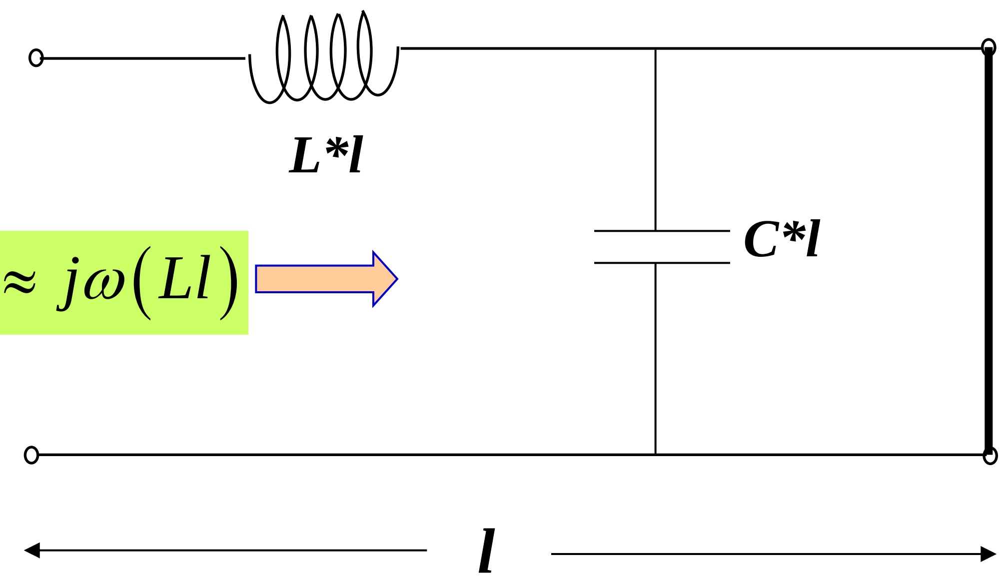
- **Short-Circuited Line**

$$Z(-l) = jZ_0 \tan(\beta l)$$

At low frequency:

$$Z(-l) \approx jZ_0(\beta l) = \sqrt{\frac{L}{C}}\left(\omega\sqrt{LC}\,l\right) = j\omega(Ll)$$

$$Z(-l) \approx j\omega(Ll)$$



Extremely useful result for RF Circuit Design

Example – 1

For a short-circuited TL of $l = 10$ cm, compute the magnitude of the input impedance when the frequency is swept from f = 1 GHz to 4 GHz. Assume the line parameters L = 209.4 nH/m and C = 119.5 pF/m.

Solution:

$$Z_0 = \sqrt{L/C} = \sqrt{(209.4*0.1)/(119.5*0.5)} = 41.86\Omega$$

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(209.4*0.1)*(119.5*0.5)} = 41.86\Omega} = 1.99*10^8 m/s$$

$$Z(z=-l) = jZ_0 \tan(\beta l) = jZ_0 \tan\left(\frac{2\pi f}{v_p} l\right)$$

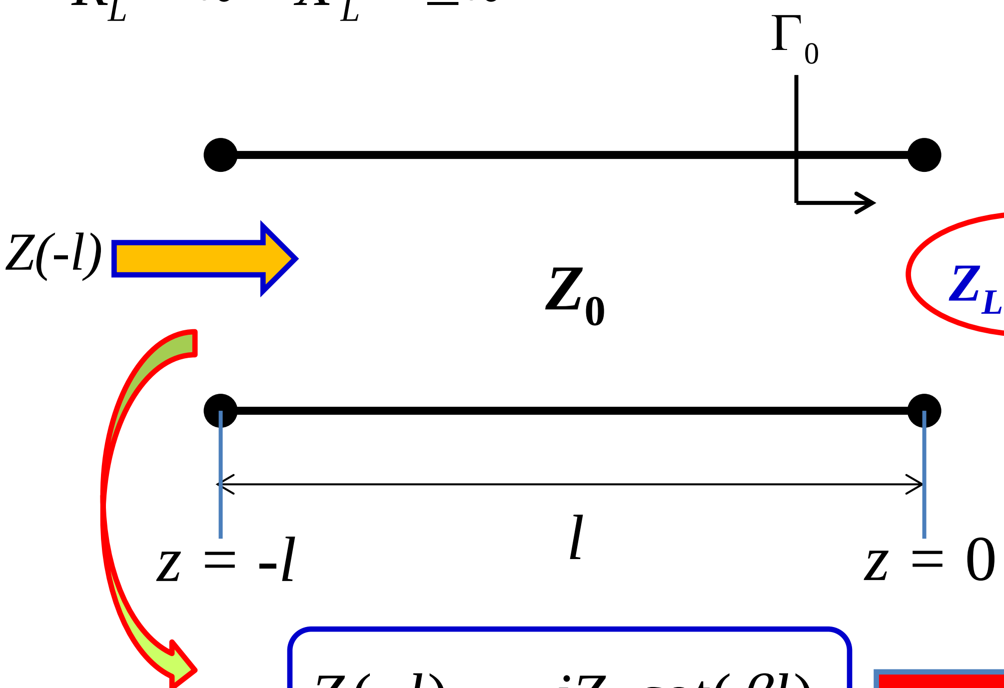
Set $l = 10$ cm and then write a MATLAB program to obtain the Z_{in} curve

Compare the MATLAB results to that obtained from ADS simulation

Special Termination Conditions (contd.)

- $Z_L \rightarrow \infty$ ⟸ **Open-Circuited Line** ⟹ A device with infinite load is called open-circuit

$R_L = \infty \quad X_L = \pm\infty$



Open-circuit entails setting this impedance to infinite: $Z_L \rightarrow \infty$

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

$$Z(-l) = -jZ_0 \cot(\beta l)$$

Alternatively

$$Z(-l) = -jZ_0 \cot\left(\frac{2\pi l}{\lambda}\right)$$

Again note that this impedance is **purely reactive**. current and voltage on the transmission line are 90° **out of phase**.

Special Termination Conditions (contd.)

- **Open-Circuited Line**

- The current and voltage along the TL is:

$$V(z) = V_0^+\left[e^{-j\beta z} + e^{+j\beta z}\right] = 2V_0^+\cos(\beta z)$$

$$I(z) = -j\frac{2V_0^+}{Z_0}\sin(\beta z)$$

- At the load, $z = 0$, therefore: $V(0) = 2V_0^+$ $I(0) = 0$

As expected, the current is zero at the end of the transmission line (i.e. the current through the open). Likewise, the voltage at the end of the line (i.e., the voltage across the open) is at a maximum!

- Finally, the reflection coefficient **function** is:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{j2\beta z}$$

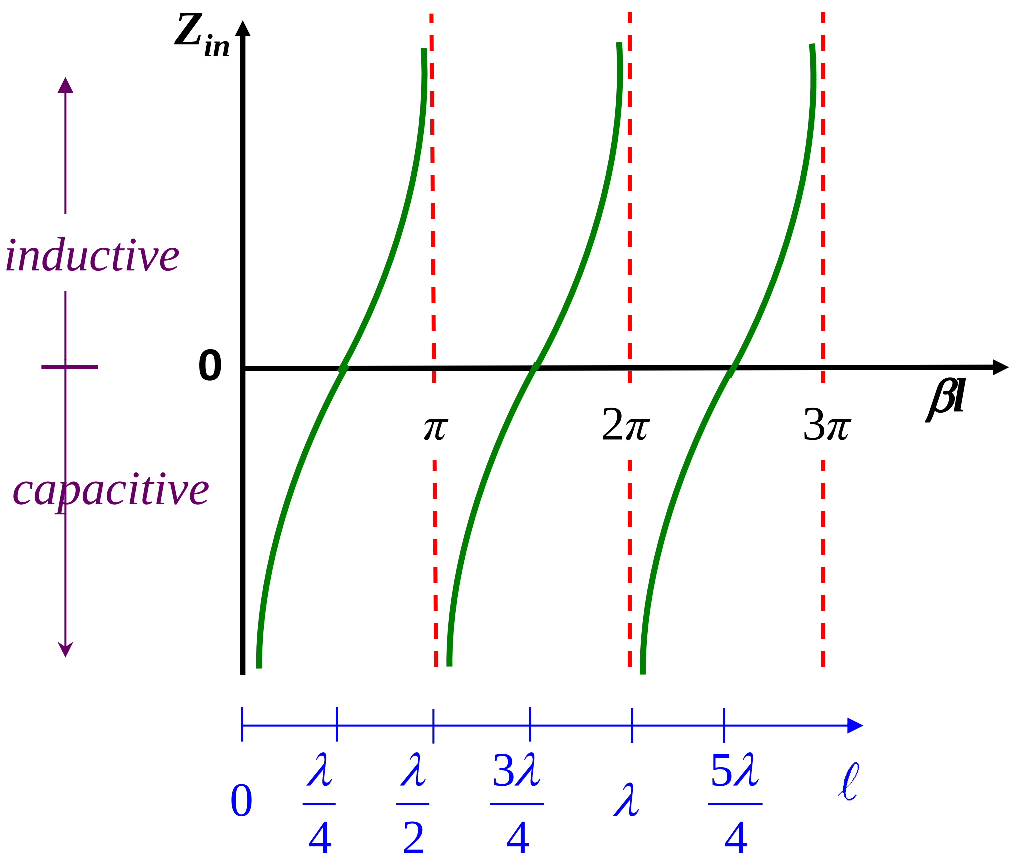
→ $|\Gamma(z)| = 1$ → $\left|V^-(z)\right| = \left|V^+(z)\right|$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

Special Termination Conditions (contd.)

- **Open-Circuited Line**

$$Z(-l) = -jZ_0 \cot(\beta l)$$



It can be observed:

- **At *l*=0,** the impedance is infinite (open-circuit condition)
- **Increase in -*l*** leads to capacitive behavior
- **At -*l* = λ/4,** the impedance equals zero (short-circuit condition)
- **Further increase in -*l*** leads to inductive behavior
- **At -*l*=λ/2,** the impedance becomes infinite (open-circuit condition)
- **The entire periodic sequence repeats for -*l* >λ/2 and so on…**

Special Termination Conditions (contd.)

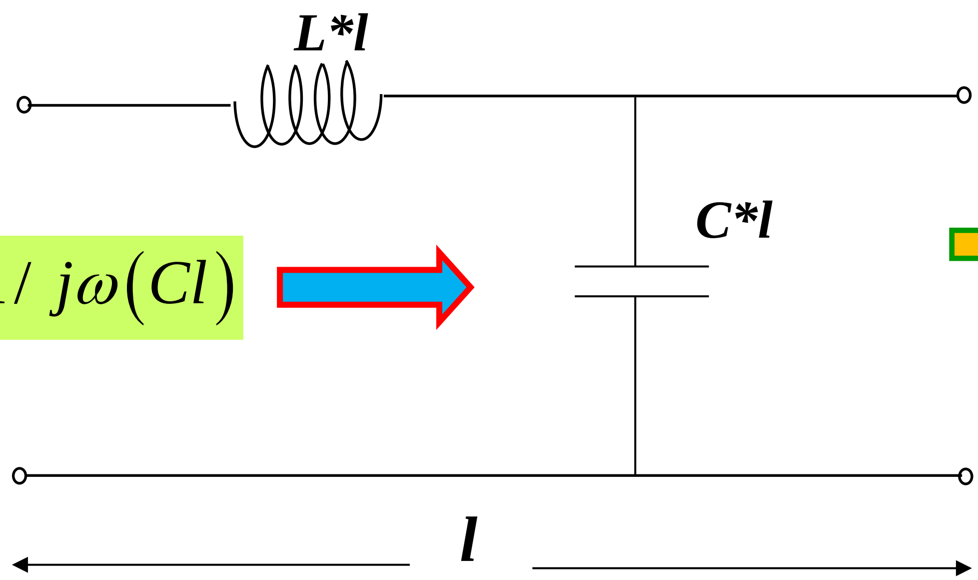
- **Open-Circuited Line**

$$Z(-l) = -jZ_0 \cot(\beta l)$$

At low frequency:

$$Z(-l) \approx -jZ_0/(\beta l) = -j\sqrt{\frac{L}{C}}\left(1/\omega\sqrt{LC}\,l\right) = \frac{1}{j\omega(Cl)}$$

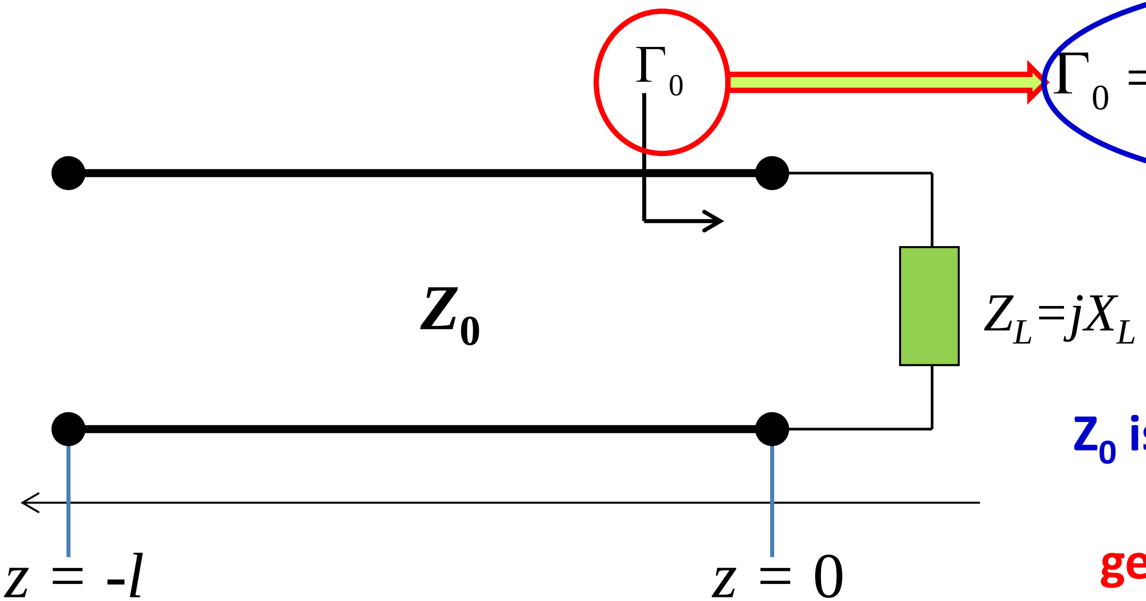
$$Z(-l) \approx 1/j\omega(Cl)$$



Extremely useful result for RF Circuit Design

Special Termination Conditions (contd.)

- Load impedance is purely reactive: $Z_L = jX_L$ ⟸ $R_L = 0$



$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Z_0 is real for a lossless TL, therefore this reflection coefficient is generally some complex number

$|\Gamma_0|^2 = 1$ ⟹ magnitude is 1 and therefore we can write

$$\Gamma_0 = e^{j\theta_\Gamma}$$

Where, $\theta_\Gamma = \tan^{-1}\left(\frac{2Z_0 X_L}{X_L^2 - Z_0^2}\right)$

Special Termination Conditions (contd.)

- **Load impedance is purely reactive:** $Z_L = jX_L$ ⟵ $R_L = 0$
- We can write voltage and current as:

$$V(z) = V_0^+\left(e^{-j\beta z} + e^{+j\theta_\Gamma}e^{+j\beta z}\right) \Rightarrow V(z) = 2V_0^+ e^{+j\frac{\theta_\Gamma}{2}} \cos\left(\beta z + \frac{\theta_\Gamma}{2}\right)$$

$$I(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - e^{+j\theta_\Gamma}e^{+j\beta z}\right) \Rightarrow I(z) = j\frac{2V_0^+}{Z_0} e^{+j\frac{\theta_\Gamma}{2}} \sin\left(\beta z + \frac{\theta_\Gamma}{2}\right)$$

- Therefore the line impedance is:

$$Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot\left(\beta z + \frac{\theta_\Gamma}{2}\right)$$

Purely Reactive

Again note that this impedance is **purely reactive**. current and voltage on the transmission line are 90° **out of phase**.

Special Termination Conditions (contd.)

- **Load impedance is purely reactive:** $Z_L = jX_L$ ← $R_L = 0$

- At the load end of the line: $Z(z=0) = Z_L = \dfrac{V(z=0)}{I(z=0)} = jZ_0 \cot\left(\dfrac{\theta_\Gamma}{2}\right)$

→ with a little trigonometry, we can show (trust me!) that: $\cot\left(\dfrac{\theta_\Gamma}{2}\right) = \dfrac{X_L}{Z_0}$

$$\therefore Z(z=0) = Z_L = jZ_0 \cot\left(\frac{\theta_\Gamma}{2}\right) = jX_L$$

← **Expected!**

- The reflection coefficient is: $\Gamma(z) = \dfrac{V^-(z)}{V^+(z)} = e^{+j2\left(\frac{\theta_\Gamma}{2} + \beta z\right)}$ ⇒ $|\Gamma(z)| = 1$ → $|V^-(z)| = |V^+(z)|$

The magnitude of forward and backward waves on TL is same → a reactive load leads to results very similar to that of an open or short circuit

Special Termination Conditions (contd.)

- **Load impedance is purely resistive:** $Z_L = R_L \neq 50\Omega$ ⟵ $X_L = 0$

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R_L - Z_0}{R_L + Z_0}$$

⟵ A real value considering that Z_0 is real valued.

The current, voltage, and thus the line impedance are complex in this case and expressions can't be simplified any further.

Q: Why is that? When the load was purely **imaginary** (reactive), we were able to **simplify** our general expressions, and likewise deduce all sorts of interesting results!

Special Termination Conditions (contd.)

- Load impedance is purely resistive: $Z_L = R_L \neq 50\Omega$ ⟵ $X_L = 0$

A: True! And here's **why**. Remember, a **lossless** transmission line has series inductance and shunt capacitance **only**. In other words, a length of lossless transmission line is a **purely reactive** device (it absorbs **no** energy!).

- If we attach a **purely reactive** load at the end of the transmission line, we still have a **completely** reactive system (load and transmission line). Because this system has **no** resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly **simplified**.
- However, if we attach a **purely real** load to our reactive transmission line, we now have a **complex** system, with **both** real and imaginary (i.e., resistive and reactive) components.

This **complex** case is exactly what our general expressions **already** describes—**no** further simplification is possible!

Special Termination Conditions (contd.)

- **Load impedance is complex (the general case):** $Z_L = R_L + jX_L$

Q: Haven't we **already** determined all the **general** expressions (e.g., $\Gamma_0, I(z), V(z), and\ \Gamma(z)$ for this general case? Is there **anything** else left to be determined?

A: There is **one** last thing we need to discuss. It seems trivial, but its ramifications are **very** important!

- For you see, the "general" case is **not**, in reality, quite so general. Although the reactive component of the load can be **either** positive or negative ($-\infty < X_L < \infty$), the resistive component of a passive load **must** be positive ($R_L > 0$)—there's **no** such thing as a (passive) **negative** resistor!
- This leads to one **very** important and useful result.
- You can find out from expression of reflection coefficient that **conservation of energy** is satisfied—the reflected wave from a passive load **cannot** be larger than the wave incident on it.

Transmission Line Input Impedance

Q: Just what do you mean by **input** impedance?

A: The input impedance is simply the line impedance seen at the **beginning** $(\boldsymbol{z} = -\boldsymbol{l})$ of the transmission line, i.e.:

$$Z_{in} = Z(z = -l) = \frac{V(z = -l)}{I(z = -l)}$$

Note Z_{in} equal to **neither** the load impedance Z_L **nor** the characteristic impedance Z_0!

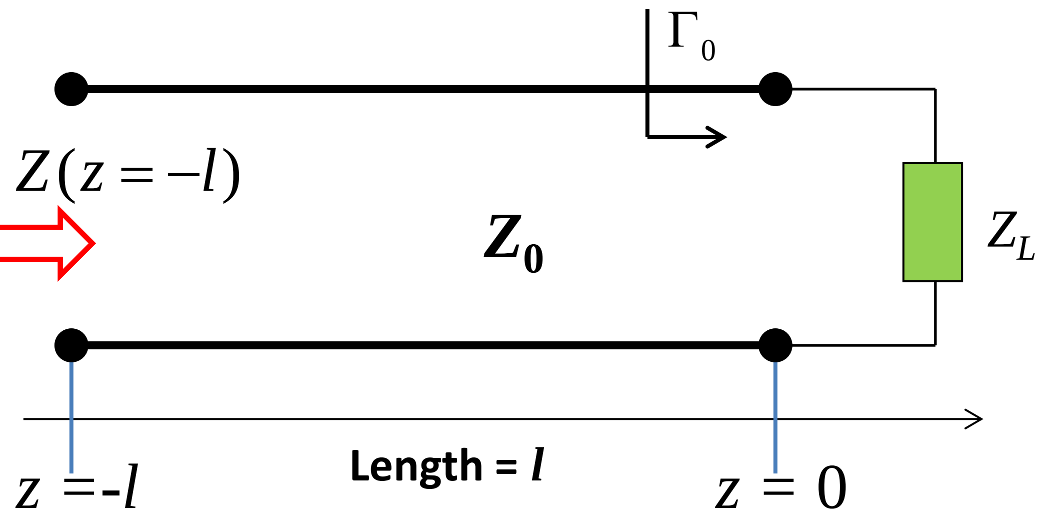
$$Z_{in} \neq Z_L \qquad\qquad Z_{in} \neq Z_0$$

Transmission Line Input Impedance (contd.)

- We know the line impedance of a lossless TL loaded with an arbitrary load impedance is:

$$Z(z) = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

$$Z_{in} = Z(z = -l)$$



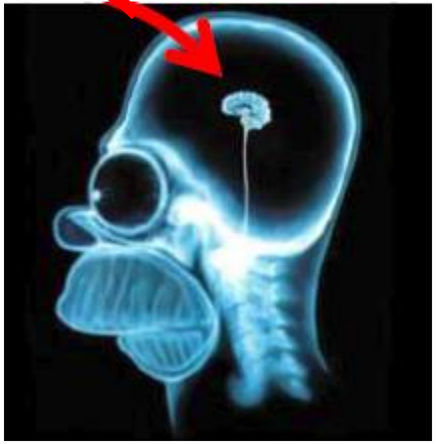
$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

This input impedance can be radically different from load impedance (Z_L) → definitely regulated by β, Z_0 and length of the line (l)

Transmission Line Input Impedance – Special Cases

- Now let us look at the input impedances for some important load impedances and line lengths

→ You should commit these results to memory

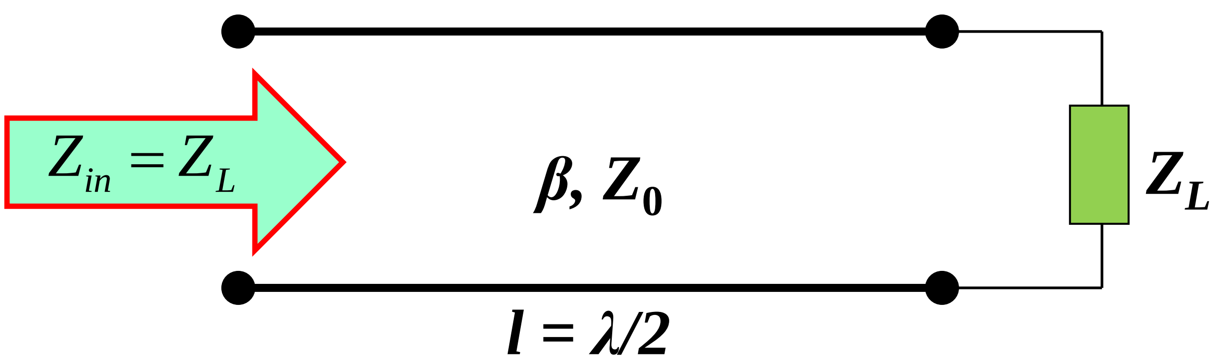


Transmission Line Input Impedance – Special Cases

1. length of the line is $l = m(\lambda/2)$

$$Z_{in} = Z(z = \lambda/2) = Z_0 \frac{Z_L + jZ_0 \tan\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{2}\right)}{Z_0 + jZ_L \tan\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{2}\right)} = Z_L$$

- For a transmission line of half wavelength long the input impedance equals the load impedance irrespective of the characteristic impedance of the line
- It means it is possible to design a circuit segment where the transmission line's characteristic impedance plays no role (obviously the length of line segment has to equal half wavelength at the operating frequency)

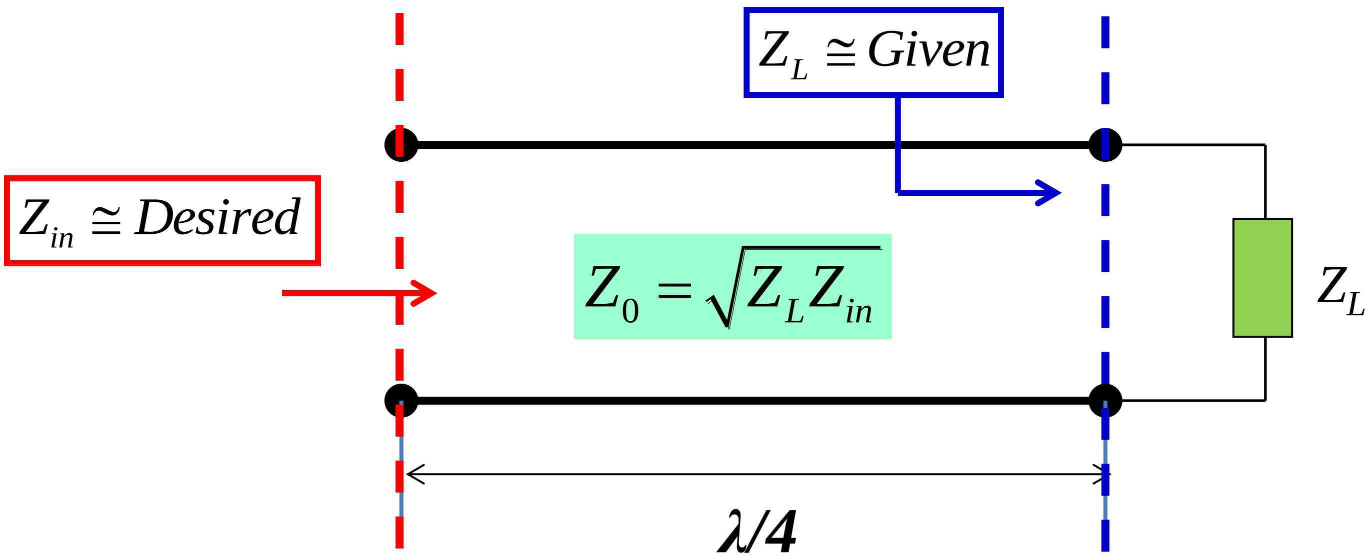


Transmission Line Input Impedance – Special Cases (contd.)

2. length of the line is $l = \lambda/4$ or $\lambda/4 + m(\lambda/2)$

$$Z_{in} = Z(l=\lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan\left(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\right)}{Z_0 + jZ_L \tan\left(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\right)} = \frac{Z_0^2}{Z_L}$$

- This result implies that a transmission line segment can be used to synthesize matching of any desired real input impedance (Z_{in}) to the specified real load impedance (Z_L)



This is known as quarter-wave impedance transformer

Transmission Line Input Impedance – Special Cases (contd.)

2. length of the line is $l = \lambda/4$ or $\lambda/4 + m(\lambda/2)$

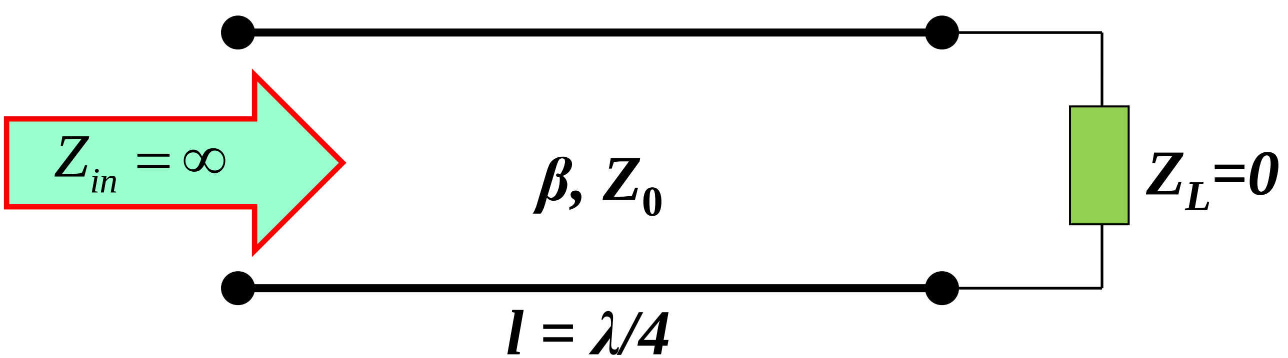
$$Z_{in} = \frac{Z_0^2}{Z_L}$$

input impedance of a quarter-wave line is inversely proportional to the load impedance

→ Think about what this means! Say the load impedance is a short circuit then:

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{Z_0^2}{0} = \infty$$

$Z_{in} = \infty$! This is an open circuit ! The quarter wave TL transforms a short-circuit into open-circuit and vice-versa



Example – 2

- Consider a load resistance $R_L = 100\Omega$ to be matched to a 50Ω line with a quarter-wave transformer. Find the characteristic impedance of the matching section and plot the magnitude of the reflection coefficient versus normalized frequency, f/f_0, where f_0 is the frequency at which the line is $\lambda/4$ long.

  - the necessary characteristic impedance is:

$$Z_0 = \sqrt{Z_L Z_{in}} \quad \Rightarrow \quad \therefore Z_0 = \sqrt{Z_L Z_{in}} = \sqrt{50 \times 100} = 70.71\Omega$$

  - The reflection coefficient magnitude is given as

$$|\Gamma_0| = \left| \frac{Z_0 - Z_{in}}{Z_0 + Z_{in}} \right|$$

$\Leftarrow$ Z_{in} is dependent on frequency

Example – 2 (contd.)

$$Z_{in} = Z(l = \lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}$$

$$\beta l = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda_0}{4}\right) = \left(\frac{2\pi f}{v_p}\right)\left(\frac{v_p}{4 f_0}\right) = \frac{\pi f}{2 f_0}$$

For higher frequencies the matching section looks electrically longer, and for lower frequencies it looks shorter.

Plot the magnitude of the reflection coefficient versus f/f_0 using these two equations

HW # 0

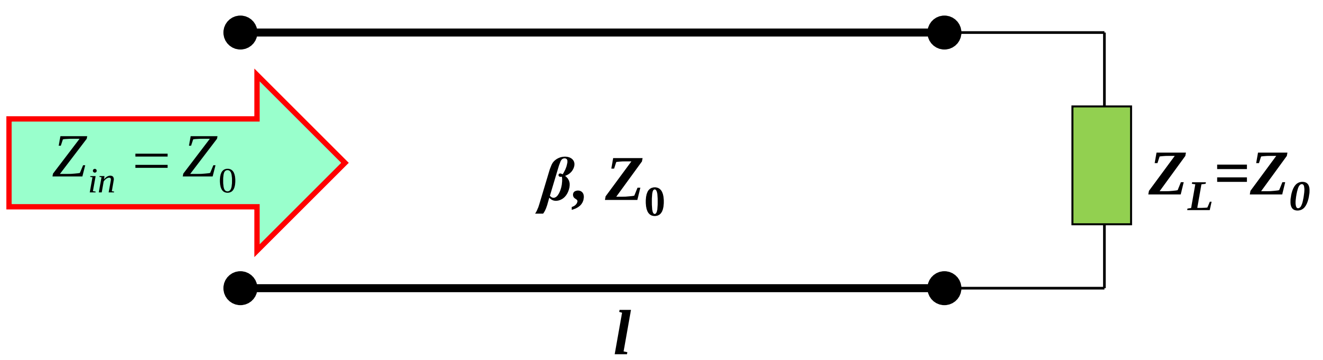
Transmission Line Input Impedance – Special Cases (contd.)

3. $Z_L = Z_0$

the load is **numerically equal** to the characteristic impedance of the transmission line (a real value).

$$Z_{in} = Z_0 \frac{Z_0 + jZ_0 \tan(\beta l)}{Z_0 + jZ_0 \tan(\beta l)} = Z_0$$

In other words, if the **load impedance (Z_L)** is **equal** to the TL **characteristic impedance (Z_0)**, the **input impedance (Z_{in})** likewise will be equal to **characteristic impedance (Z_0)** of the TL **irrespective of its length**



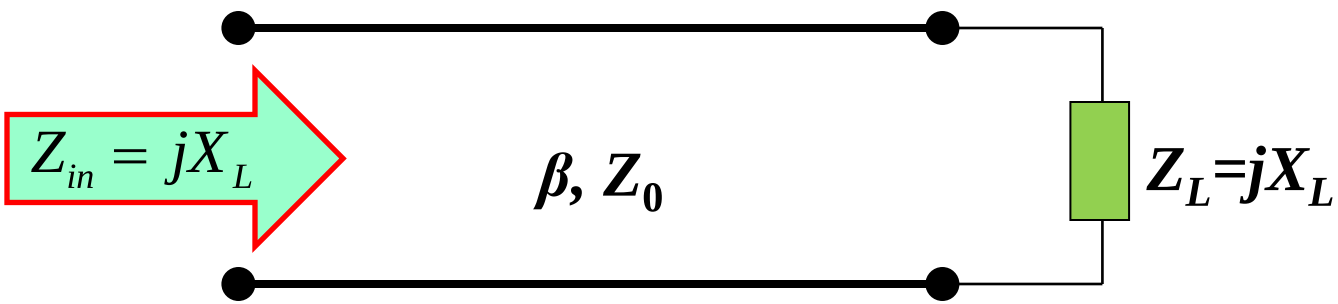
Transmission Line Input Impedance – Special Cases (contd.)

4. $Z_L = jX_L$ ⟵ the load is **purely reactive** (i.e., the resistive component is zero)

$$Z_{in} = Z(z=-l) = Z_0 \frac{jX_L + jZ_0 \tan(\beta l)}{Z_0 - X_L \tan(\beta l)} = jZ_0 \frac{X_L + Z_0 \tan(\beta l)}{Z_0 - X_L \tan(\beta l)}$$

⟶ **Purely Reactive**

In other words, if the **load impedance (Z_L) is purely reactive** then the **input impedance likewise will be purely reactive irrespective of the line length (*l*)**



Note that the **opposite is not true: even if the load is purely resistive (Z_L = R), the input impedance will be complex** (both resistive and reactive components).

Transmission Line Input Impedance – Special Cases (contd.)

4. $Z_L = jX_L$

Note that the **opposite** is **not** true: even if the load is **purely resistive** $(Z_L = R_L)$, the input impedance will be **complex** (both resistive and reactive components).

Q: Why is this?

A: ??

Transmission Line Input Impedance – Special Cases (contd.)

5. $l \ll \lambda$ ← the transmission line is **electrically small**

- If length l is small with respect to signal wavelength λ then:

$$\beta l = \frac{2\pi}{\lambda} l = 2\pi \frac{l}{\lambda} \approx 0$$

- Therefore: $\cos(\beta l) = 1$ $\quad$ $\sin(\beta l) = 0$

- Thus the input impedance is:

$$Z_{in} = Z_0 \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)} = Z_0 \frac{Z_L(1) + jZ_0(0)}{Z_0(1) + jZ_L(0)} = Z_0$$

In other words, if the transmission line length is much smaller than a wavelength, the **input** impedance Z_{in} will **always** be equal to the **load** impedance Z_L.

Transmission Line Input Impedance – Special Cases (contd.)

5. $l << \lambda$ ⟵ the transmission line is **electrically small**

This is the assumption we used in all previous circuits courses (e.g., Linear Circuits, Digital Circuits, Integrated Electronics, Analog Circuit Design etc.)! In those courses, we assumed that the signal frequency ω is relatively **low**, such that the signal wavelength λ is **very large** $(\lambda \gg l)$.

- Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the **same**!

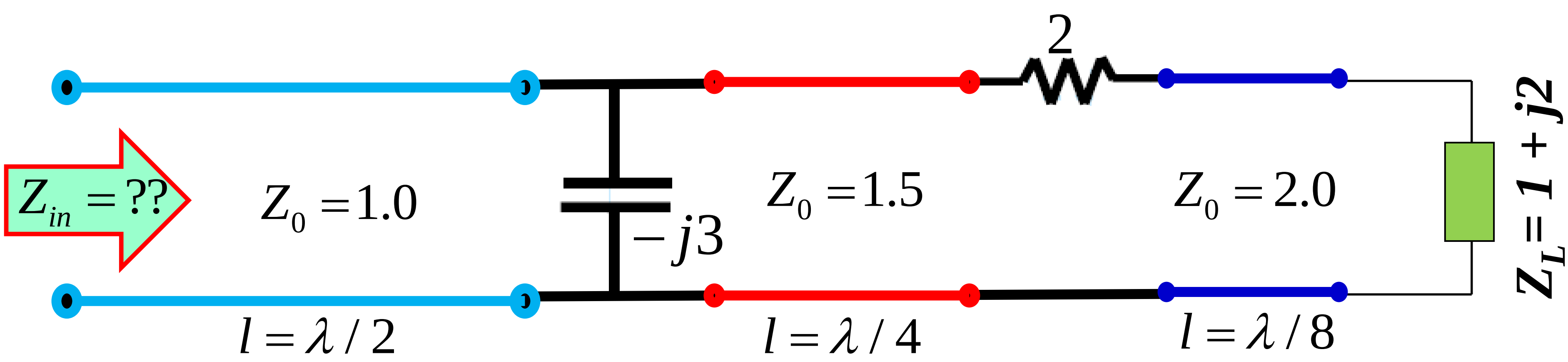
$$V(z=-l) \approx V(z=0)$$

$$I(z=-l) \approx I(z=0)$$

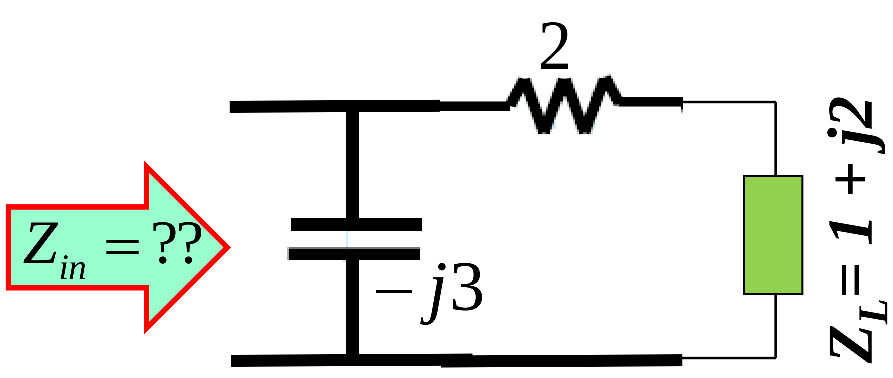
If $l \ll \lambda$, our "wire" behaves **exactly** as it did in *Linear Circuits* course!

Example – 3

Determine the input impedance of the following circuit:



How about the following solution?

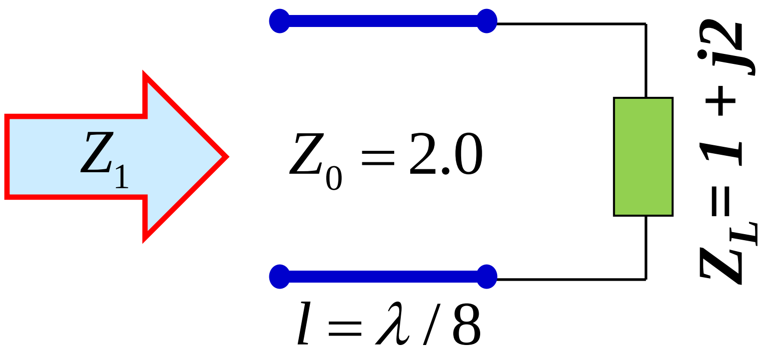


$$Z_{in} = \frac{-j3*(2+1+j2)}{-j3+(2+1+j2)} = 2.7 - j2.1$$

Where are the contributions of the TL??

Example – 3 (contd.)

- **Let us define Z_1 as the input impedance of the last section as:**



Then the impedance Z_1 is:

$$Z_1 = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)}$$

Where: $\beta d = \frac{2\pi}{\lambda} * \frac{\lambda}{8} = \frac{\pi}{4}$

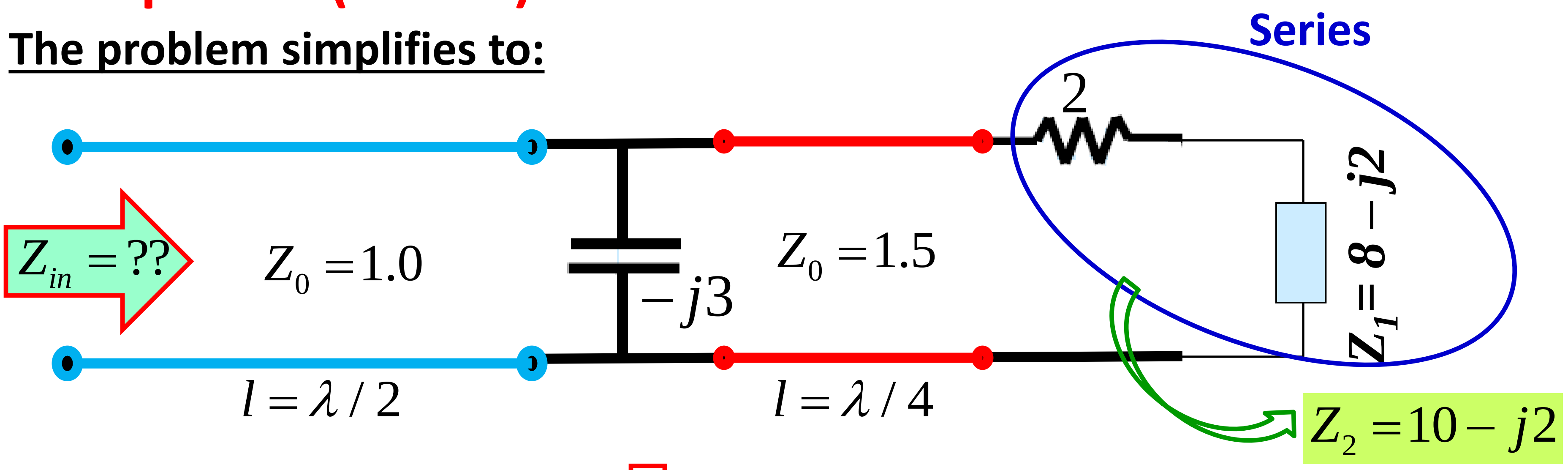
Therefore:

$$Z_1 = 2\left(\frac{(1 + j2) + j2\tan(\pi/4)}{2 + j(1 + j2)\tan(\pi/4)}\right) \Rightarrow Z_1 = 2\left(\frac{1 + j4}{j}\right)$$

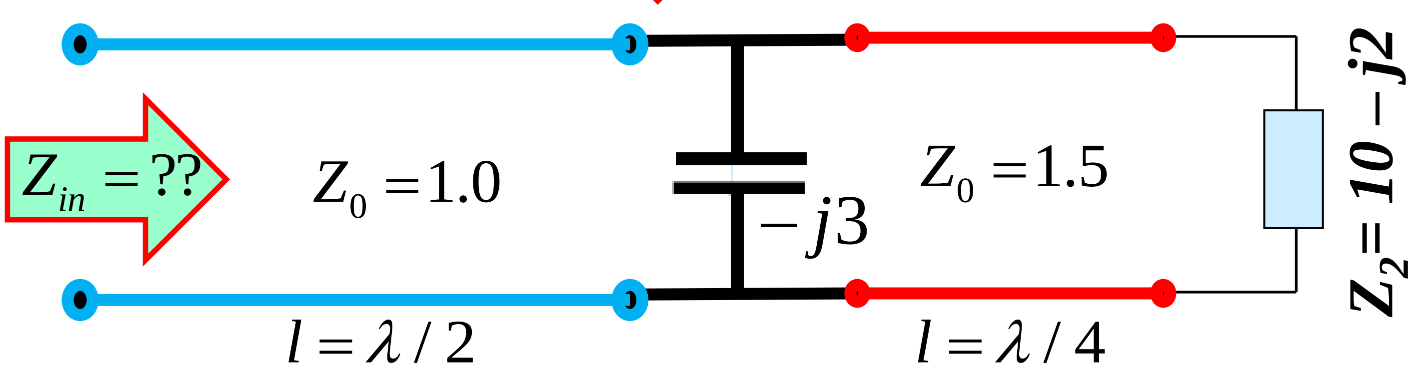
$$\therefore Z_1 = 8 - j2$$

Example – 3 (contd.)

The problem simplifies to:

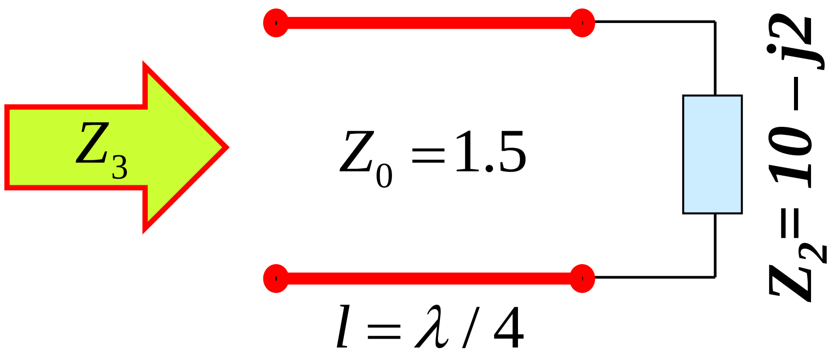


Simplification of the problem



Example – 3 (contd.)

- **Now let us define the input impedance of the middle TL as Z_3:**

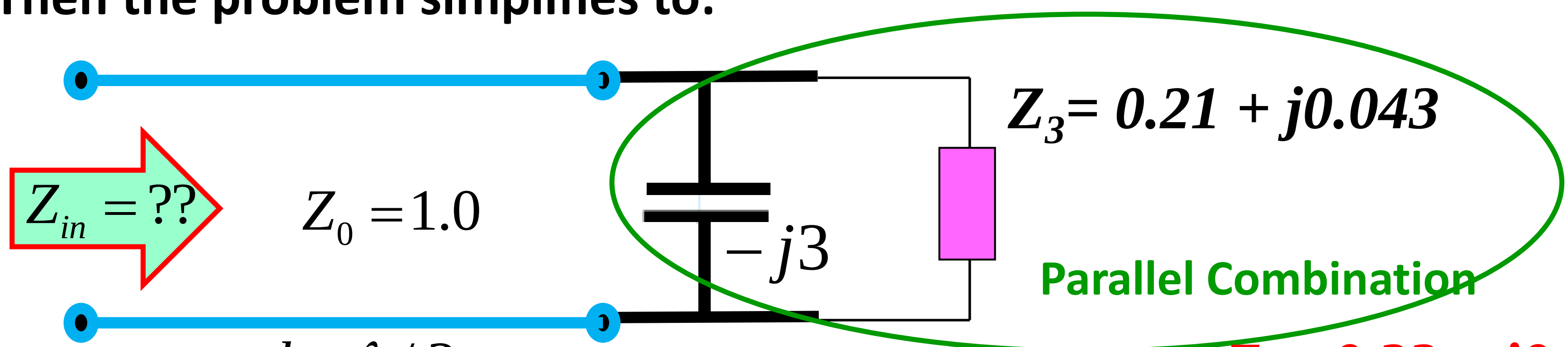


This is a quarter-wave TL → one of the special cases we considered earlier → where the input impedance is:

$$Z_3 = \frac{Z_0^2}{Z_2}$$

Therefore: $Z_3 = \dfrac{(1.5)^2}{10 - j2}$ → $\therefore Z_3 = 0.21 + j0.043$

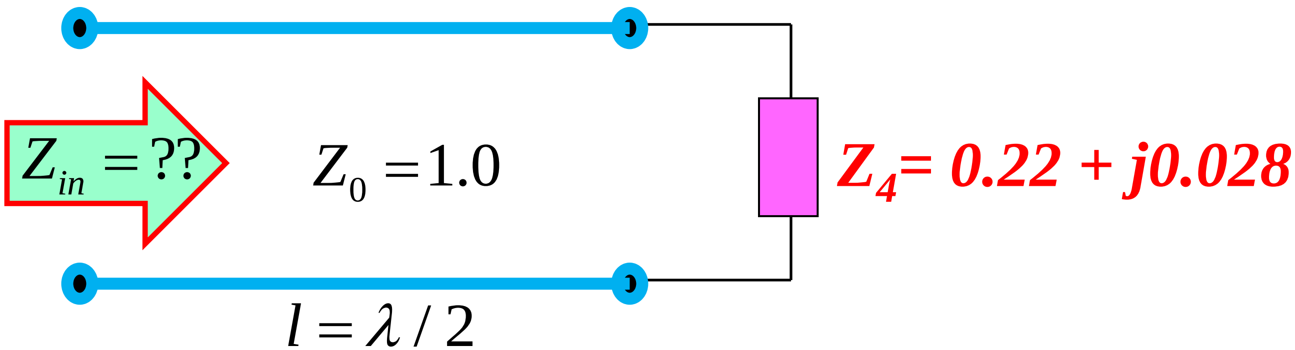
- **Then the problem simplifies to:**



$Z_4 = 0.22 + j0.028$

Example – 3 (contd.)

- **Finally the simplified problem is:**



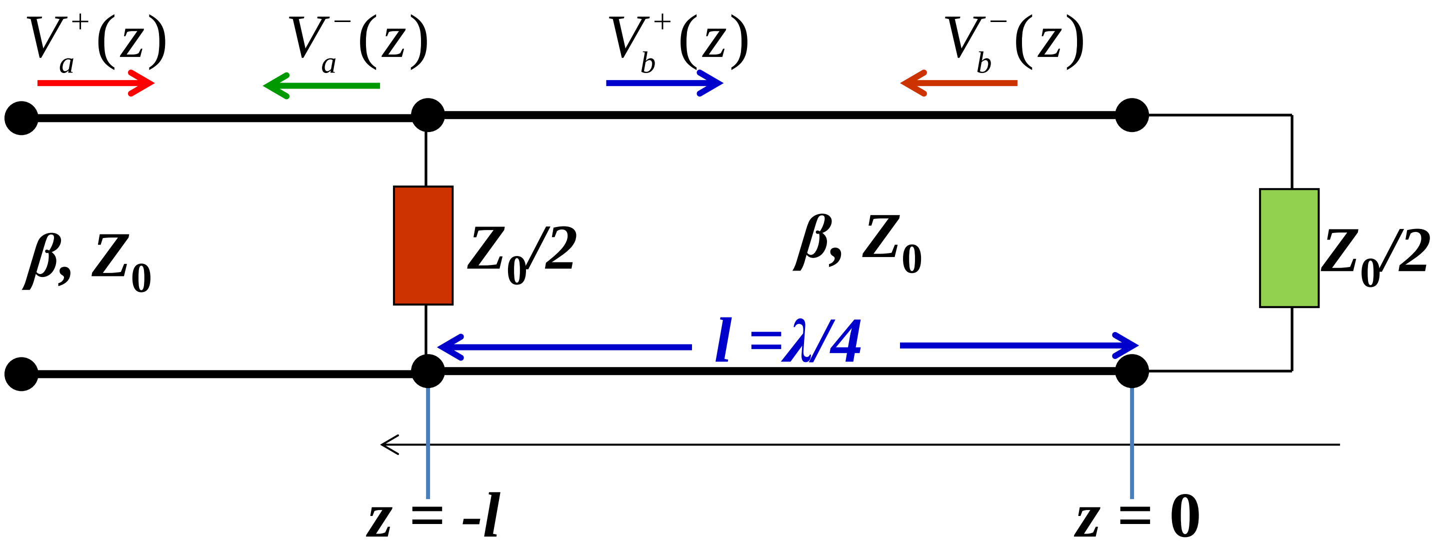
TL is a half wavelength → special case we discussed earlier → input impedance equals the load impedance

$$\therefore Z_{in} = Z_4 = 0.22 + j0.028$$

Example – 4

For the following circuit determine:

$$\frac{V_a^-}{V_a^+} \qquad \frac{V_b^+}{V_a^+} \qquad \frac{V_b^-}{V_a^+}$$



Given:

$$V(z) = V_a^+(z) + V_a^-(z) = V_a^+ e^{-j\beta z} + V_a^- e^{+j\beta z} \qquad \text{For } z < -l$$

$$V(z) = V_b^+(z) + V_b^-(z) = V_b^+ e^{-j\beta z} + V_b^- e^{+j\beta z} \qquad \text{For } -l < z < 0$$