Lecture – 5

Date: 19.01.2017

- Smith Chart Fundamentals

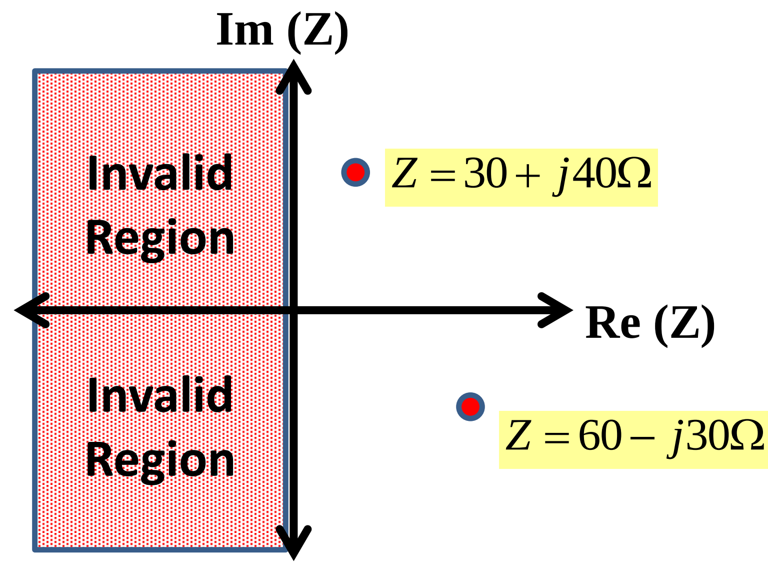
Smith Chart

- Smith chart – what?
- The Smith chart is a very convenient graphical tool for analyzing and studying TLs behavior.
- It is mapping of impedance in standard complex plane into a suitable complex reflection coefficient plane.
- It provides graphical display of reflection coefficients.
- The impedances can be directly determined from the graphical display (ie, from Smith chart)
- Furthermore, Smith charts facilitate the analysis and design of complicated circuit configurations.

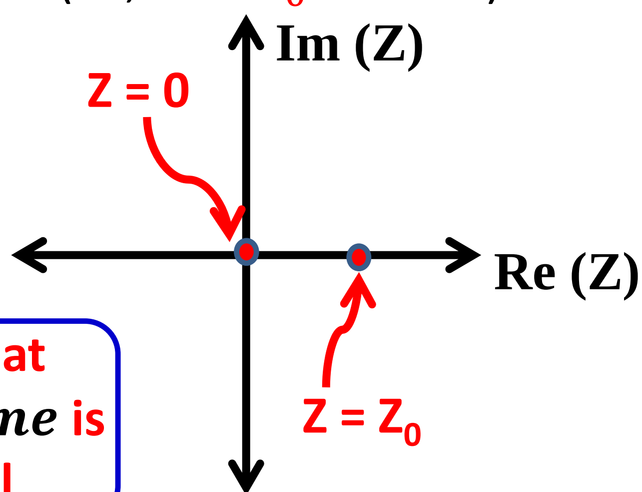
The Complex Γ- Plane

- Let us first display the impedance Z on complex Z-plane

each dimension is defined by a single real line: the horizontal line (axis) indicates the real component of Z, and the vertical line (axis) indicates the imaginary component of Z → Intersection of these lines indicate the complex impedance



- How do we plot an **open circuit** (i.e, $Z = \infty$), **short circuit** (i.e, $Z = 0$), and **matching condition** (i.e, $Z = Z_0 = 50\Omega$) on the complex Z-plane



It is apparent that complex $Z - plane$ is not very useful

The Complex Γ-Plane (contd.)

- The **limitations** of complex Z-plane can be **overcome** by complex Γ-plane
- We know **Z ↔ Γ** (i.e, if you know **one**, you know the **other**).
- We can define a complex Γ-plane similar to a complex Z-plane.
- Let us revisit the reflection coefficient in complex form:

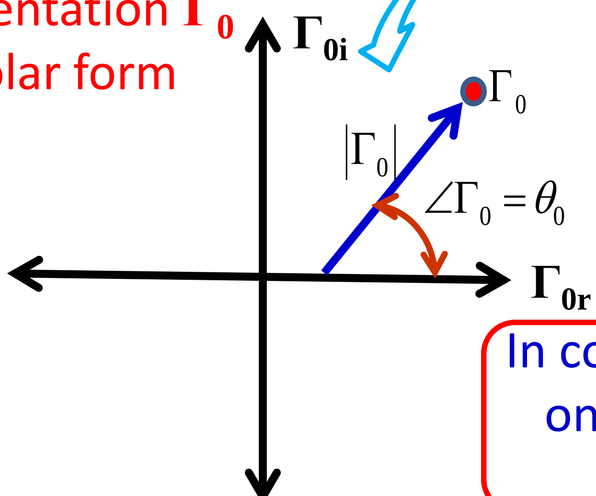
$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + \Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

Where,

$$\theta_0 = \tan^{-1}\left(\frac{\Gamma_{0i}}{\Gamma_{0r}}\right)$$

In the special terminated conditions of pure short-circuit and pure open-circuit conditions the corresponding Γ_0 are -1 and +1 located on the real axis in the complex Γ-plane.

Representation Γ_0 in polar form

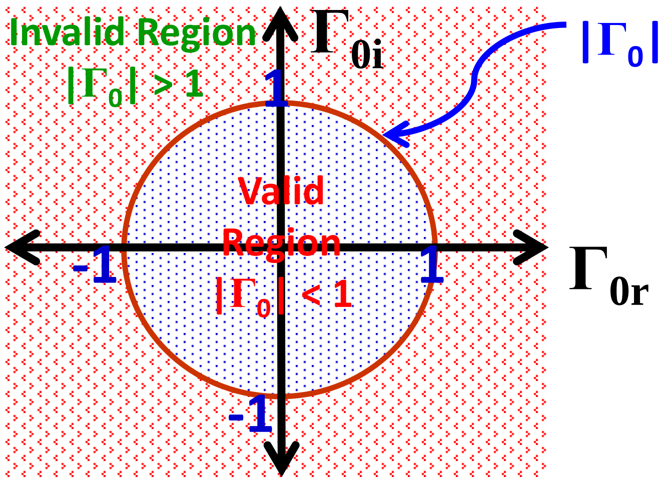


the reflection coefficient has a valid region that encompasses all the four quadrants in the complex Γ-plane within the -1 to +1 bounded region

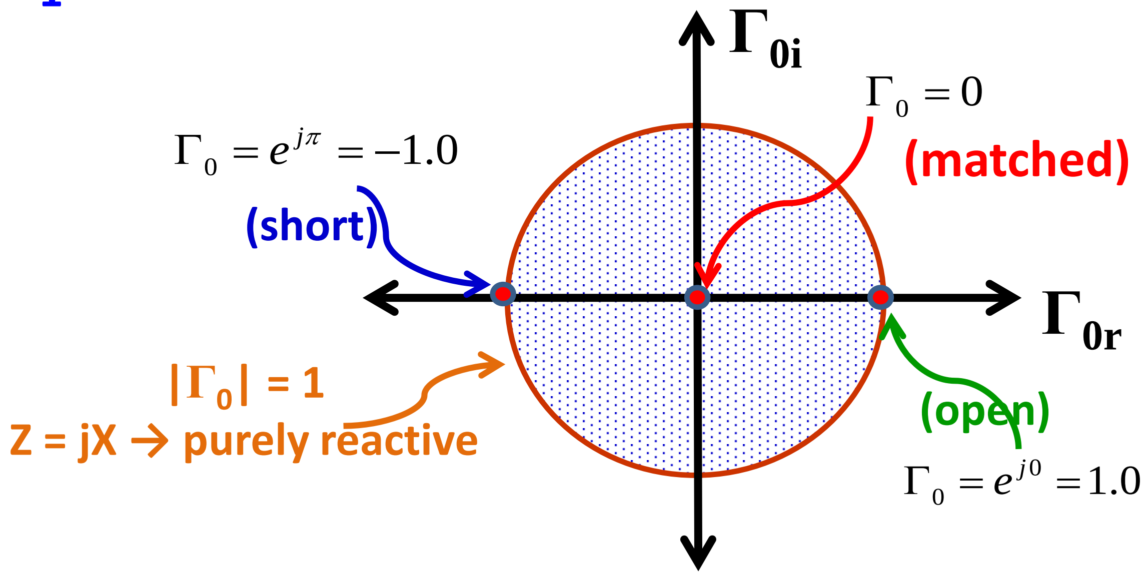
In complex Z-plane the valid region was unbounded on the right half of the plane → as a result many important impedances could not be plotted

The Complex Γ-Plane (contd.)

- **Validity Region**



- We can plot all the valid impedances (i.e R > 0) within this bounded region.



Example – 1

- A TL with a characteristic impedance of Z_0 = 50Ω is terminated into following load impedances:
  - (a) Z_L = 0 (Short Circuit)
  - (b) $Z_L \rightarrow \infty$ (Open Circuit)
  - (c) Z_L = 50Ω
  - (d) Z_L = (16.67 – j16.67)Ω
  - (e) Z_L = (50 + j50)Ω

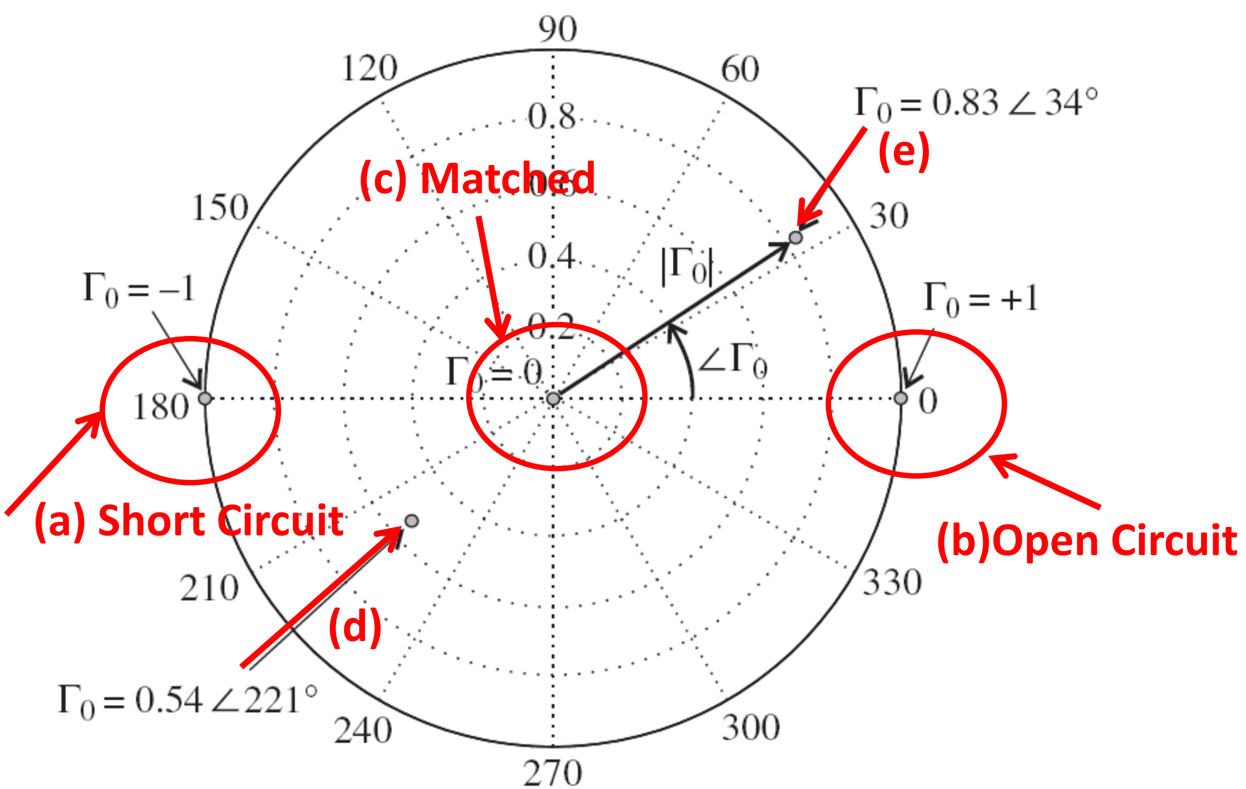
Display the respective reflection coefficients in complex Γ-plane

Example – 1 (contd.)

- **Solution:** We know the relationship between Z and Γ:

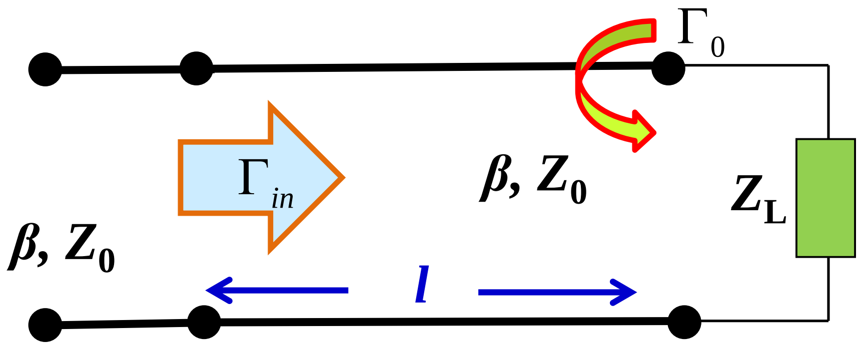
$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + \Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

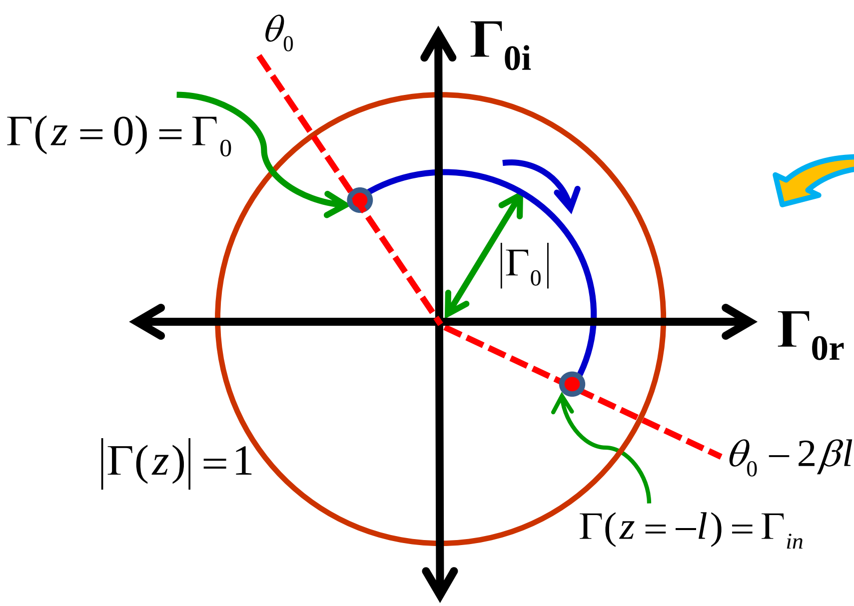
(a) Γ_0 = -1 (Short Circuit)
(b) Γ_0 = 1 (Open Circuit)
(c) Γ_0 = 0 (Matched)
(d) Γ_0 = 0.54<221°
(e) Γ_0 = 0.83<34°



Transformations on the Complex Γ-Plane

- Lets consider the terminated lossless TL.





- At $z = 0$, Γ_0 describes the mismatch between Z_L and Z_0.
- The move away from the load (or towards the input/source) in the negative z-direction (clockwise rotation) requires multiplication of Γ_0 by a factor $\exp(+j2\beta z)$ in order to explicitly define the mismatch at location 'z' known as $\Gamma(z)$.
- This transformation of Γ_0 to $\Gamma(z)$ is the key ingredient in Smith chart as a graphical design/display tool.

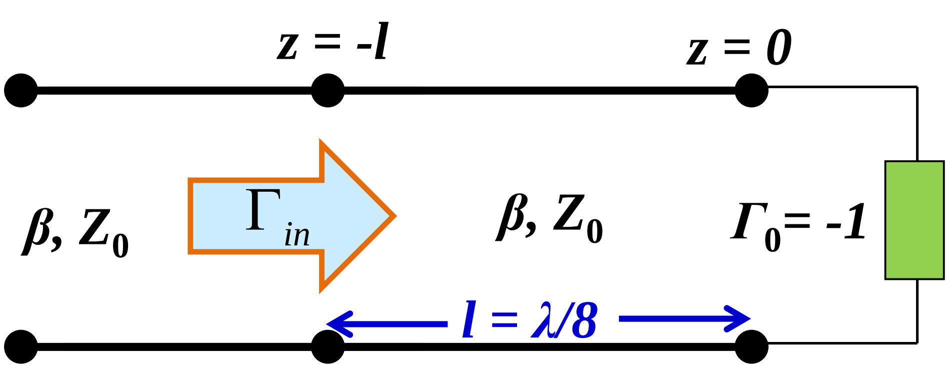
Graphical interpretation of

$$\Gamma(z) = \Gamma_0 e^{+2j\beta z}$$

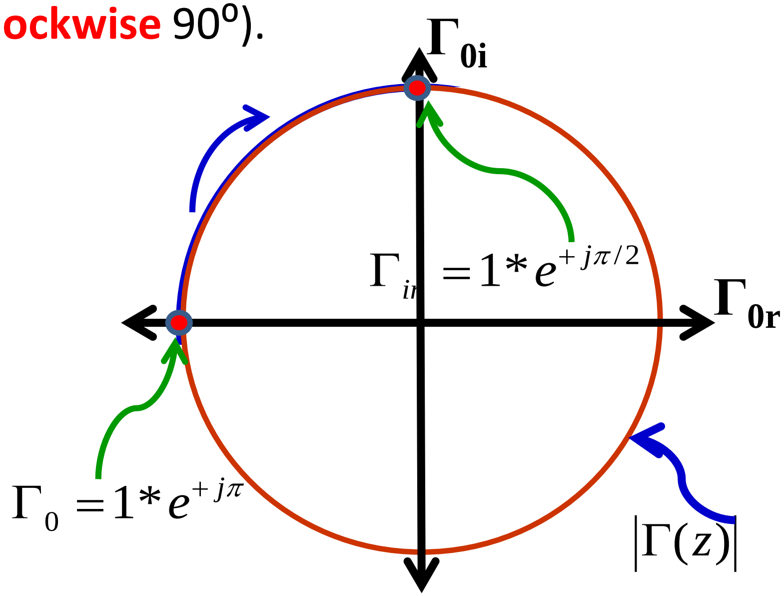
Transformations on the Complex Γ-Plane (contd.)

- It is clear that addition of a length of TL to a load Γ_0 **modifies** the **phase** θ_0 but **not** the **magnitude** Γ_0, we trace a **circular arc** as we parametrically plot $\Gamma(z)$! This arc has a **radius** Γ_0 and an **arc angle** $2\beta l$ radians.

- We can therefore **easily** solve many interesting TL problems **graphically**—using the complex Γ-plane! For **example,** say we wish to determine Γ_{in} for a transmission line length l = λ/8 and terminated with a **short** circuit.



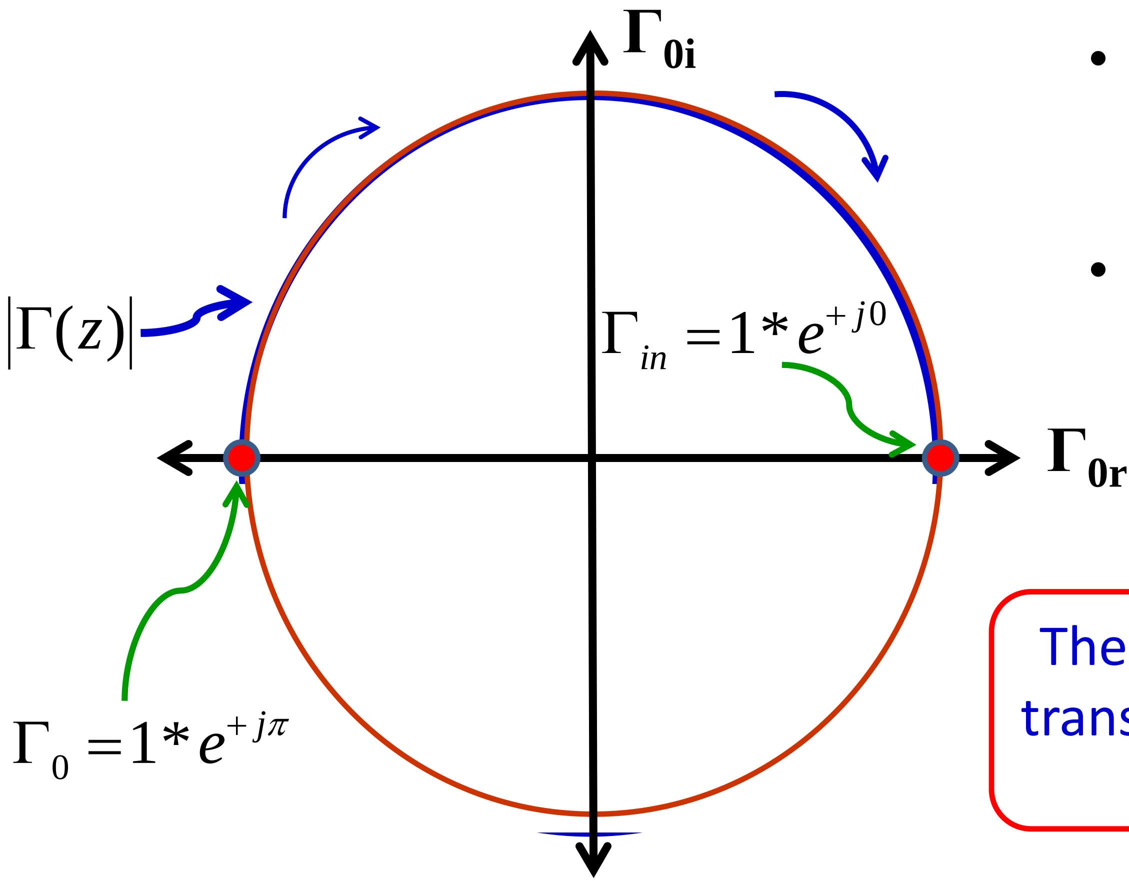
- The reflection coefficient of a **short** circuit is Γ_0 = −1 =1*e(jπ), and therefore we **begin** at the leftmost point on the complex Γ-plane. We then move along a **circular arc** $-2\beta l$ = −2(π/4) = −π/2 radians (i.e., rotate **clockwise** 90^o).



When we stop, we find we are at the point for Γ_{in}; in this case Γ_{in} = 1*e(jπ/2)

Transformations on the Complex Γ-Plane (contd.)

- Now let us consider the same problem, only with a new transmission line length $l = \lambda/4$.
- Now we rotate clockwise $2\beta l = \pi$ radians.

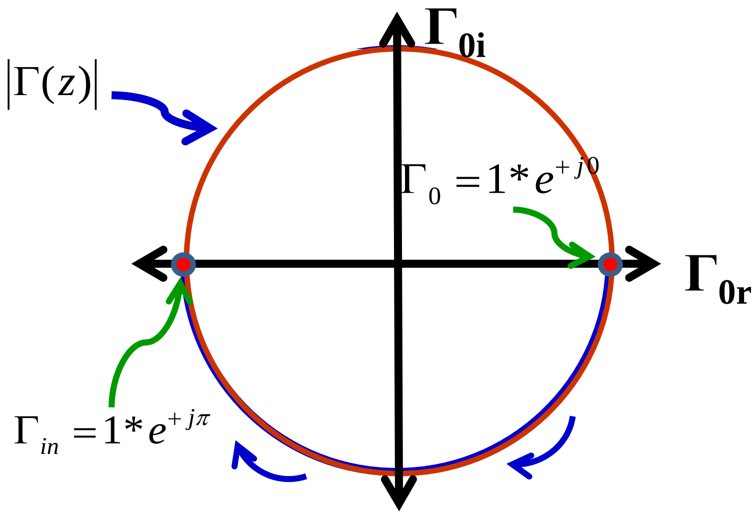


- In this case the input reflection coefficient is $\Gamma_{in} = 1*e(j0) = 1$
- The reflection coefficient of an open circuit

The short circuit load has been transformed into an open circuit with a quarter-wave TL

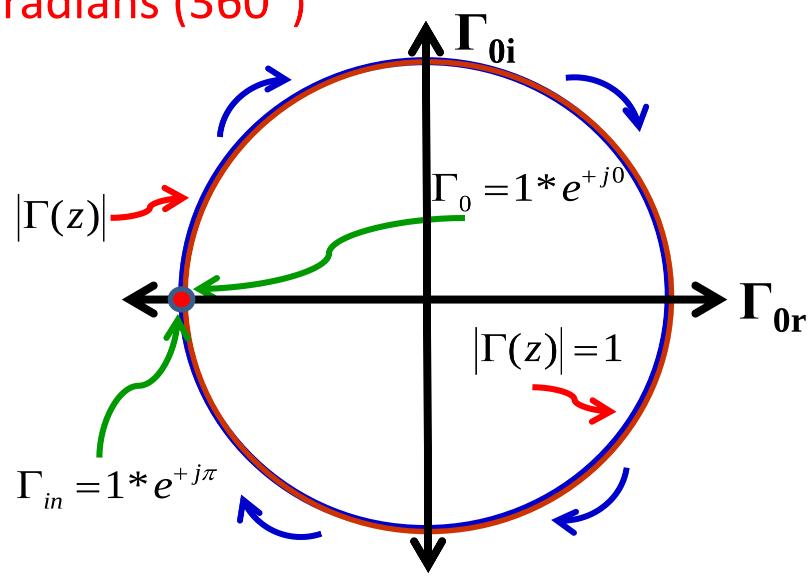
Transformations on the Complex Γ-Plane (contd.)

- We also know that a quarter-wave TL transforms an open-circuit into short-circuit → graphically it can be shown as:



- We came clear around to where we **started!**
- Thus we conclude that $\Gamma_{in} = \Gamma_0$

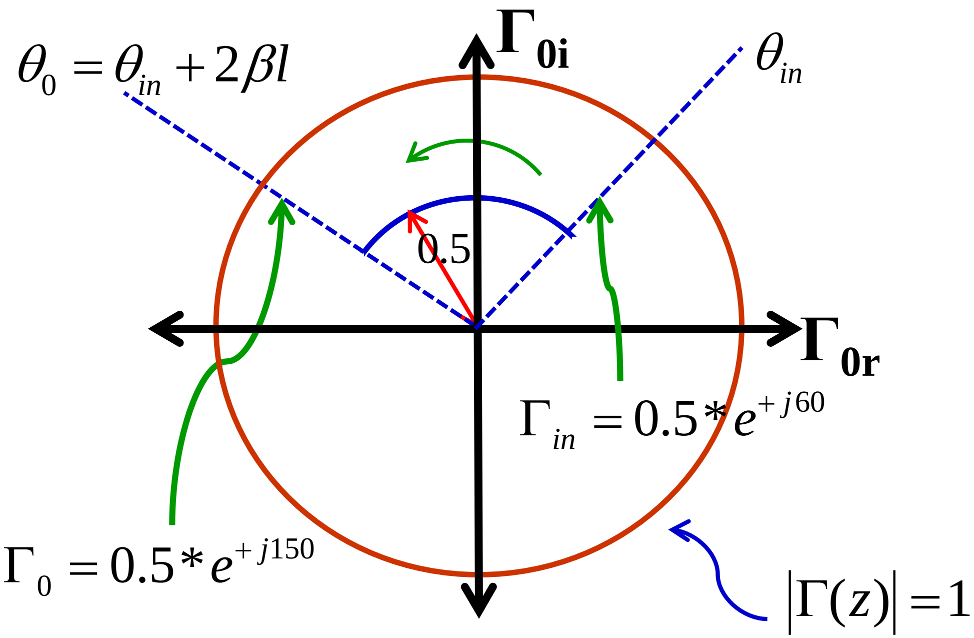
- Now let us consider the same problem again, only with a new transmission line length $l = \lambda/2$.
- Now we rotate clockwise $2\beta l = 2\pi$ radians (360^o)



It comes from the fact that **half-wavelength** TL is a special case, where we know that $\mathbf{Z_{in} = Z_L}$ → eventually it leads to $\mathbf{\Gamma_{in} = \Gamma_0}$

Transformations on the Complex Γ-Plane (contd.)

- Now let us consider the **opposite** problem. Say we know that the **input** reflection coefficient at the beginning of a TL with length l = λ/8 is: $\boldsymbol{\Gamma_{in} = 0.5e(j60°)}$.
- What is the reflection coefficient at the **load**?
- In this case we rotate counter-clockwise along a circular arc (radius =0.5) by an amount 2βl = π/2 radians (90^o).
- In essence, we are removing the phase associated with the TL.



The reflection coefficient at the load is:

$$\Gamma_0 = 0.5 * e^{+j150}$$

Mapping Z to Γ

- the line impedance and reflection coefficient are **equivalent** – either one can be expressed in terms of the other.

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \quad \Leftrightarrow \quad Z(z) = Z_0\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

- The expressions depend on Z_0 of the TL. To generalize, we first define a **normalized** impedance value **z'** as:

$$z'(z) = \frac{Z(z)}{Z_0} = \frac{R(z)}{Z_0} + j\frac{X(z)}{Z_0} = r(z) + jx(z)$$

therefore

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{\left(Z(z)/Z_0\right) - 1}{\left(Z(z)/Z_0\right) + 1} = \frac{z'(z) - 1}{z'(z) + 1}$$

$$z'(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

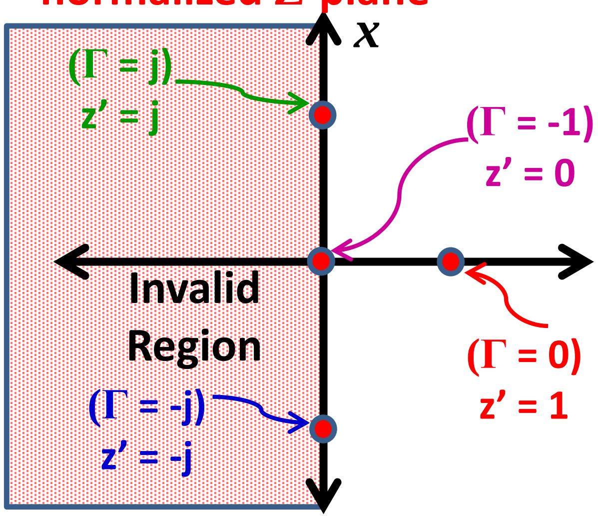
These equations describe a mapping between z' and Γ. That means that each and every normalized impedance value likewise corresponds to one specific point on the complex Γ-plane

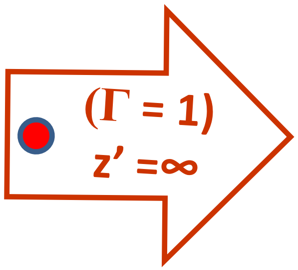
Mapping Z to Γ (contd.)

- Lets indicate values of some common normalized impedances (shown below) on the complex Γ-plane and vice-versa.

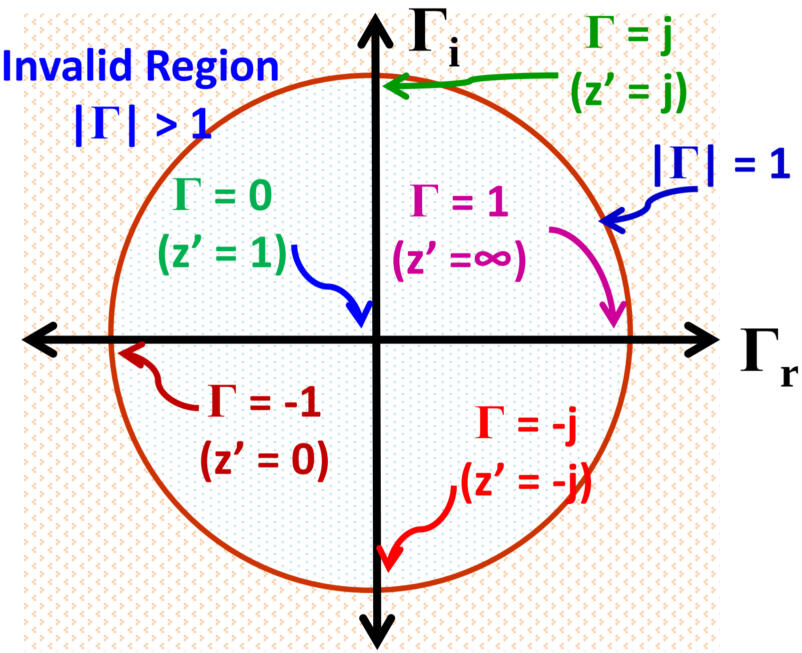
Case	Z	z'	Γ
1	∞	∞	1
2	0	0	-1
3	Z_0	1	0
4	jZ_0	j	j
5	$-jZ_0$	-j	-j

- **These map onto five points on the normalized Z-plane**





- The five normalized impedances map five specific points on the complex Γ-plane.

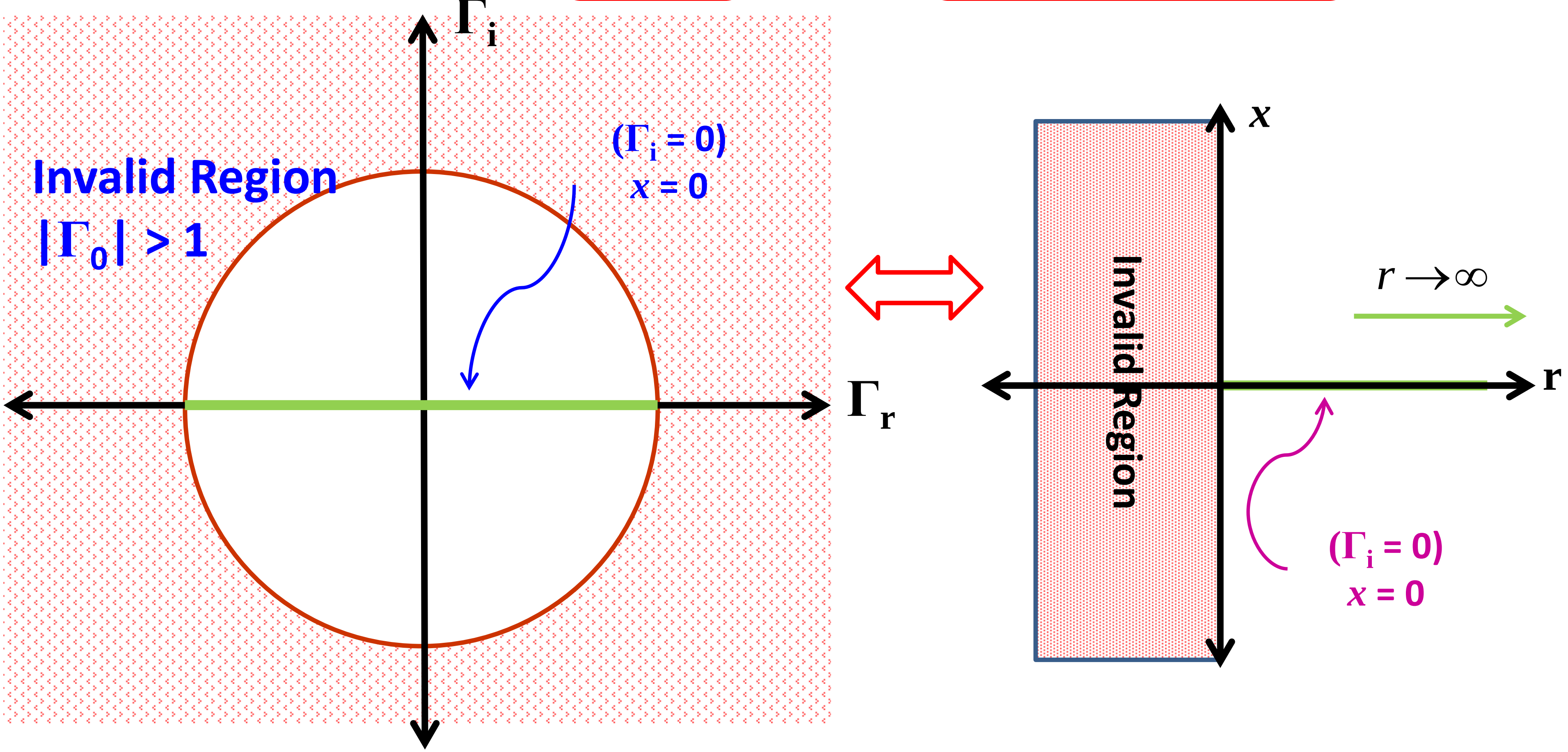


Apparently the normalized impedances can be mapped on complex Γ-plane and vice versa and gives us a clue that whole impedance contours (i.e, set of points) can be mapped to complex Γ-plane

Mapping Z to Γ (contd.)

Case-I: Z = R → impedance is purely real

$z' = r + j0$ → $\Gamma = \dfrac{r-1}{r+1}$ → $\Gamma_r = \dfrac{r-1}{r+1} \qquad \Gamma_i = 0$



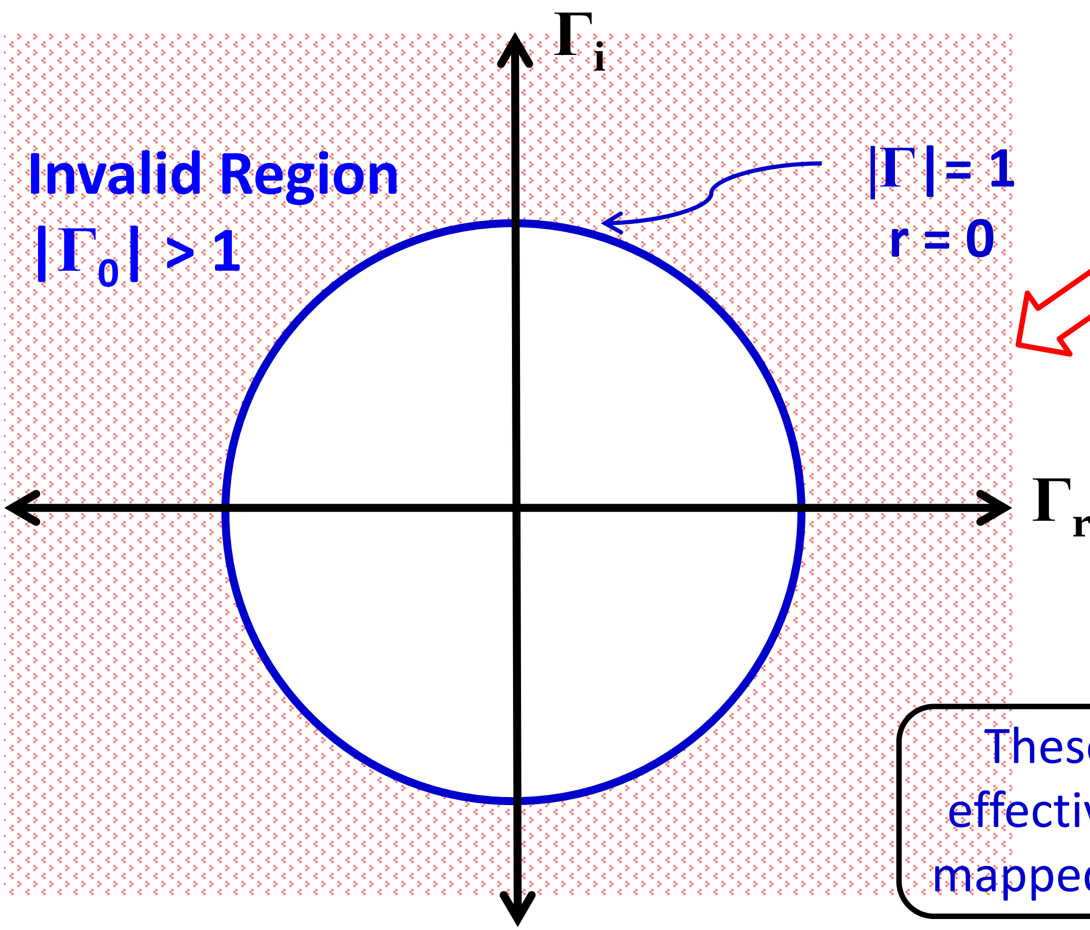
Mapping Z to Γ (contd.)

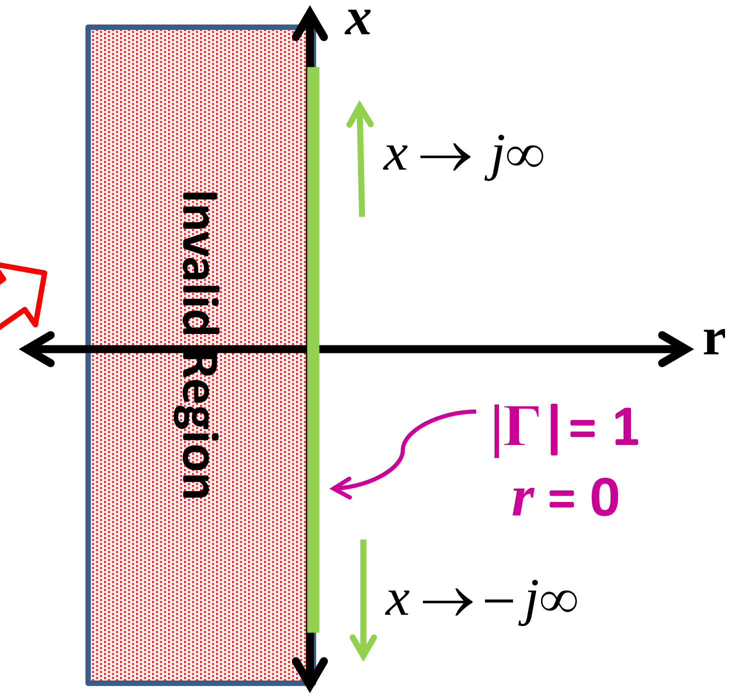
Case-II: Z = jX → impedance is purely imaginary

$z' = 0 + jx$

Purely reactive impedance results in a reflection coefficient with unity magnitude

$|\Gamma| = 1$





These cases (I and II) demonstrate that effectively any complex impedance can be mapped to complex Γ-plane → **Smith Chart**

Mapping Z to Γ (contd.)

In summary

- A vertical line $r = 0$ on complex Z-plane maps to a circle $|\Gamma| = 1$ on the complex Γ-plane
- A horizontal line $x = 0$ on complex Z-plane maps to the line $\Gamma_i = 0$ on the complex Γ-plane

Very fascinating in an academic sense, but are not relevant considering that actual values of impedance generally have both a real and imaginary component

Mappings of more general impedance contours (e.g, $r = 0.5$ and $x = -1.5$ corresponding to normalized impedance $0.5 - j1.5$) can also be mapped

Smith Chart

The Smith Chart (contd.)

- Let us revisit the generalized reflection coefficient formulation:

$$\Gamma(z)=|\Gamma_0|e^{j\theta_0}e^{j2\beta z}=\Gamma_r+j\Gamma_i$$

- Therefore, the normalized impedance can be formulated as:

$$z'(z)=r+jx=\frac{Z(z)}{Z_0}=\frac{1+\Gamma(z)}{1-\Gamma(z)}=\frac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r-j\Gamma_i}$$

$$\Rightarrow\left((1-\Gamma_r)-j\Gamma_i\right)(r+jx)=(1+\Gamma_r)+j\Gamma_i$$

- The separation of real and imaginary part results in:

$$r(1-\Gamma_r)+x\Gamma_i=(1+\Gamma_r) \quad \leftarrow \text{Real}$$

$$x(1-\Gamma_r)-r\Gamma_i=\Gamma_i \quad \leftarrow \text{Imaginary}$$

- Simplification and then elimination of reactance (x) from these two give:

$$\left(\Gamma_r-\frac{r}{1+r}\right)^2+\Gamma_i^2=\frac{1}{(1+r)^2}$$

⟺

Similar equation to circle of radius l, centered at (p,q):

$$(\Gamma_r-p)^2+(\Gamma_i-q)^2=l^2$$

The Smith Chart (contd.)

center: $(p,q) = \left(\frac{r}{1+r}, 0\right)$ and radius: $l = \frac{1}{1+r}$

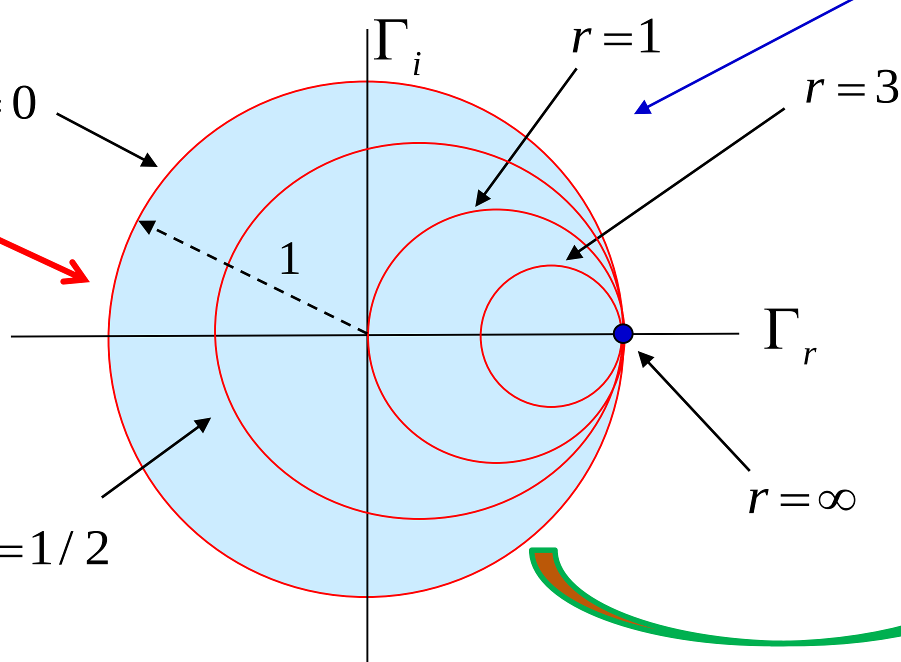
Observations:

- **For r =0:** $p^2 + q^2 = 1$; (p, q) = (0, 0) and l = 1
- **For r =1/2:** $(p - 1/3)^2 + q^2 = (2/3)^2$; (p, q) = (1/3, 0) and l = 2/3
- **For r =1:** $(p - 1/2)^2 + q^2 = (1/2)^2$; (p, q) = (1/2, 0) and l = 1/2
- **For r =3:** $(p - 3/4)^2 + q^2 = (1/4)^2$; (p, q) = (3/4, 0) and l = 1/4

Circles of distinct centre and radii

Note: $p + l = 1$

Because of $(q - 0)^2$ term, all the constant resistance (r) circles have centers on this line



This approach enables mapping of any realizable vertical line (representing **r**) in the complex Γ-plane

The Smith Chart (contd.)

- For the mapping of horizontal lines of the normalized impedance plane to Γ-plane, let us simplify and eliminate resistance (***r***) from these:

$$r(1-\Gamma_r)+x\Gamma_i=(1+\Gamma_r)$$ ← **Real**

$$x(1-\Gamma_r)-r\Gamma_i=\Gamma_i$$ ← **Imaginary**

$$(\Gamma_r-1)^2+\left(\Gamma_i-\frac{1}{x}\right)^2=\left(\frac{1}{x}\right)^2$$

center: $(p,q)=(1,1/x)$

radius: $l=\frac{1}{|x|}$

Note:

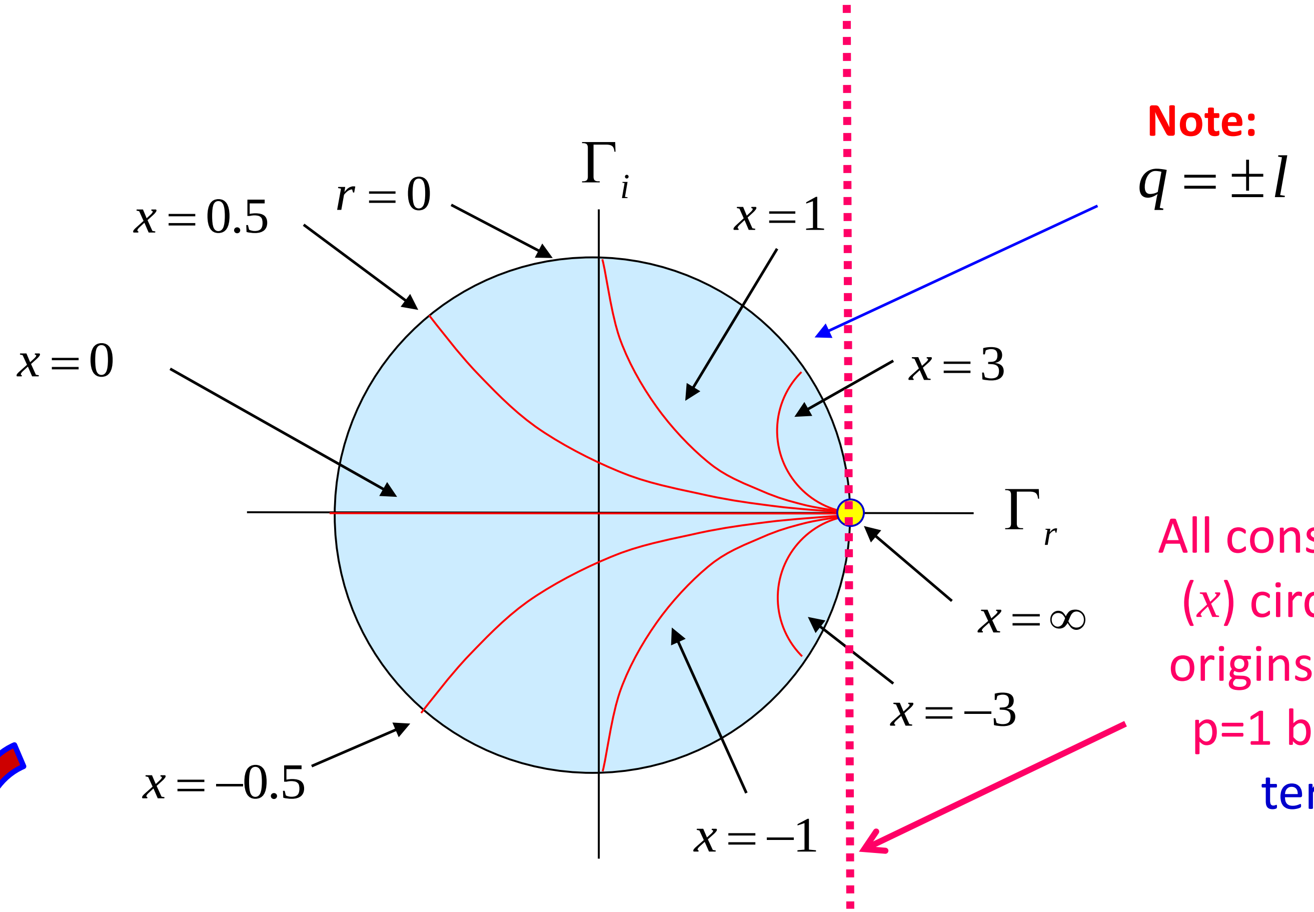
$$q=\pm l$$

Observations:

- **For x =1:** (p – 1)^2 + (q – 1)^2 = (1)^2; (p, q) = (1, 1) and l = 1
- **For x =-1:** (p – 1)^2 + (q + 1)^2 = (1)^2; (p, q) = (1, -1) and l = 1
- **For x =1/2:** (p – 1)^2 + (q – 2)^2 = (2)^2; (p, q) = (1, 2) and l = 2
- **For x =-1/2:** (p – 1)^2 + (q + 2)^2 = (2)^2; (p, q) = (1, -2) and l = 2

→ **Circles of distinct centre and radii**

The Smith Chart (contd.)



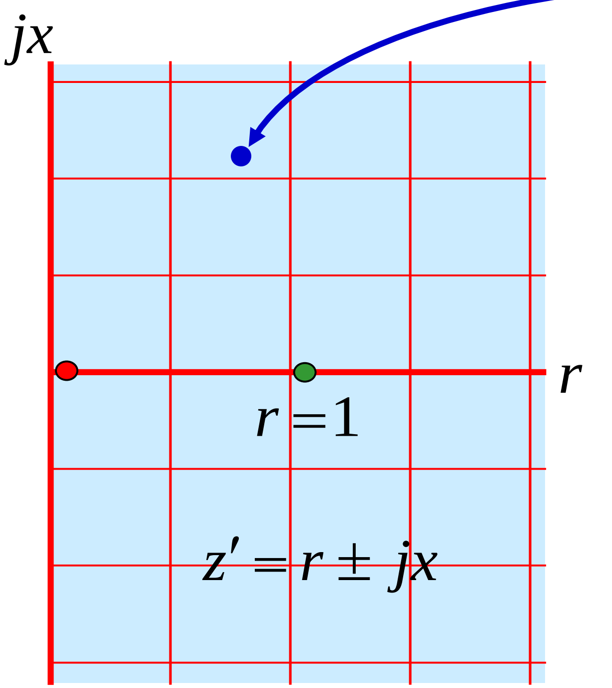
All constant reactance (x) circles have their origins along this line p=1 because of the term (p – 1)²

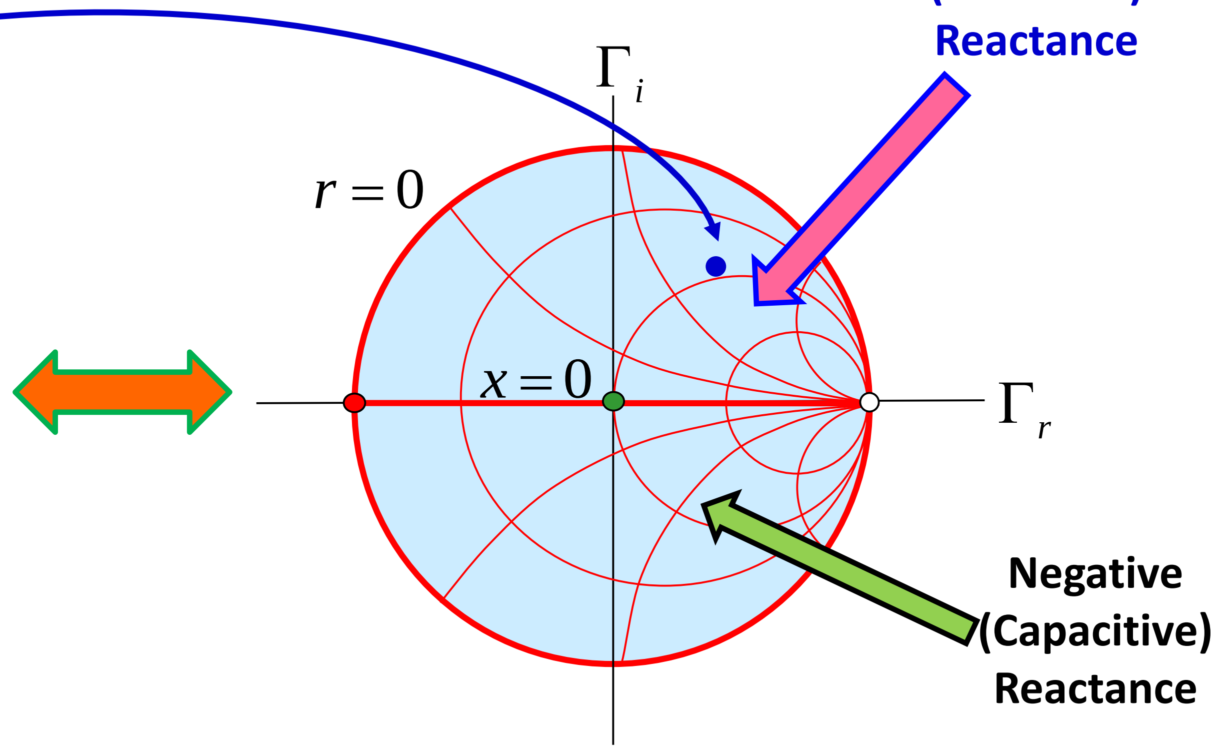
This approach enables mapping of any realizable horizontal line (representing x) in the complex Γ-plane

The Smith Chart (contd.)

- Combination of these constant resistance and reactance circles define the mappings from normalized impedance (z') plane to Γ-plane and is called as Smith chart.

$$z'(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} \quad \Longleftrightarrow \quad \Gamma(z) = \frac{z'(z)-1}{z'(z)+1}$$



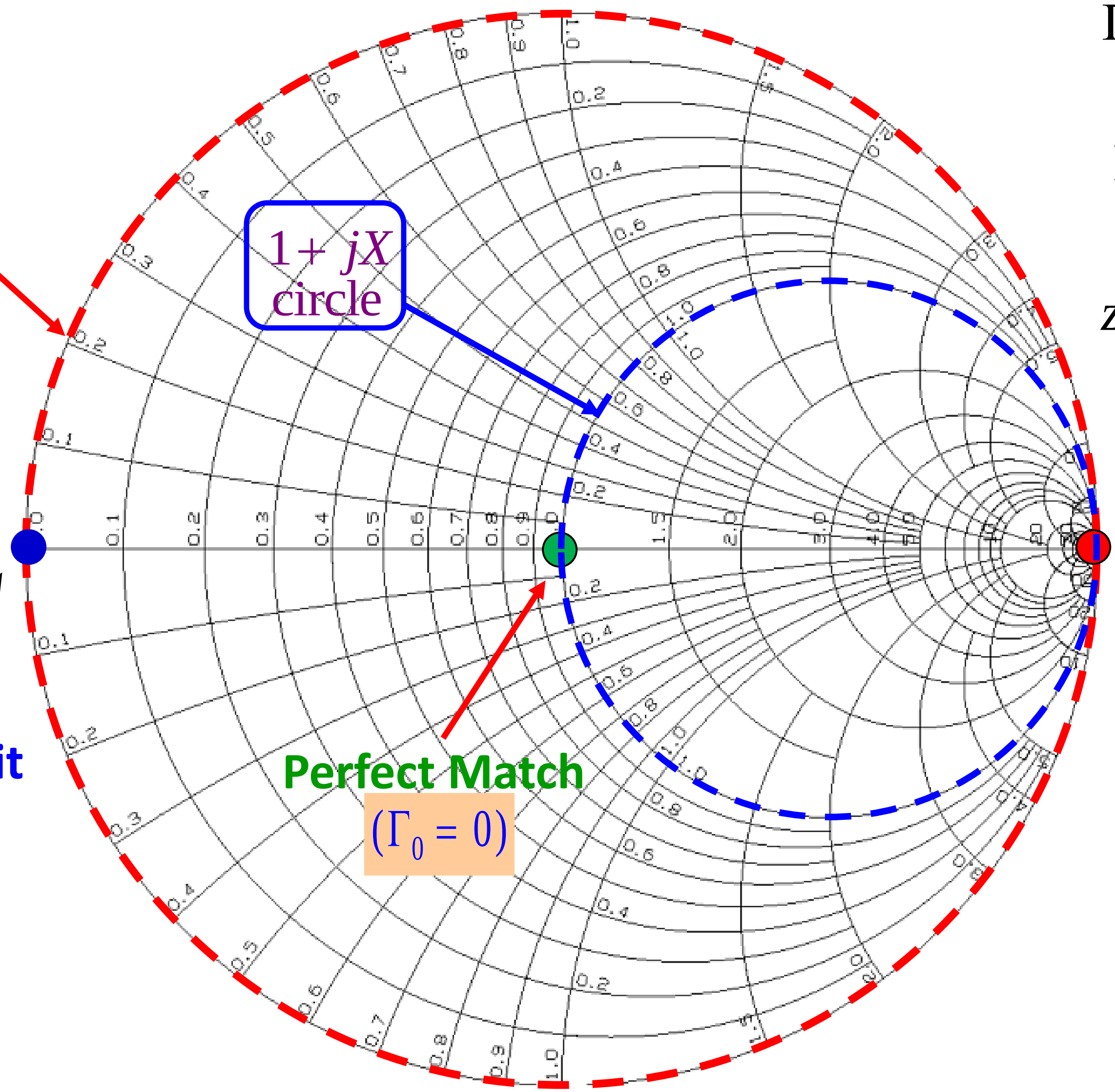


The Smith Chart – Important Points

$$\Gamma(z) = \frac{z'(z)-1}{z'(z)+1}$$

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

$$z'(z) = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

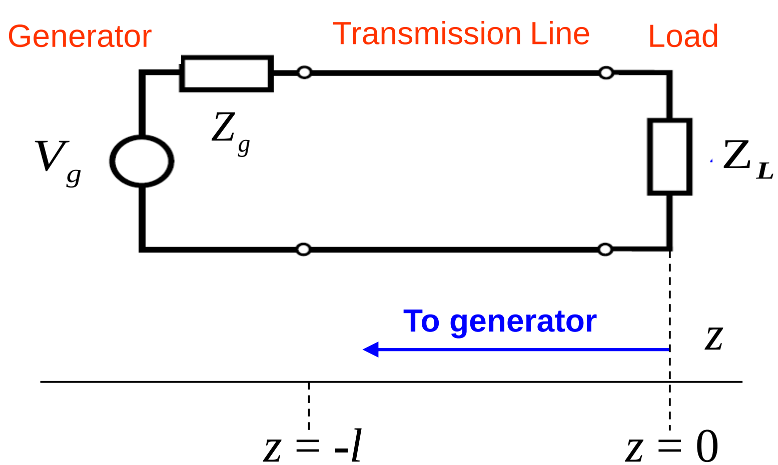


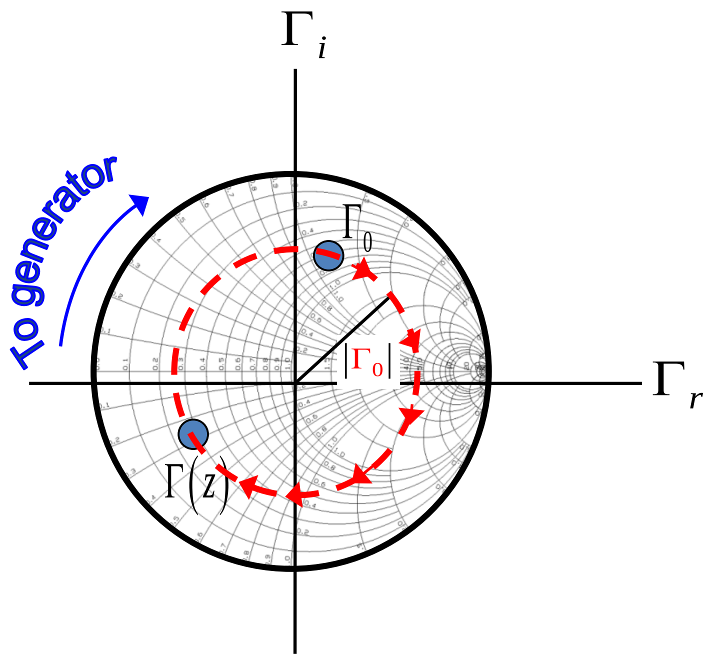
The Smith Chart (contd.)

$$z'(z) = \frac{1+\Gamma_0 e^{+2j\beta z}}{1-\Gamma_0 e^{+2j\beta z}}$$

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

movement in negative z direction (toward generator) $\Leftrightarrow$ clockwise motion on circle of constant $|\Gamma_0|$



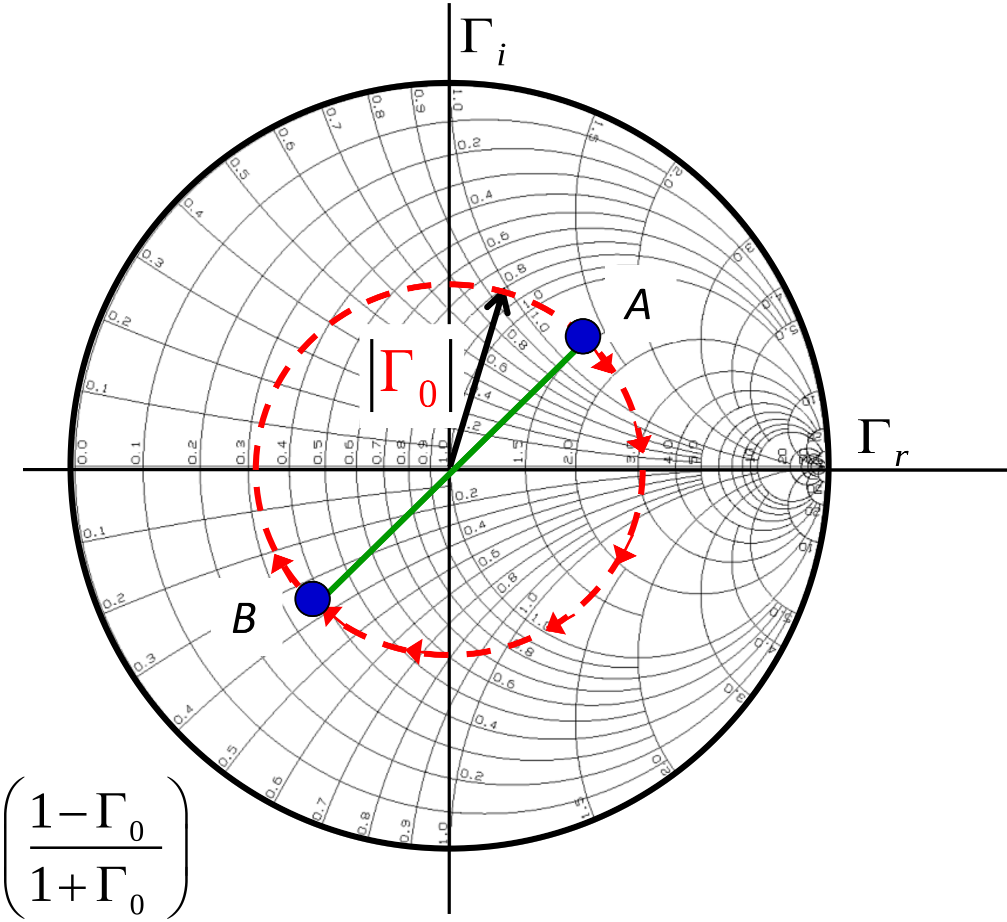


angle change = $2\beta z$

The Smith Chart (contd.)

Reciprocal Property

$$z'(z) = \left(\frac{1 + \Gamma_0 e^{+j2\beta z}}{1 - \Gamma_0 e^{+j2\beta z}} \right)$$



- Go half-way around the Smith chart:

$$-l = \lambda / 4$$

$$2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\left(-\frac{\lambda}{4}\right) = -\pi$$

$$z'(z=0) = \left(\frac{1+\Gamma_0}{1-\Gamma_0}\right) \qquad z'(z=-l) = \left(\frac{1-\Gamma_0}{1+\Gamma_0}\right)$$

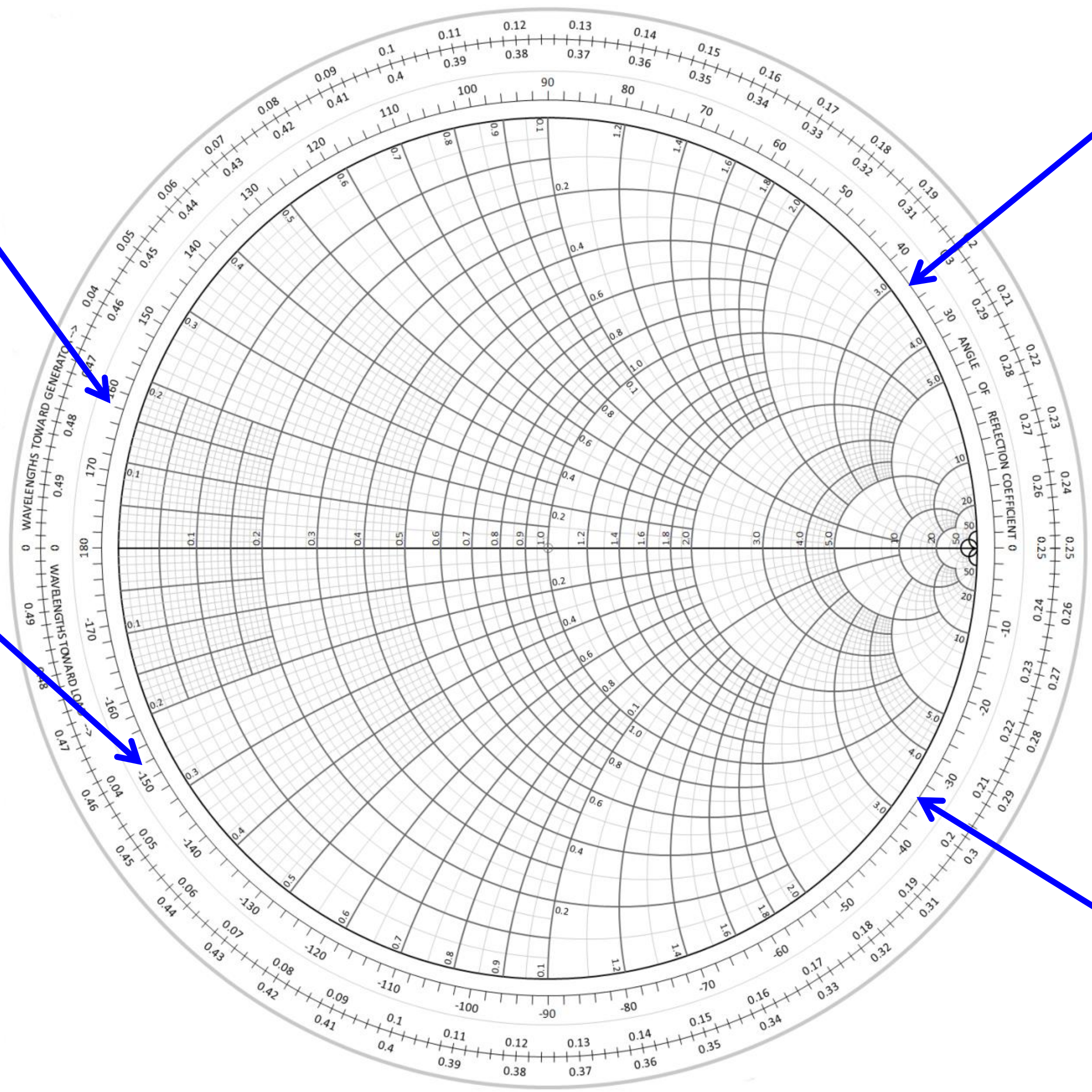
$$z'(z=0) = \frac{1}{z'(z=-l)}$$

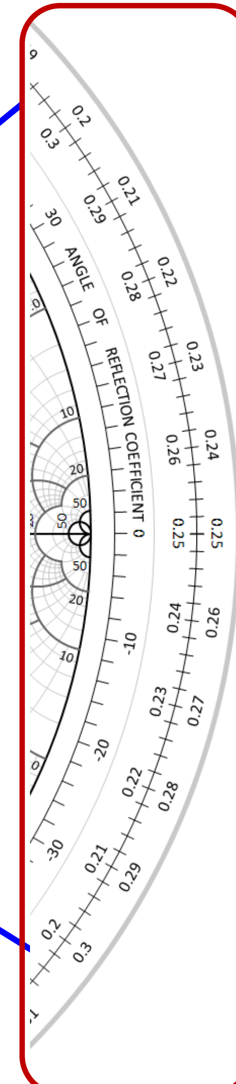
$$z'(A) = \frac{1}{z'(B)}$$

$$z'(A) = y'(B)$$

The Smith Chart – Outer Scale

Note that around the **outside** of the Smith Chart there is a scale indicating the **phase angle**, from 180^o to -180^o.





The Smith Chart – Outer Scale (contd.)

- Recall however, for a **terminated** transmission line, the reflection coefficient function is:

$$\Gamma(z)=\Gamma_0\, e^{+j2\beta z}=|\Gamma_0|\, e^{+j(2\beta z+\theta_0)}$$

- Thus, the **phase** of the reflection coefficient function depends on transmission line **position z** as:

$$\theta_\Gamma(z)=2\beta z+\theta_0 \;=2\left(\frac{2\pi}{\lambda}\right)z+\theta_0=4\pi\left(\frac{z}{\lambda}\right)+\theta_0$$

- As a result, a **change** in line position z (i.e., Δz) results in a **change** in reflection coefficient phase θ_Γ (i.e., $\Delta\theta_\Gamma$):

$$\Delta\theta_\Gamma=4\pi\left(\frac{\Delta z}{\lambda}\right)$$

- E.g., a change of position equal to one-quarter wavelength **Δz =λ/4** results in a phase change of π radians—we rotate **half-way** around the complex Γ-plane (otherwise known as the Smith Chart).

The Smith Chart – Outer Scale (contd.)

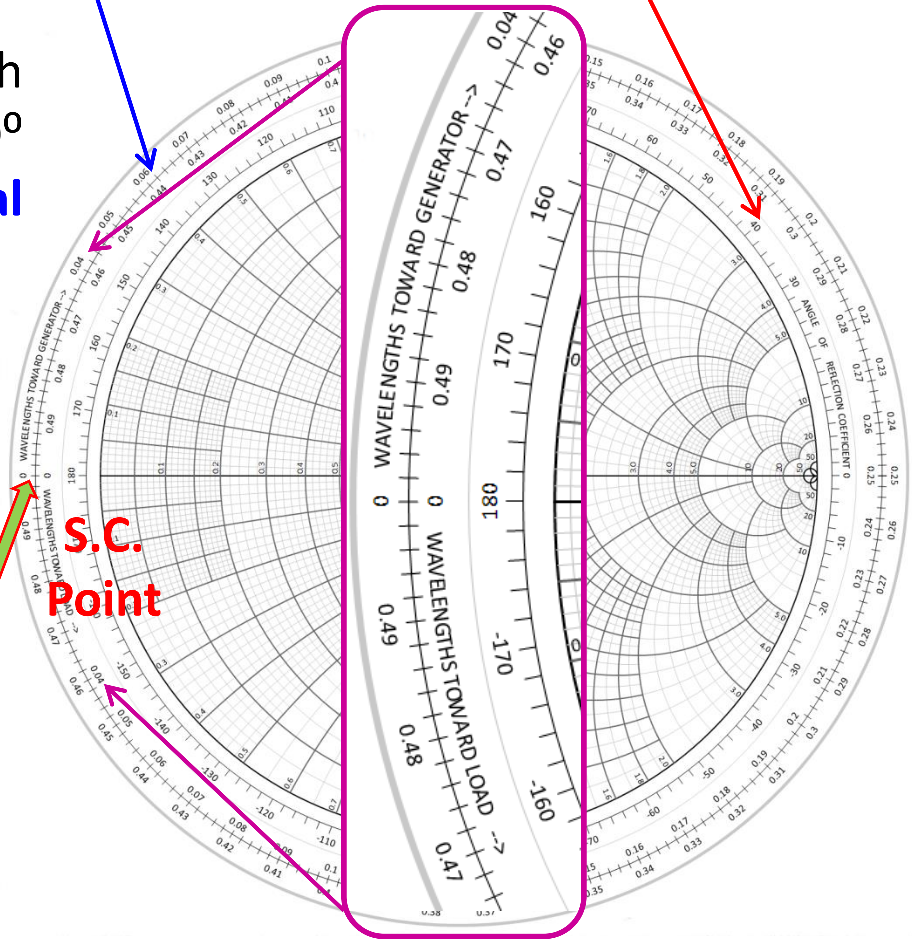
- The Smith Chart then has a **second scale** (besides θ_Γ) that surrounds it —one that relates TL position in **wavelengths** ($\Delta z/\lambda$) to the θ_Γ:

- Since the phase scale on the Smith Chart extends from $-180^o < \theta_\Gamma < 180^o$ (i.e., $-\pi < \theta_\Gamma < \pi$), this **electrical length scale** extends from:

$$0 < z/\lambda < 0.5$$

- Note, for this mapping the reflection coefficient phase at location z = 0 is $\theta_\Gamma = -\pi$. Therefore, $\theta_0 = -\pi$, and we find that:

$$\Gamma_0 = |\Gamma_0| e^{+j\theta_0} = |\Gamma_0| e^{-j\pi} = -|\Gamma_0|$$

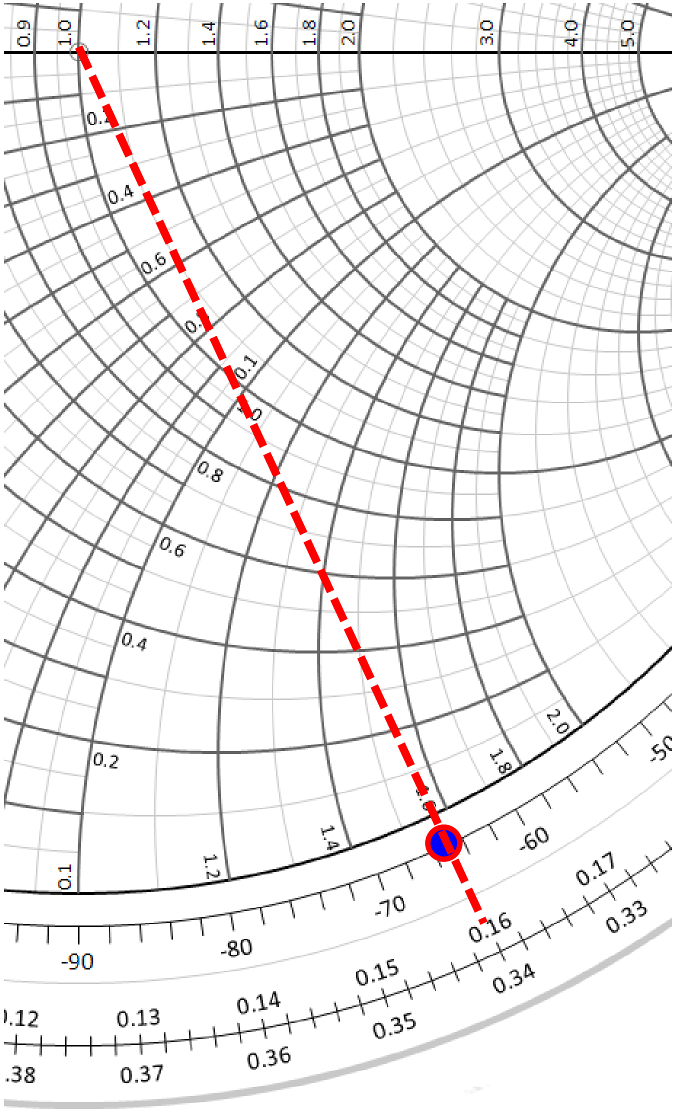


The Smith Chart – Outer Scale (contd.)

- **Example:** say you're at some location $z = z_1$ along a TL. The value of the **reflection coefficient** at that point happens to be:

$$\Gamma(z = z_1) = 0.685e^{-j65^\circ}$$

- Finding the **phase angle** of $\theta_\Gamma = -65^o$ on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is: 0.160λ



Note: this tells us **nothing** about the location $z = z_1$. This does **not** mean that $z_1 = 0.160\lambda$, for example!

The Smith Chart – Outer Scale (contd.)

- Now, say we **move a short distance** Δz (i.e., a distance less than λ/2) along the transmission line, to a **new location** denoted as $z = z_2$ and find that the **reflection coefficient** has a value of:

$$\Gamma(z = z_2) = 0.685e^{j74^\circ}$$

- Now finding the **phase angle** of $\theta_\Gamma = 74^o$ on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

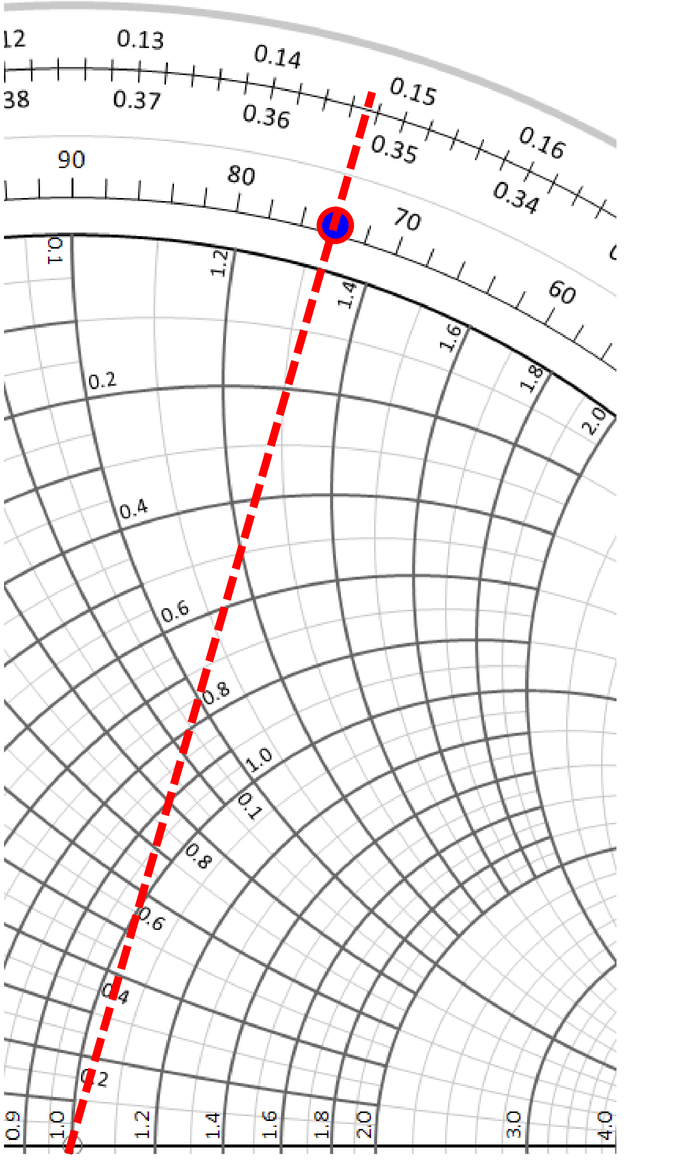
$$0.353\lambda$$

Note: this tells us **nothing** about the location $z = z_2$. This does **not** mean that $z_1 = 0.353\lambda$, for example!

Q: So what do the values 0.160λ and 0.353λ tell us?

A: They allow us to determine the **distance between** points z_2 and z_1 on the transmission line.

$$\Delta z = z_2 - z_1 = 0.353\lambda - 0.160\lambda = 0.193\lambda$$



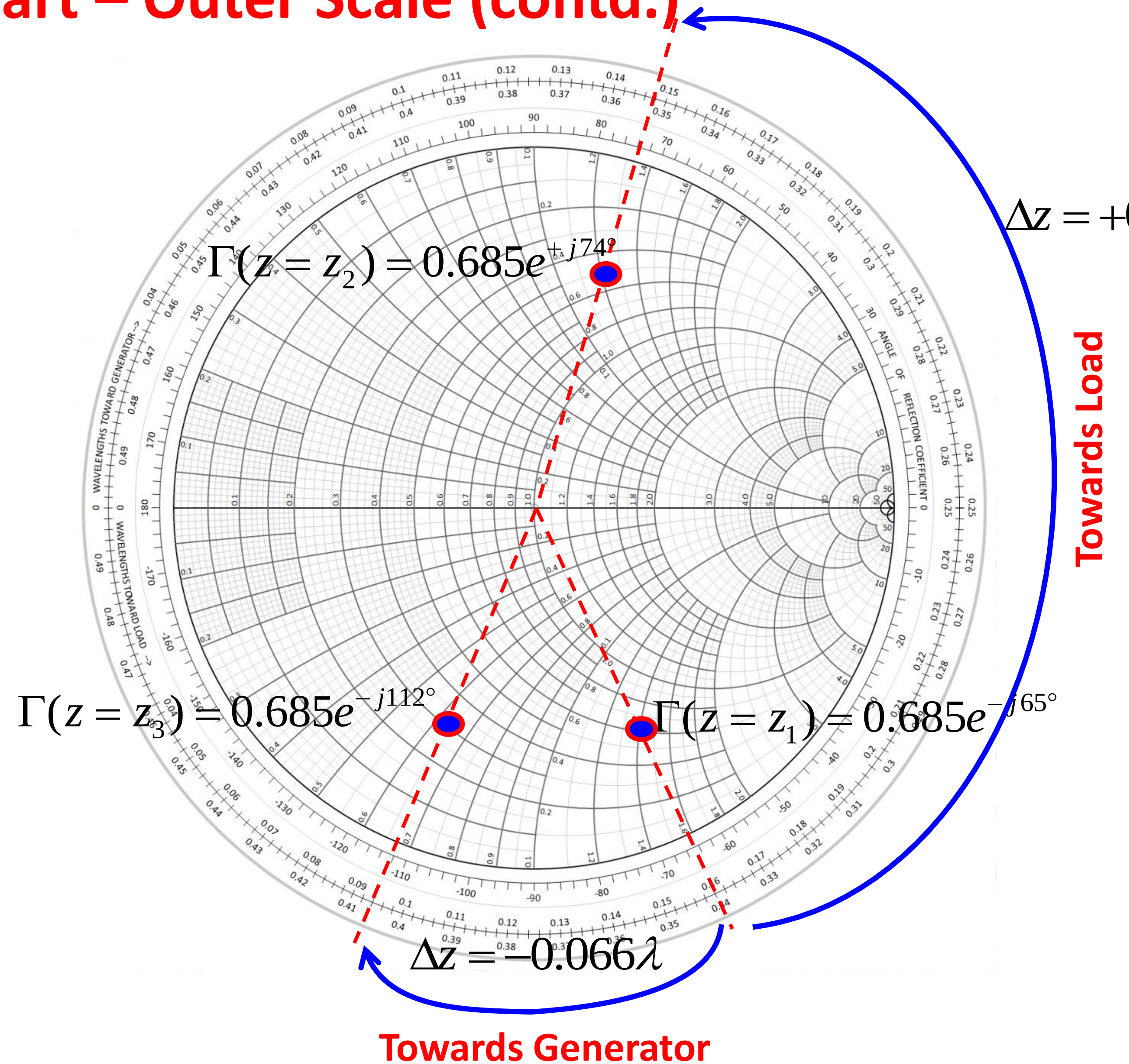
The Smith Chart – Outer Scale (contd.)

The transmission line location z_2 is a distance of 0.193λ from location z_1!

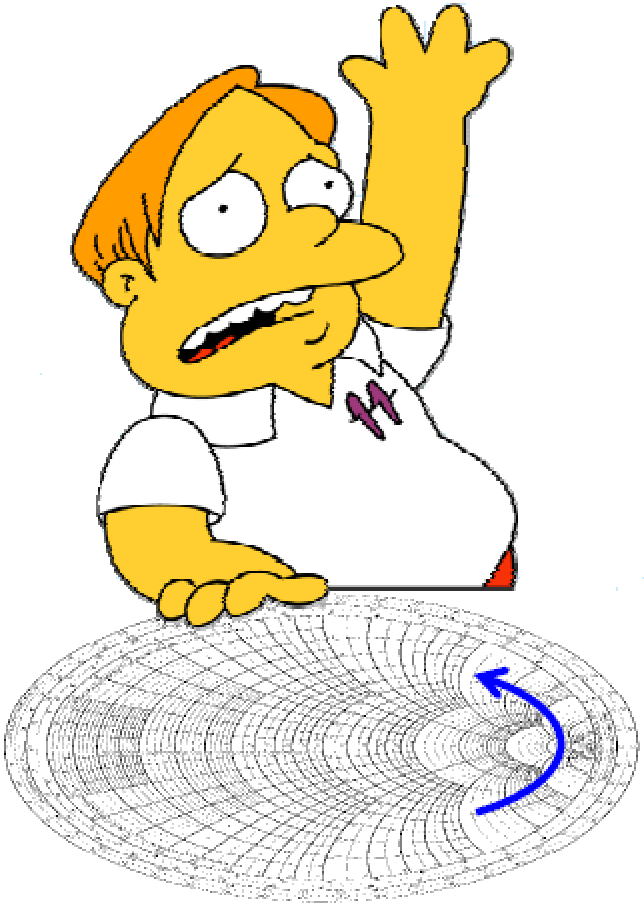
Q: But, say the reflection coefficient at some point z_3 has a phase value of $\theta_\Gamma = -112^o$, which maps to a value of 0.094λ on the outer scale of Smith chart. It gives $\Delta z = z_3 - z_1 = 0.094\lambda - 0.160\lambda = -0.066\lambda$. What does the **–ve** value mean?

- **In the first example,** $\Delta z > 0$, meaning $z_2 > z_1$ → the location z_2 is closer to the load than is location z_1
  - the **positive** value Δz maps to a phase change of $74^o - (-65^o) = 139^o$
  - In other words, as we **move toward the load** from location z_1 to location z_2, we rotate **counter-clockwise** around the Smith chart
- **In the second example,** $\Delta z < 0$, meaning $z_3 < z_1$ → the location z_3 is closer to the beginning of the TL (i.e., farther from the load) than is location z_1
  - the **negative** value Δz maps to a phase change of $-112^o - (-65^o) = -47^o$
  - In other words, as we **move away from the load (i.e, towards the generator)** from location z_1 to location z_3, we rotate **clockwise** around the Smith chart

The Smith Chart – Outer Scale (contd.)



The Smith Chart – Outer Scale (contd.)



Q: Wait! I just used a Smith Chart to analyze a TL problem in the manner you have just explained. At one point on my transmission line the phase of the reflection coefficient is $\theta_\Gamma = +170^o$, which is denoted as 0.486λ on the **"wavelengths toward load"** scale.

- I then moved a short distance along the line **toward the load**, and found that the reflection coefficient phase was $\theta_\Gamma = -144^o$, which is denoted as 0.050λ on the **"wavelengths toward load"** scale.
- According to **your** "instruction", the distance between these two points is:

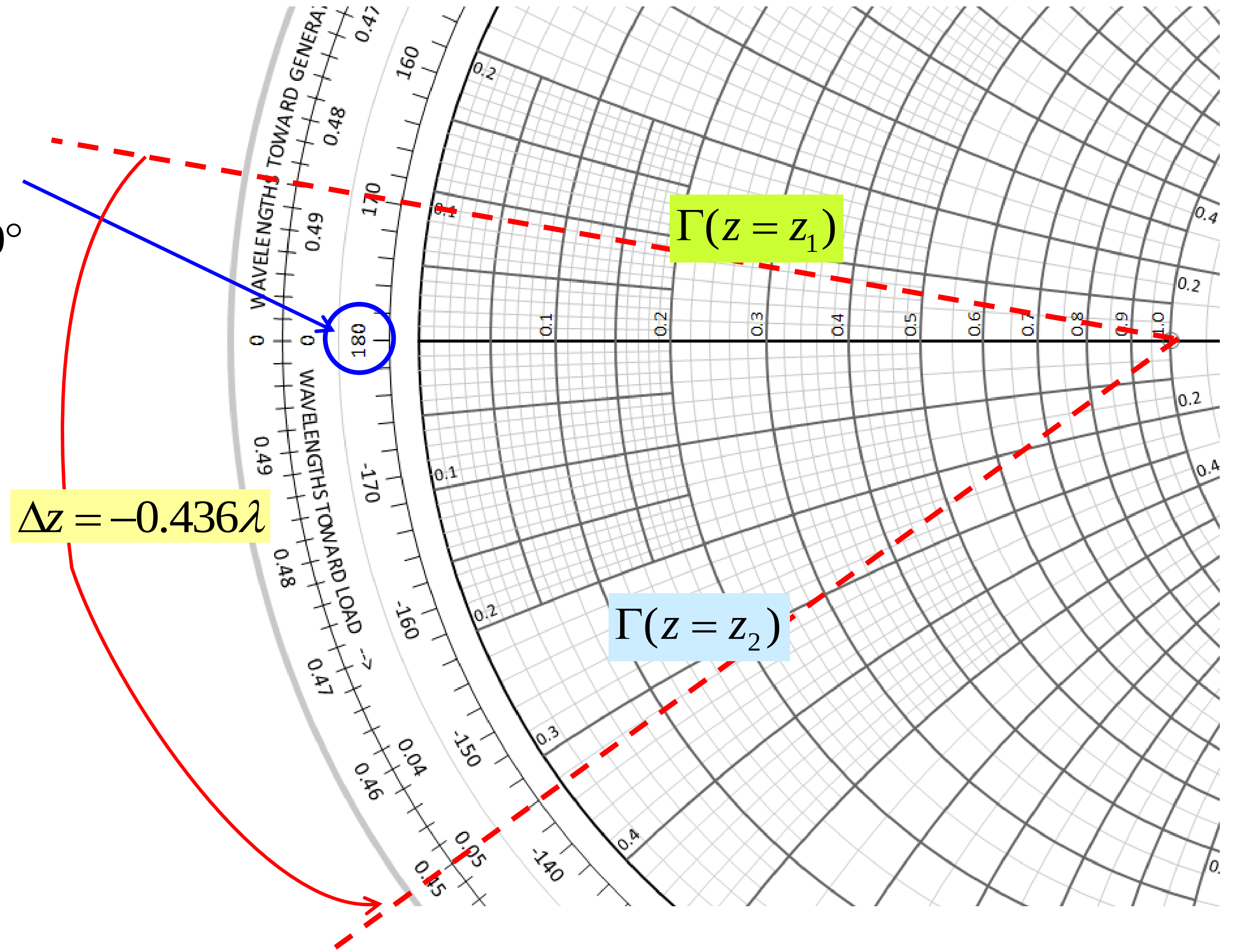
$$\Delta z = 0.050\lambda - 0.486\lambda = -0.436\lambda$$

A large **negative** value! This says that I moved nearly a half wavelength **away** from the load, but I know that I moved just a short distance **toward** the load! **What happened?**

The Smith Chart – Outer Scale (contd.)

The electrical length scales on the Smith chart begin and end where $\theta_\Gamma = \pm 180°$

In your example, when rotating counter-clockwise (i.e., moving toward the load) you **passed by this transition**. This makes the calculation of **Δz** a bit more problematic.



The Smith Chart – Outer Scale (contd.)

- As you rotate counter-clockwise around the Smith Chart, the "wavelengths toward load" scale increases in value, until it reaches a **maximum** value of 0.5λ (at $\theta_\Gamma = \pm \pi$)
- At that point, the scale "resets" to its **minimum** value of **zero**
- Thus, in such a situation, we must divide the problem into **two steps**:
- **Step 1:** Determine the electrical length from the **initial** point to the **"end"** of the scale at 0.5λ
- **Step 2:** Determine the electrical distance from the **"beginning"** of the scale (i.e., 0) and the **second location** on the transmission line
- **Add** the results of steps 1 and 2, and you have your answer!

For example, let's look at the case that originally gave us the erroneous result. The distance from the initial location to the **end of the scale** is:

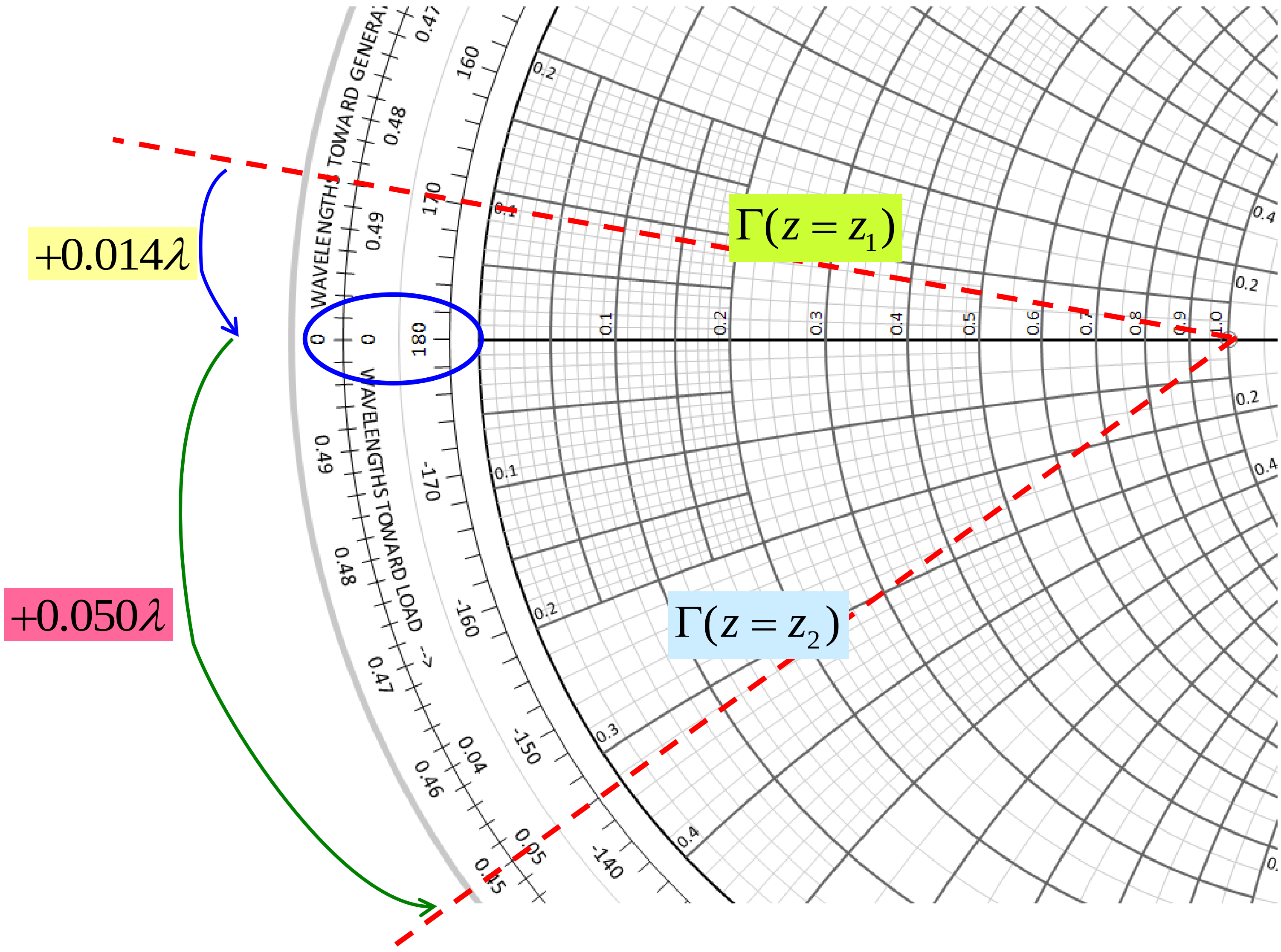
$$0.500\lambda - 0.486\lambda = +0.014\lambda$$

And the distance from the **beginning of the scale** to the second point is:

$$0.050\lambda - 0.000\lambda = +0.050\lambda$$

Thus the distance between the two points is: $+0.014\lambda + 0.050\lambda = +0.064\lambda$

The Smith Chart – Outer Scale (contd.)

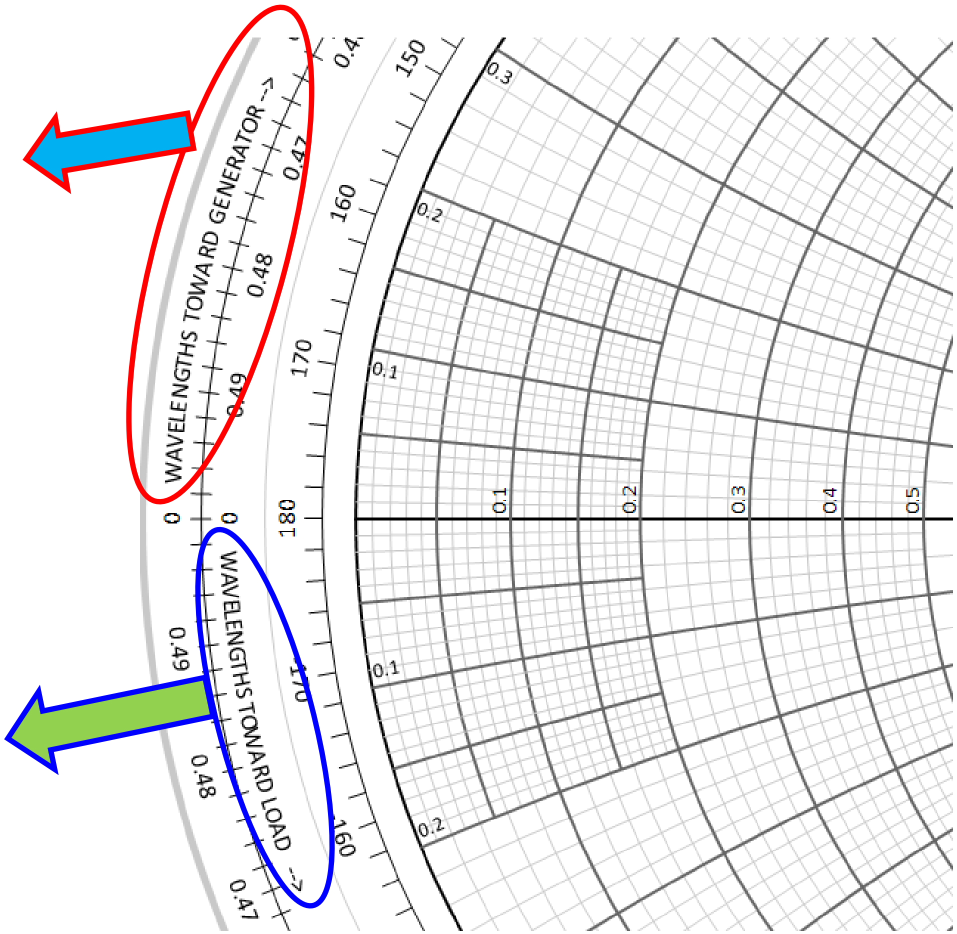


The Smith Chart – Outer Scale (contd.)

- The Δz towards generator could also be mentioned as a +ve term if we consider the upper metric in the “Outer Scale”

Clockwise Rotation

- **gives +ve distance when moving towards generator**
- **gives –ve distance when moving towards load**



Counter-clockwise Rotation

- **gives -ve distance when moving towards generator**
- **gives +ve distance when moving towards load**