Lecture – 3 Date: 12.01.2017

- Lossless Transmission Lines
- Special Termination Conditions

Lossless Transmission Line

- For a lossless transmission line:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$\beta = \omega\sqrt{LC}$$

- Similarly the current phasor for a lossless line can be described:

$$\therefore I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

$$Z_0 = \frac{\omega L}{\beta} = \frac{\omega L}{\omega\sqrt{LC}} = \sqrt{\frac{L}{C}}$$

Q: Z_0 and β are determined from L, C, and ω. How do we find V_0^+ *and* V_0^-?

A: Apply **Boundary Conditions**!

Every transmission line has **2** "boundaries":

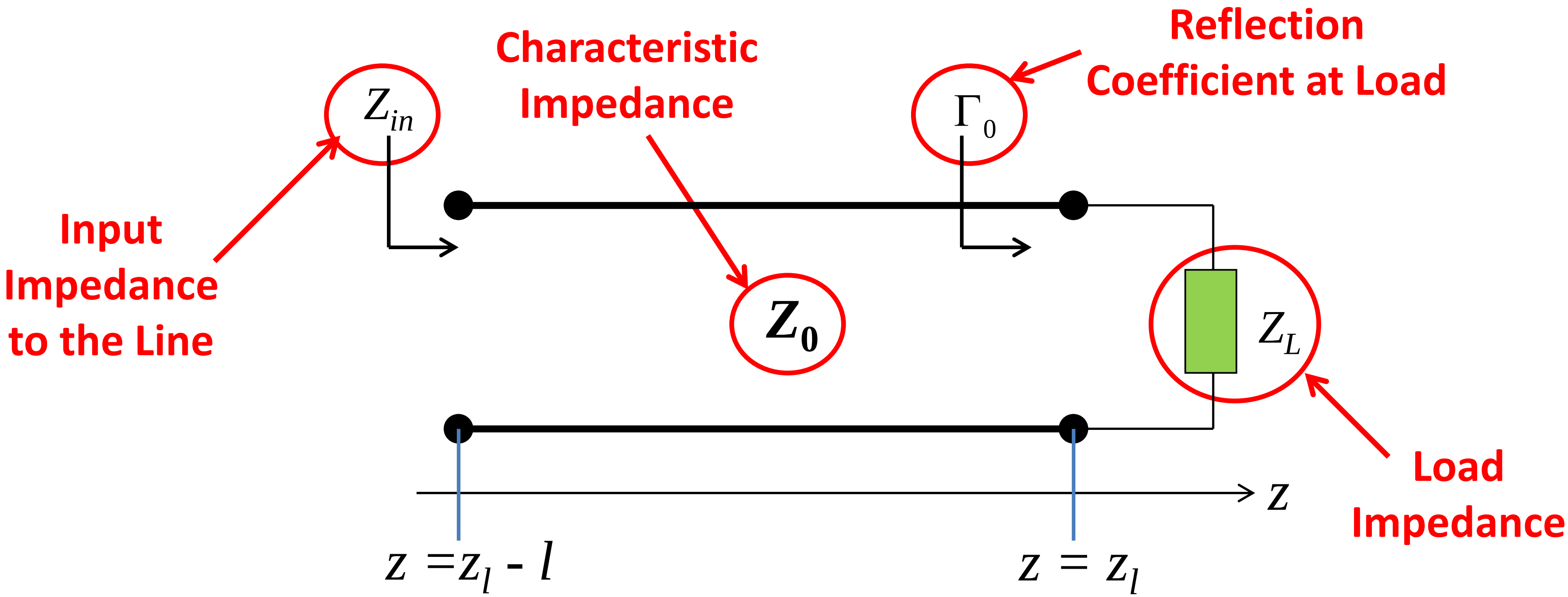
1) At one end of the transmission line.

2) At the **other** end of the trans line!

Typically, there is a **source** at one end of the line, and a **load** at the other.

Terminated Lossless Transmission Line

- Now let's **attach** something to our transmission line. Consider a **lossless** line, length l, terminated with a **load** Z_l.

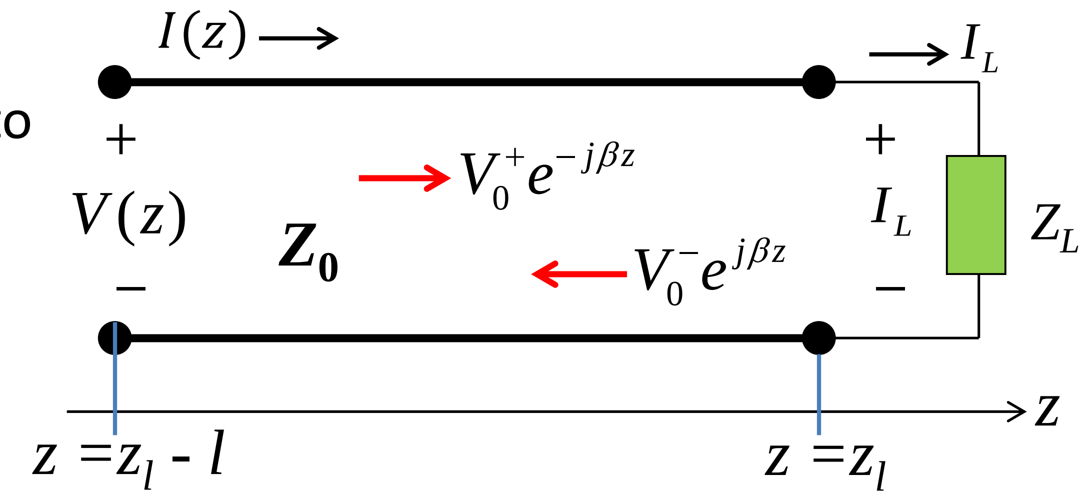


Q: What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points z where $z_l - l < z < z_l$.

A: To find out, we must apply **boundary conditions**!

Terminated Lossless Transmission Line (contd.)

- The load is assumed at $z = z_l$
- The voltage wave couples into the line at $z = z_l - l$



$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

Incident Wave   Reflected Wave

- **At the load:** $$V(z = z_l) = V^+(z = z_l) + V^-(z = z_l) = V_0^+ e^{-j\beta z_l} + V_0^- e^{j\beta z_l}$$

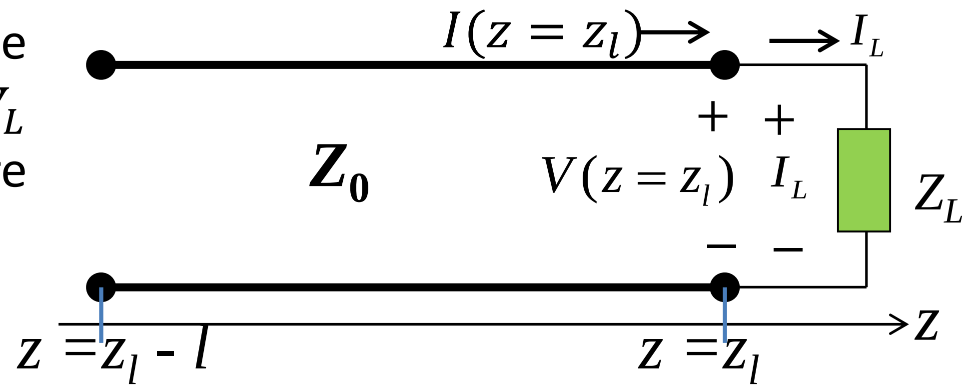
$$I(z = z_l) = \frac{V^+(z = z_l)}{Z_0} - \frac{V^-(z = z_l)}{Z_0} = \frac{V_0^+}{Z_0} e^{-j\beta z_l} - \frac{V_0^-}{Z_0} e^{j\beta z_l}$$

- Furthermore, the load voltage and current must be related by **Ohm's law**:

$$V_L = Z_L I_L$$

Terminated Lossless Transmission Line (contd.)

- Most importantly, recognize that the values $I(z=z_l), V(z=z_l)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!



$$I(z=z_l) = I_L \qquad V(z=z_l) = V_L$$

So now we have the **boundary conditions** for **this** particular problem.

Careful! Different transmission line problems lead to **different** boundary conditions—**you** must assess each problem **individually** and **independently**!

- **Combining** these equations and boundary conditions, we find that:

$$V(z=z_l) = V_L = Z_L I_L = Z_L I(z=z_l)$$

$$V^+(z=z_l) + V^-(z=z_l) = \frac{Z_L}{Z_0}\left(V^+(z=z_l) - V^-(z=z_l)\right)$$

Terminated Lossless Transmission Line (contd.)

- Rearranging, we can conclude:

$$\frac{V^{-}(z=z_l)}{V^{+}(z=z_l)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

also holds true for current wave but with opposite sign

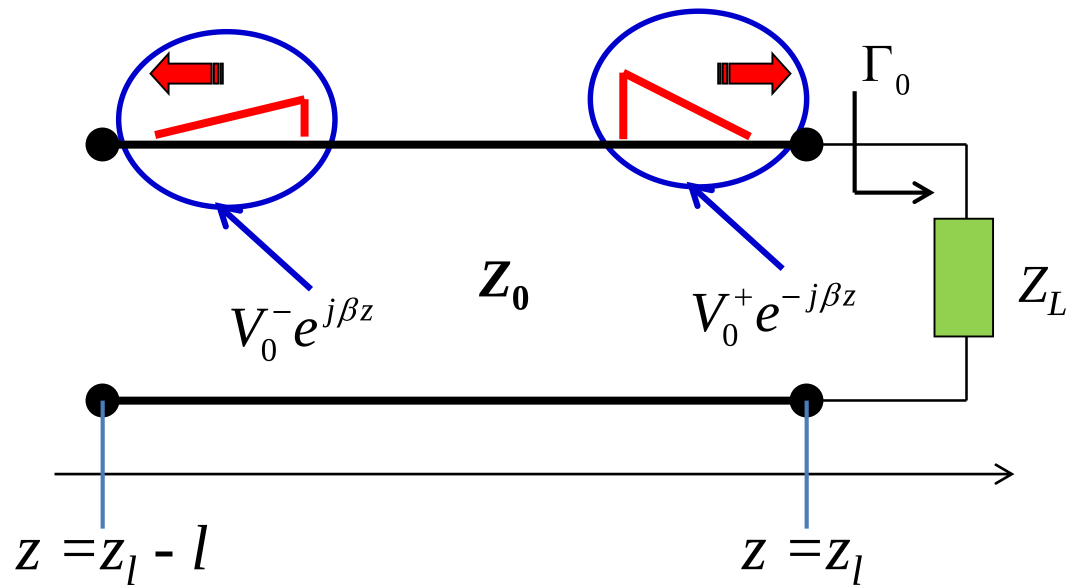
Voltage Reflection Coefficient $\Gamma(z = z_l)$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol (Γ_0)!

$$\Gamma_0 = \frac{V^{-}(z=z_l)}{V^{+}(z=z_l)}$$

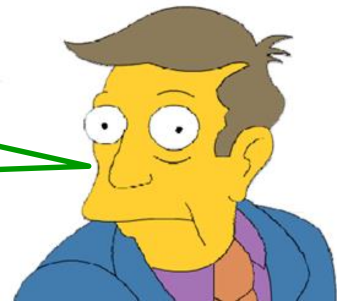
$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

More useful representation as it involves known circuit/system quantities



Terminated Lossless Transmission Line (contd.)

Q: I'm confused! Just what are we trying to accomplish in this handout?



A: We are trying to find V(z) and I(z) when a lossless transmission line is terminated by a load Z_L!

- We can express the reflected voltage wave as:

$$\Gamma_0 = \frac{V^-(z=z_l)}{V^+(z=z_l)} = \frac{V_0^- e^{+j\beta z_l}}{V_0^+ e^{-j\beta z_l}} \quad \Rightarrow \quad V_0^- = \Gamma_0 V_0^+ e^{-j2\beta z_l}$$

- Therefore:

$$V^-(z) = \left(\Gamma_0 V_0^+ e^{-j2\beta z_l}\right) e^{+j\beta z}$$

$$V(z) = V^+(z) + V^-(z) = V_0^+ \left[e^{-j\beta z} + \left(\Gamma_0 e^{-j2\beta z_l}\right) e^{+j\beta z} \right]$$

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0} = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \left(\Gamma_0 e^{-j2\beta z_l}\right) e^{+j\beta z} \right]$$

Terminated Lossless Transmission Line (contd.)

- **Simplify** by **arbitrarily** assigning the end point a **zero** value (i.e., $z_l = 0$)

$$V(z=0) = V^+(z=0) + V^-(z=0) = V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} = V_0^+ + V_0^-$$

$$I(z=0) = \frac{V_0^+ - V_0^-}{Z_0}$$

$$Z(z=0) = \frac{V(z=0)}{I(z=0)} = Z_0\left[\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right] = Z_L$$

- The current and voltage along the line in this case are:

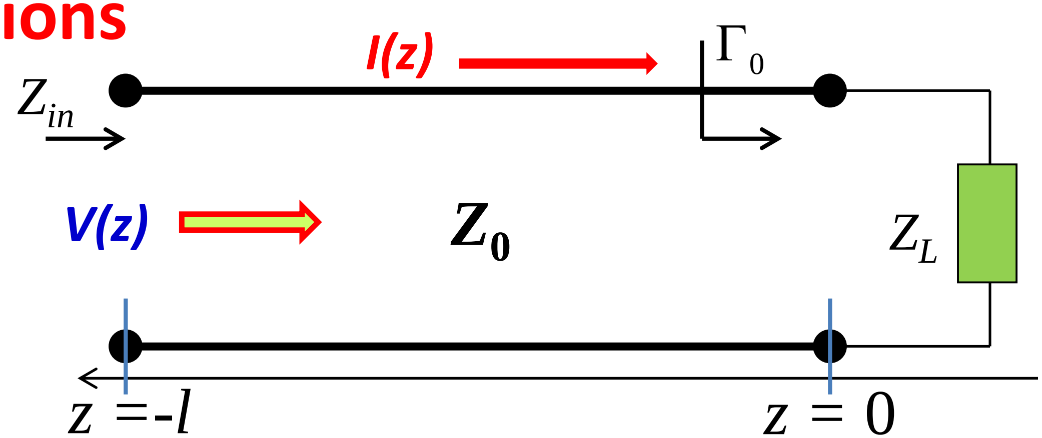
$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma_0 e^{+j\beta z}\right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma_0 e^{+j\beta z}\right]$$

Q: But, how do we determine V_0^+??

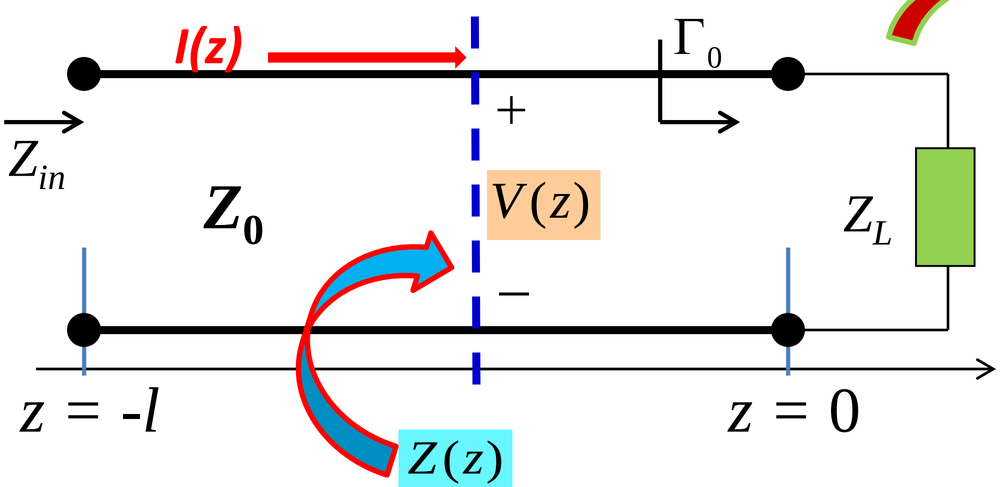
Special Termination Conditions

- Let us once again consider a generic TL terminated in arbitrary impedance Z_L



- It's interesting to note that **Z_L** enforces a boundary condition that explicitly determines neither **V(z)** nor **I(z)**—but **completely** specifies **line impedance** Z(z)!

$$Z(z) = \frac{V(z)}{I(z)}$$



$$Z(z) = \frac{V(z)}{I(z)} = \frac{V^+ e^{+j\beta z}\left(1+\Gamma_0 e^{-j(2\beta z)}\right)}{\frac{V^+ e^{+j\beta z}}{Z_0}\left(1-\Gamma_0 e^{-j(2\beta z)}\right)}$$

Special Termination Conditions (contd.)

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_0 e^{+j\beta z}}{e^{-j\beta z} - \Gamma_0 e^{+j\beta z}} = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

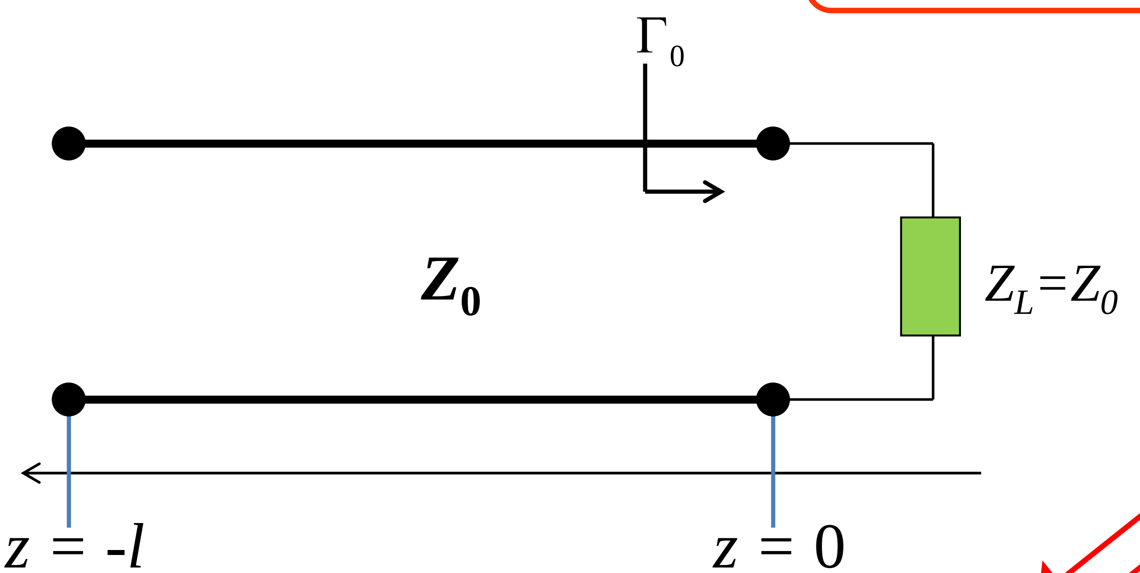
- Likewise, the load boundary condition leaves $V^+(z)$ and $V^-(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(\boldsymbol{z})$!

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \Gamma_0 e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L - Z_0} e^{+j2\beta z}$$

Let's look at some **specific** values of load impedance $Z_L = R_L + jX_L$ and see what functions $Z(z)$ and $\Gamma(\boldsymbol{z})$ result!

Special Termination Conditions (contd.)

- $Z_L = Z_0$ ⇐ **Matched Line** ⇒ the load impedance equals the characteristic impedance of the TL



The load reflection coefficient:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$

means no reflected wave $V^-(z)$

reflection coefficient is zero at all points along the line

The impedance at position z:

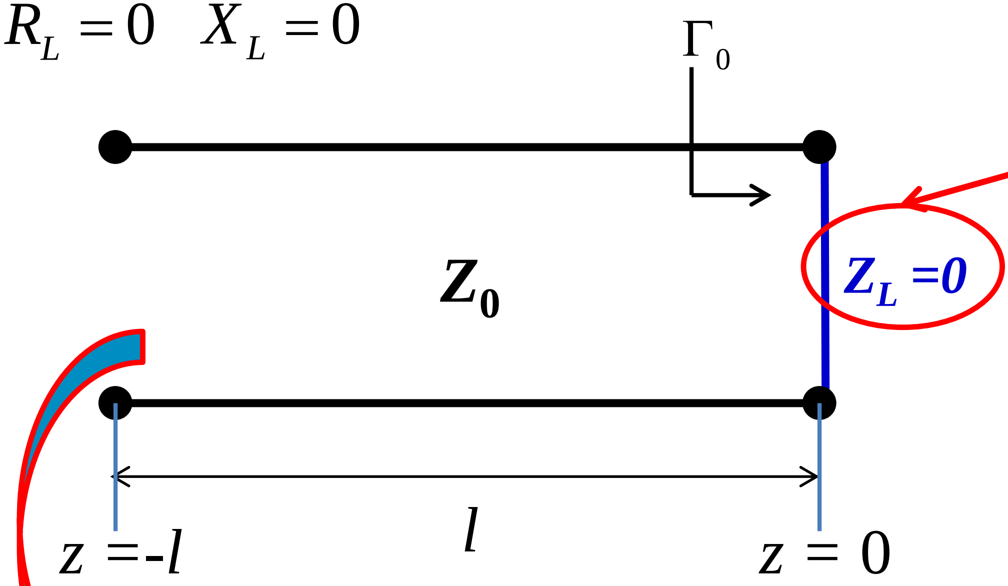
$$Z(z) = Z_0$$

The line impedance equals Z_0 → matched condition

Special Termination Conditions (contd.)

- $Z_L = 0$ ⇐ **Short-Circuited Line** ⇒ A device with no load is called short circuit

$R_L = 0 \quad X_L = 0$



Short-circuit entails setting this impedance to zero

$$\Gamma_0 = \frac{0 - Z_0}{0 + Z_0} = -1$$

$$Z(z) = -jZ_0 \tan(\beta z)$$

Alternatively ⇒

$$Z(z) = -jZ_0 \tan\left(\frac{2\pi z}{\lambda}\right)$$

Note that this impedance is **purely reactive**. This means that the current and voltage on the transmission line will be everywhere 90° **out of phase**.

Special Termination Conditions (contd.)

- **Short-Circuited Line**

- The current and voltage along the TL is:

$$V(z) = V_0^+\left[e^{-j\beta z} - e^{+j\beta z}\right] = -j2V_0^+\sin(\beta z)$$

$$I(z) = \frac{2V_0^+}{Z_0}\cos(\beta z)$$

- Finally, the reflection coefficient **function** is:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{-V_0^+e^{+j\beta z}}{V_0^+e^{-j\beta z}} = -e^{j2\beta z} \Rightarrow |\Gamma(z)| = 1 \Rightarrow \left|V^-(z)\right| = \left|V^+(z)\right|$$

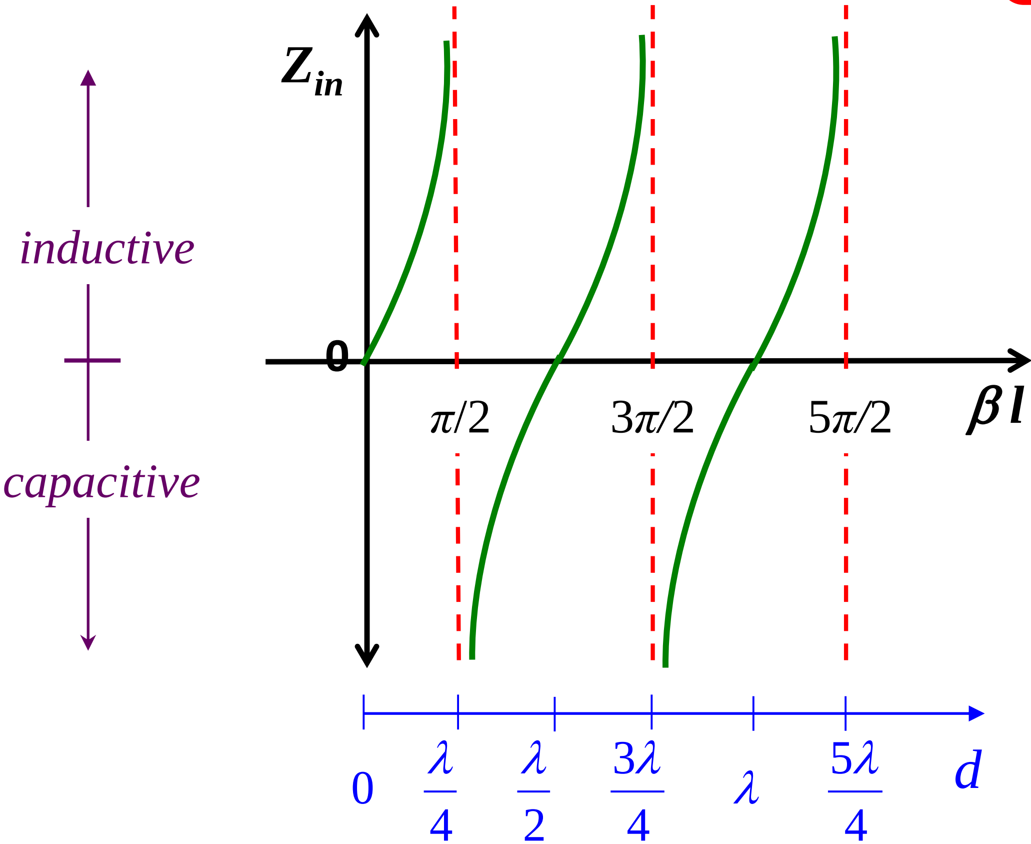
In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

- **Short-Circuited Line:** $Z(-l) = jZ_0\tan(\beta l)$

Special Termination Conditions (contd.)

- **Short-Circuited Line**

$$Z(-l) = jZ_0 \tan(\beta l)$$



It can be observed:

- **At -*l*=0,** the impedance is zero (short-circuit condition)
- **Increase in -*l*** leads to inductive behavior
- **At -*l*=λ/4,** the impedance equals infinity (open-circuit condition)
- **Further increase in -*l*** leads to capacitive behavior
- **At -*l*=λ/2,** the impedance becomes zero (short-circuit condition)
- **The entire periodic sequence repeats for -*l*>λ/2 and so on…**

HW#1: Demonstrate this behavior using ADS

Example – 1

For a short-circuited TL of l = 10 cm, compute the magnitude of the input impedance when the frequency is swept from f = 1 GHz to 4 GHz. Assume the line parameters L = 209.4 nH/m and C = 119.5 pF/m.

Solution:

HW # 1

$$Z_0 = \sqrt{L/C} = \sqrt{(209.4*0.1)/(119.5*0.5)} = 41.86\Omega$$

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(209.4*0.1)*(119.5*0.5)} = 41.86\Omega} = 1.99*10^8 m/s$$

$$Z(z=-l) = jZ_0\tan(\beta l) = jZ_0\tan\left(\frac{2\pi f}{v_p}l\right)$$

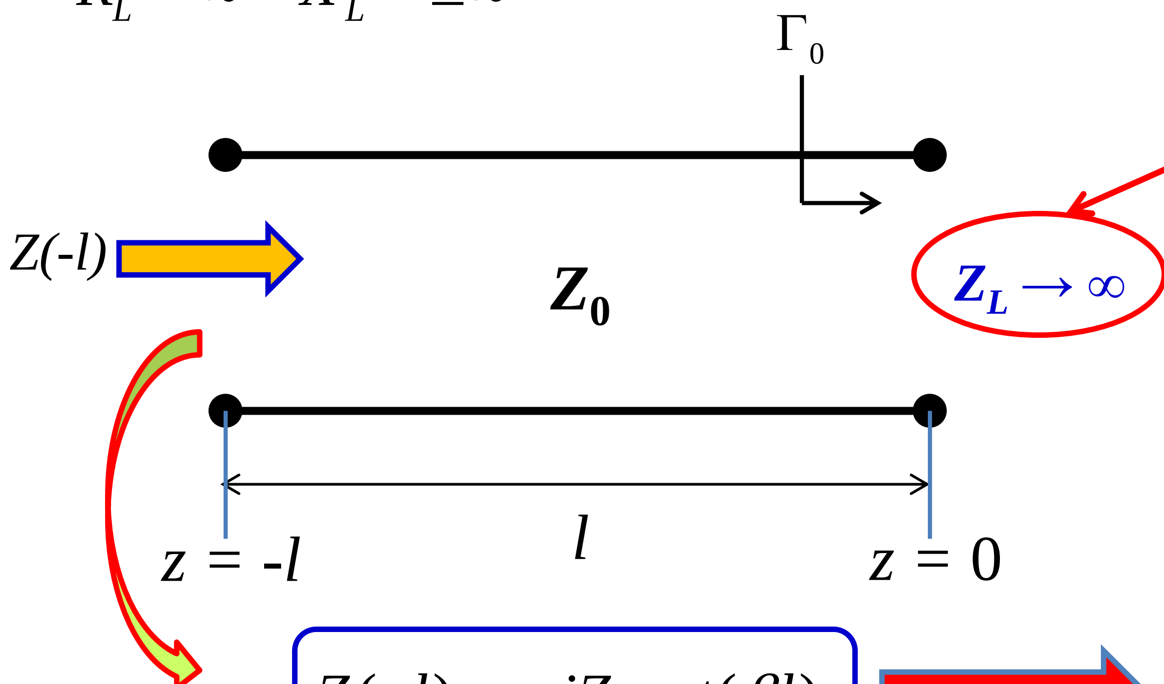
Set l = 10 cm and then write a MATLAB program to obtain the Z_{in} curve

Compare the MATLAB results to that obtained from ADS simulation

Special Termination Conditions (contd.)

- $Z_L \to \infty$ ⟸ **Open-Circuited Line** ⟹ A device with infinite load is called open-circuit

$$R_L = \infty \quad X_L = \pm\infty$$



Open-circuit entails setting this impedance to infinite

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

$$Z(-l) = -jZ_0 \cot(\beta l)$$

Alternatively

$$Z(-l) = -jZ_0 \cot\left(\frac{2\pi l}{\lambda}\right)$$

Again note that this impedance is **purely reactive**. current and voltage on the transmission line are 90° **out of phase**.

Special Termination Conditions (contd.)

- **Open-Circuited Line**

- The current and voltage along the TL is:

$$V(z) = V_0^+\left[e^{-j\beta z} + e^{+j\beta z}\right] = 2V_0^+\cos(\beta z)$$

$$I(z) = -j\frac{2V_0^+}{Z_0}\sin(\beta z)$$

- At the load, $z = 0$, therefore:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{j2\beta z}$$

$$I(0) = 0$$

As expected, the current is zero at the end of the transmission line (i.e. the current through the open). Likewise, the voltage at the end of the line (i.e., the voltage across the open) is at a maximum!

- Finally, the reflection coefficient **function** is:

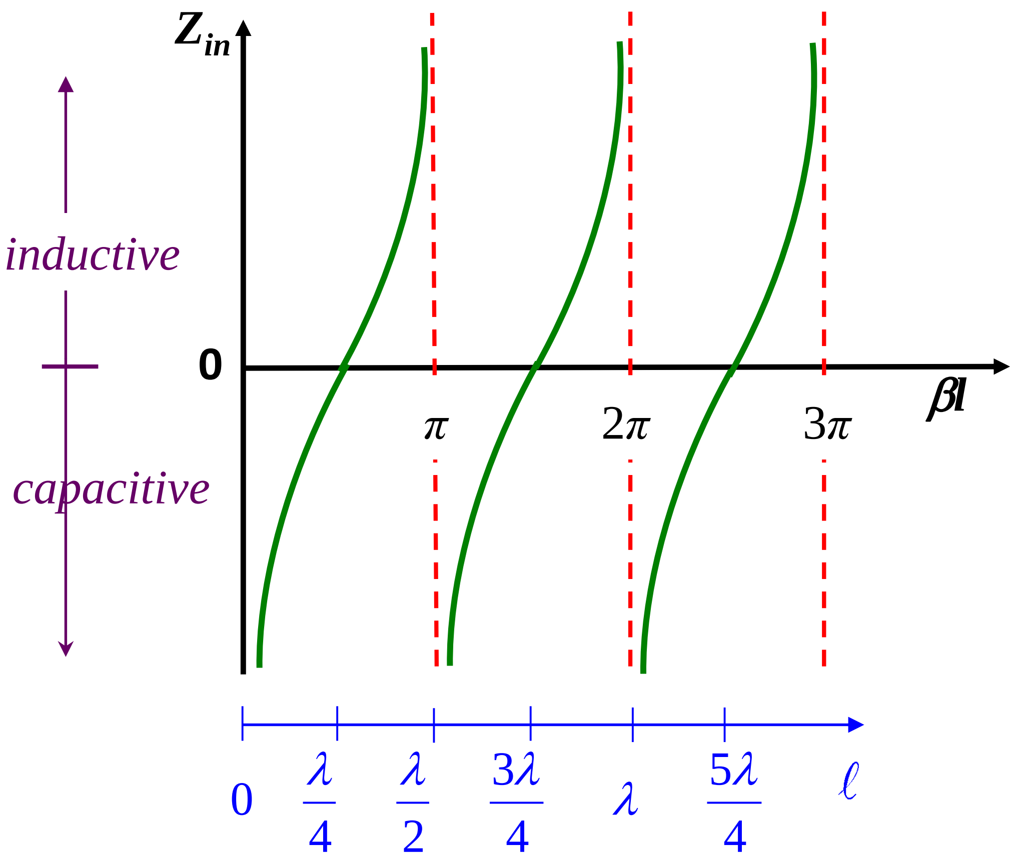
$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{j2\beta z}$$

$\Rightarrow$ $|\Gamma(z)| = 1$ $\Rightarrow$ $\left|V^-(z)\right| = \left|V^+(z)\right|$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

Special Termination Conditions (contd.)

- **Open-Circuited Line**



$$Z(-l) = -jZ_0 \cot(\beta l)$$

It can be observed:

- **At -*l*=0,** the impedance is infinite (open-circuit condition)
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- **The entire periodic sequence repeats for -*l* >λ/2 and so on…**

HW#1: Demonstrate this behavior using ADS

Transmission Line Input Impedance

Q: Just what do you mean by **input** impedance?

A: The input impedance is simply the line impedance seen at the **beginning** $(\boldsymbol{z = -l})$ of the transmission line, i.e.:

$$Z_{in} = Z(z=-l) = \frac{V(z=-l)}{I(z=-l)}$$

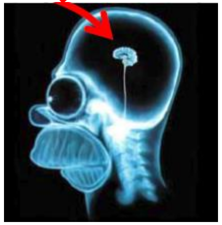
Note Z_{in} equal to **neither** the load impedance Z_L **nor** the characteristic impedance Z_0!

$$Z_{in} \neq Z_L \qquad\qquad Z_{in} \neq Z_0$$

- We know the line impedance of a lossless TL loaded with an arbitrary load impedance is:

$$Z(z) = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

The input impedance can be radically different from load impedance (Z_L) and you should commit it to memory

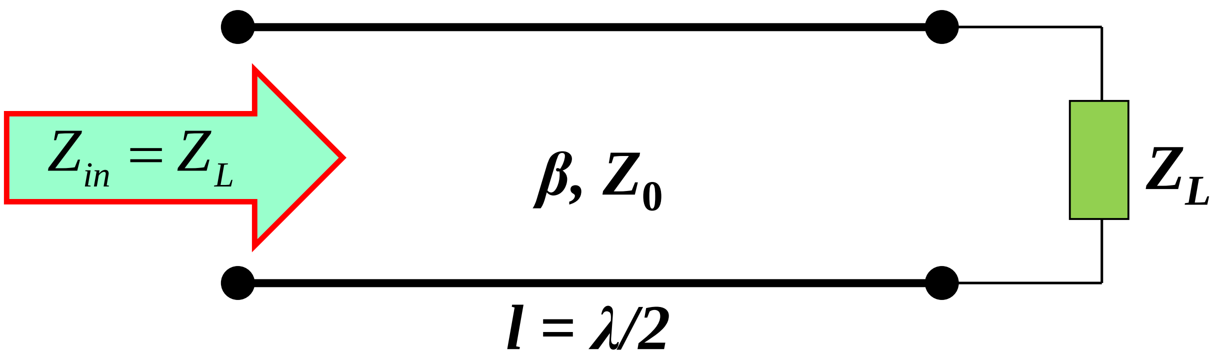


Transmission Line Input Impedance – Special Cases

1. length of the line is $l = m(\lambda/2)$

$$Z_{in} = Z(z = \lambda/2) = Z_0 \frac{Z_L + jZ_0 \tan\left(\frac{2\pi}{\lambda}.\frac{\lambda}{2}\right)}{Z_0 + jZ_L \tan\left(\frac{2\pi}{\lambda}.\frac{\lambda}{2}\right)} = Z_L$$

- For a transmission line of half wavelength long the input impedance equals the load impedance irrespective of the characteristic impedance of the line
- It means it is possible to design a circuit segment where the transmission line's characteristic impedance plays no role (obviously the length of line segment has to equal half wavelength at the operating frequency)

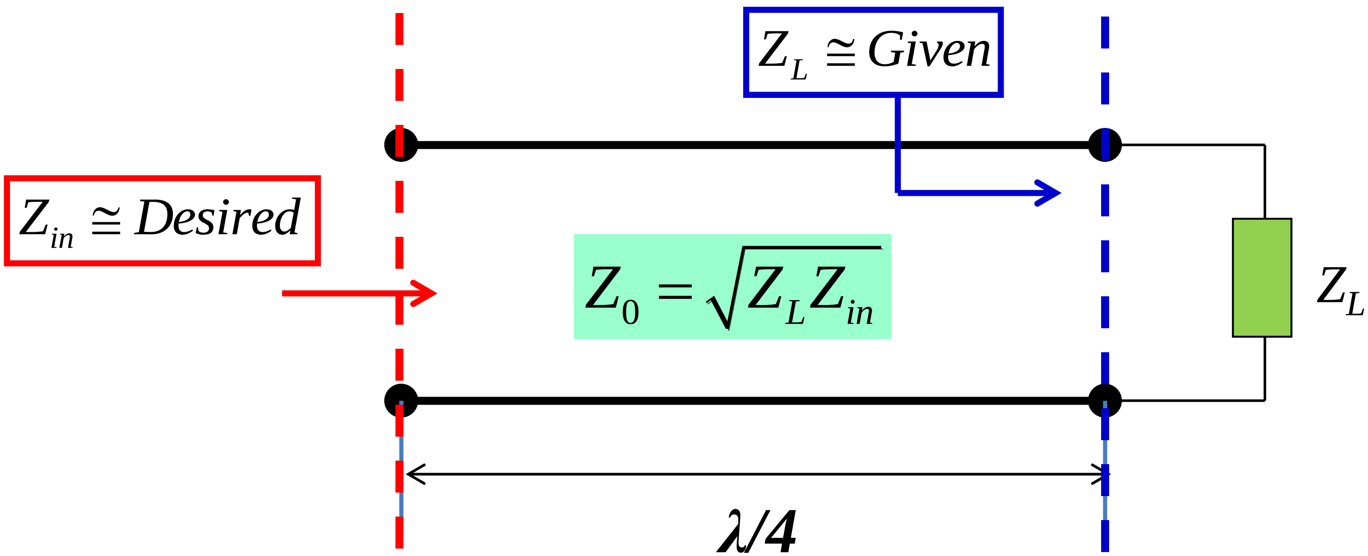


Transmission Line Input Impedance – Special Cases (contd.)

2. **length of the line is $l = \lambda/4$ or $\lambda/4 + m(\lambda/2)$**

$$Z_{in} = Z(l = \lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\right)}{Z_0 + jZ_L \tan\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\right)} = \frac{Z_0^2}{Z_L}$$

- This result implies that a transmission line segment can be used to synthesize matching of any desired real input impedance (Z_{in}) to the specified real load impedance (Z_L)



This is known as quarter-wave impedance transformer

Transmission Line Input Impedance – Special Cases (contd.)

2. length of the line is $l = \lambda/4$ or $\lambda/4 + m(\lambda/2)$

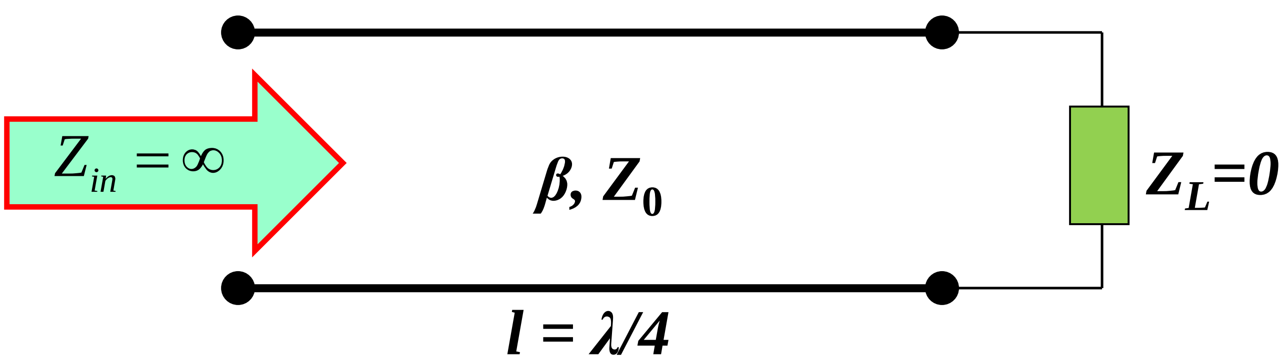
$$Z_{in} = \frac{Z_0^2}{Z_L}$$

input impedance of a quarter-wave line is inversely proportional to the load impedance

→ Think about what this means! Say the load impedance is a short circuit then:

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{Z_0^2}{0} = \infty$$

$Z_{in} = \infty$! This is an open circuit ! The quarter wave TL transforms a short-circuit into open-circuit and vice-versa



Example – 2

- Consider a load resistance $R_L = 100\Omega$ to be matched to a 50Ω line with a quarter-wave transformer. Find the characteristic impedance of the matching section and plot the magnitude of the reflection coefficient versus normalized frequency, f/f_0, where f_0 is the frequency at which the line is $\lambda/4$ long.

  - the necessary characteristic impedance is:

$$Z_0 = \sqrt{Z_L Z_{in}} \quad \Rightarrow \quad \therefore Z_0 = \sqrt{Z_L Z_{in}} = \sqrt{50 \times 100} = 70.71\Omega$$

  - The reflection coefficient magnitude is given as

$$|\Gamma_0| = \left| \frac{Z_0 - Z_{in}}{Z_0 + Z_{in}} \right|$$

$\Leftarrow$ Z_{in} is dependent on frequency

Example – 2 (contd.)

$$Z_{in} = Z(l = \lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

$$\beta l = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda_0}{4}\right) = \left(\frac{2\pi f}{v_p}\right)\left(\frac{v_p}{4 f_0}\right) = \frac{\pi f}{2 f_0}$$

For higher frequencies the matching section looks electrically longer, and for lower frequencies it looks shorter.

Plot the magnitude of the reflection coefficient versus f/f_0 using these two equations

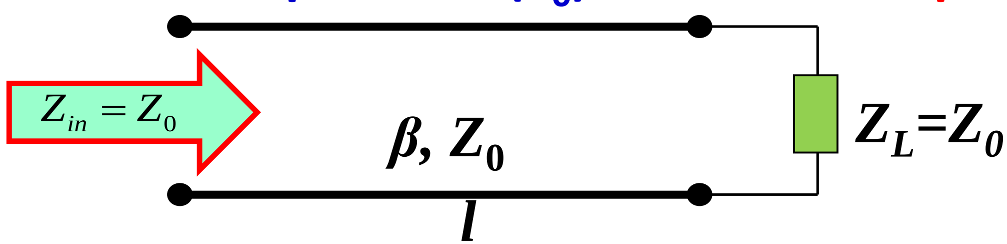
HW # 1

Transmission Line Input Impedance – Special Cases (contd.)

- $Z_L = Z_0$ — the load is **numerically equal** to the characteristic impedance of the transmission line (a real value).

$$Z_{in} = Z_0 \frac{Z_0 + jZ_0 \tan(\beta l)}{Z_0 + jZ_0 \tan(\beta l)} = Z_0$$

In other words, if the **load impedance (Z_L)** is **equal** to the TL **characteristic impedance (Z_0)**, the **input impedance (Z_{in})** likewise will be equal to **characteristic impedance (Z_0)** of the TL **irrespective of its length**



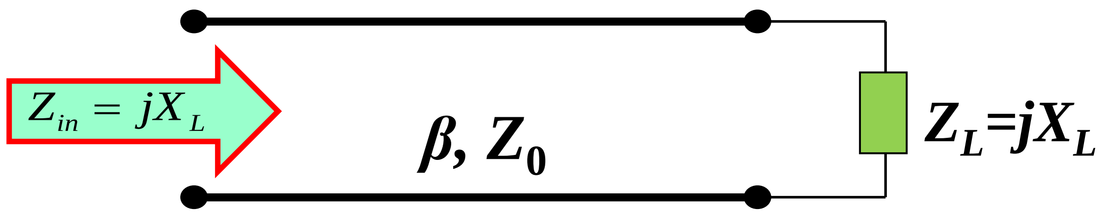
- $Z_L = jX_L$ — the load is **purely reactive** (i.e., the resistive component is zero)

$$Z_{in} = Z(z=-l) = Z_0 \frac{jX_L + jZ_0 \tan(\beta l)}{Z_0 - X_L \tan(\beta l)} = jZ_0 \frac{X_L + Z_0 \tan(\beta l)}{Z_0 - X_L \tan(\beta l)}$$

→ **Purely Reactive**

Transmission Line Input Impedance – Special Cases (contd.)

In other words, if the **load impedance (Z_L) is purely reactive** then the **input impedance likewise will be purely reactive irrespective of the line length (*l*)**



Note that the **opposite is not true: even if the load is purely resistive (Z_L = R), the input impedance will be complex** (both resistive and reactive components).

- ***l* << λ** ⟵ the transmission line is **electrically small**

- If length l is small with respect to signal wavelength λ then:

$$\beta l = \frac{2\pi}{\lambda} l = 2\pi \frac{l}{\lambda} \approx 0$$

- Therefore: $\cos(\beta l) = 1$, $\sin(\beta l) = 0$

- Thus the input impedance is:

$$Z_{in} = Z_0 \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)} = Z_0 \frac{Z_L(1) + jZ_0(0)}{Z_0(1) + jZ_L(0)} = Z_0$$

Transmission Line Input Impedance – Special Cases (contd.)

In other words, if the transmission line length is much smaller than a wavelength, the **input** impedance Z_{in} will **always** be equal to the **load** impedance Z_L.

This is the assumption we used in all previous circuits courses (e.g., Linear Circuits, Digital Circuits, Integrated Electronics, Analog Circuit Design etc.)! In those courses, we assumed that the signal frequency ω is relatively **low**, such that the signal wavelength λ is **very large** $(\lambda \gg l)$.

- Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the **same**!

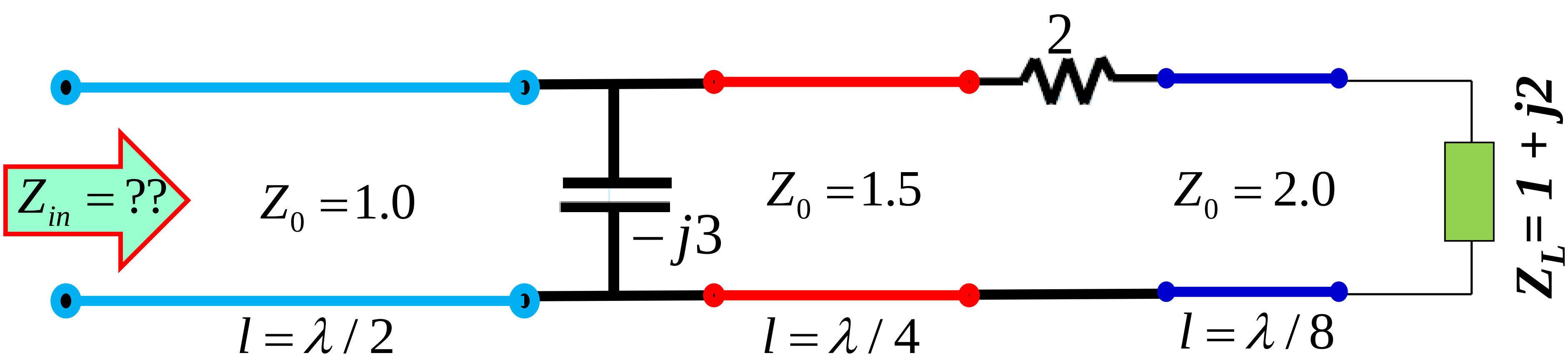
$$V(z=-l) \approx V(z=0)$$

$$I(z=-l) \approx I(z=0)$$

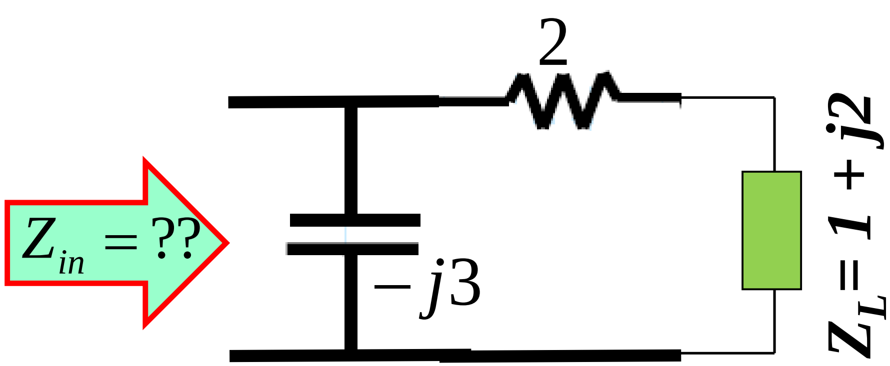
If $l \ll \lambda$, our "wire" behaves **exactly** as it did in *Linear Circuits* course!

Example – 3

Determine the input impedance of the following circuit:



How about the following solution?

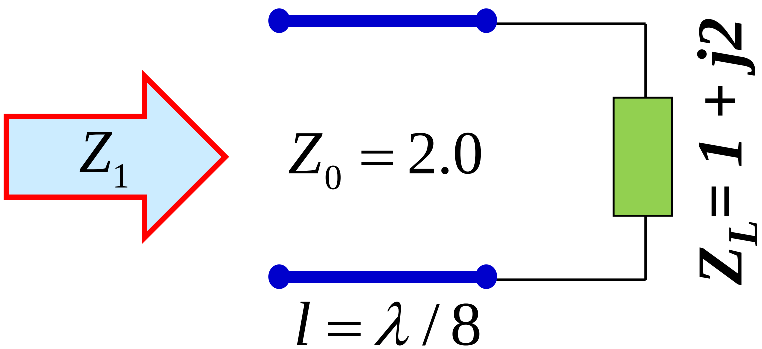


$$Z_{in} = \frac{-j3*(2+1+j2)}{-j3+(2+1+j2)} = 2.7 - j2.1$$

Where are the contributions of the TL??

Example – 3 (contd.)

- **Let us define Z_1 as the input impedance of the last section as:**



Then the impedance Z_1 is:

$$Z_1 = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)}$$

Where: $\beta d = \frac{2\pi}{\lambda} * \frac{\lambda}{8} = \frac{\pi}{4}$

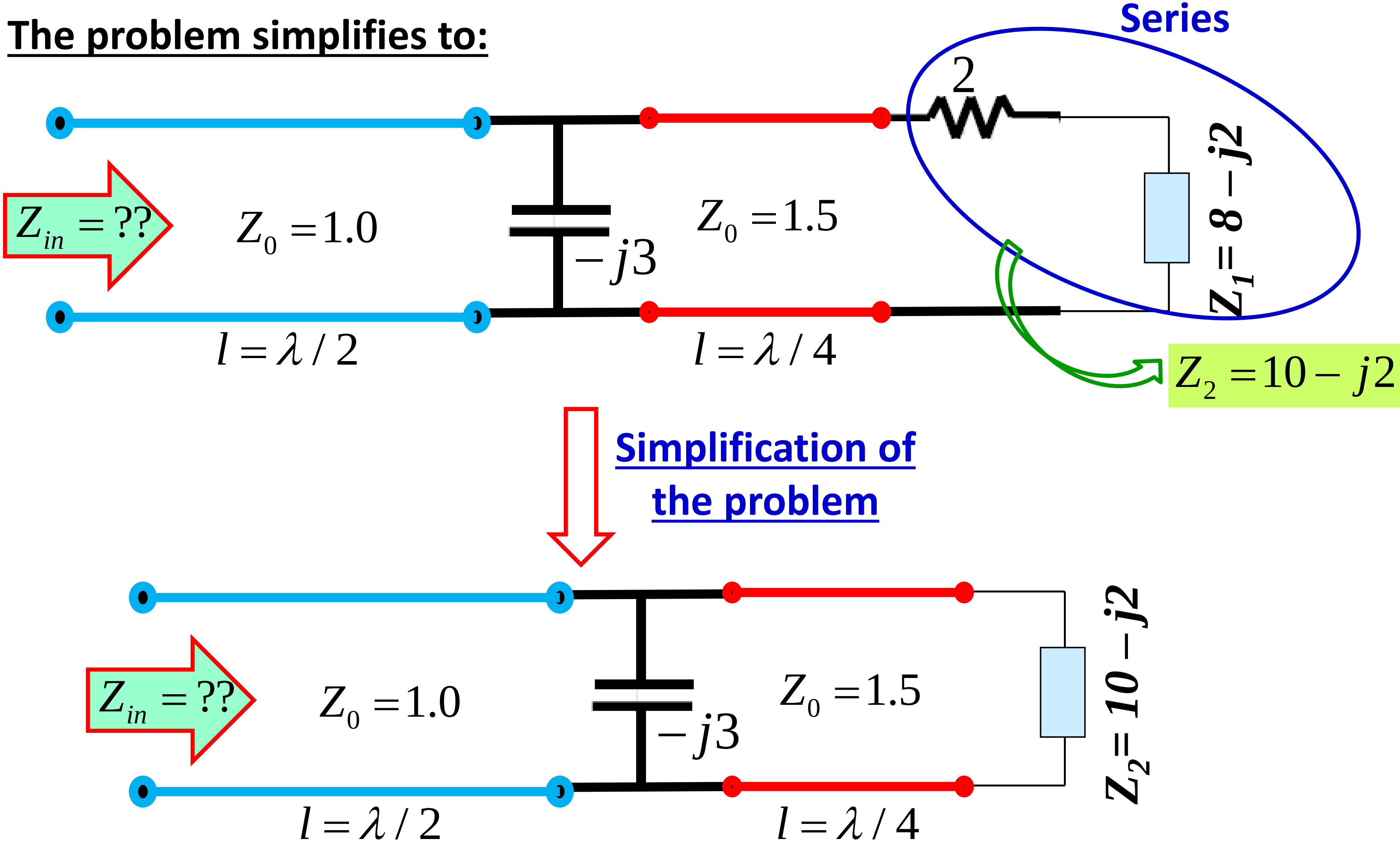
Therefore:

$$Z_1 = 2\left(\frac{(1 + j2) + j2\tan(\pi/4)}{2 + j(1 + j2)\tan(\pi/4)}\right) \Rightarrow Z_1 = 2\left(\frac{1 + j4}{j}\right)$$

$$\therefore Z_1 = 8 - j2$$

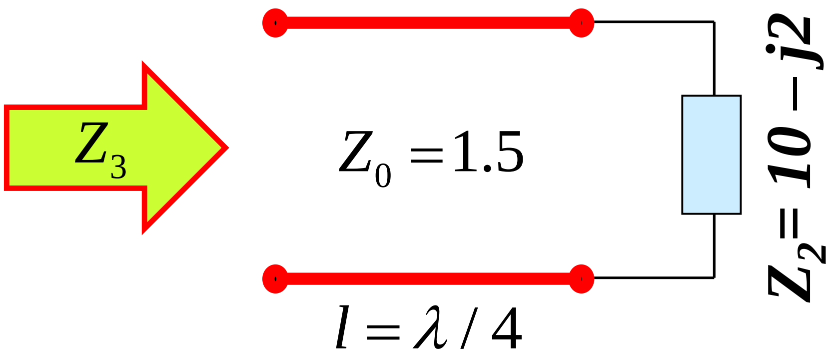
Example – 3 (contd.)

The problem simplifies to:



Example – 3 (contd.)

- Now let us define the input impedance of the middle TL as Z_3:

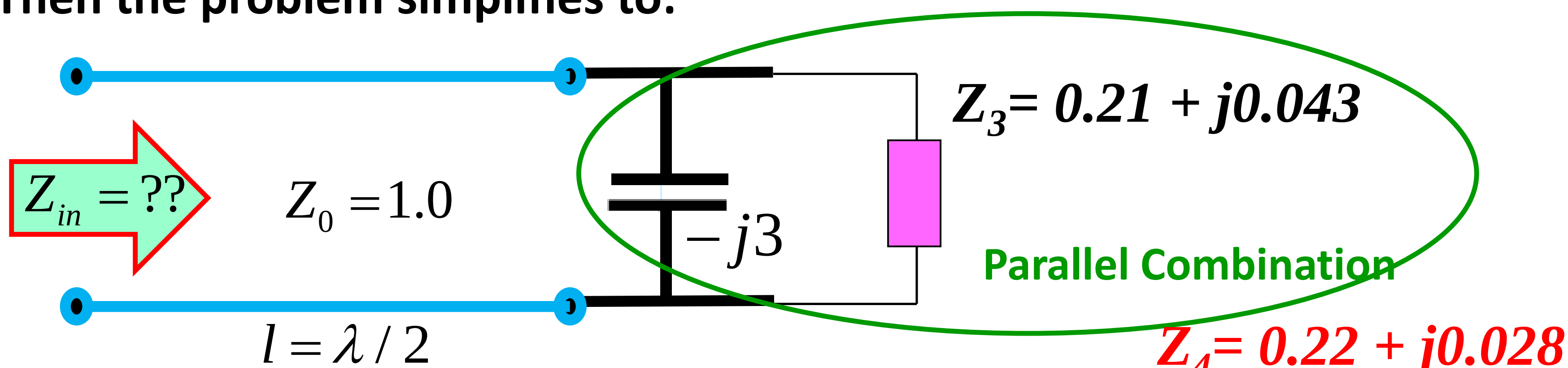


This is a quarter-wave TL → one of the special cases we considered earlier → where the input impedance is:

$$Z_3 = \frac{Z_0^2}{Z_2}$$

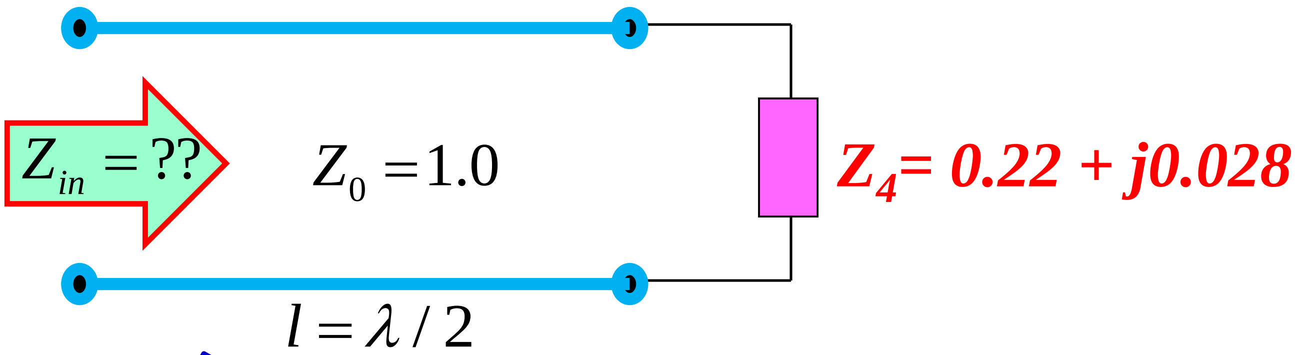
Therefore: $Z_3 = \dfrac{(1.5)^2}{10 - j2}$ $\Rightarrow$ $\therefore Z_3 = 0.21 + j0.043$

- Then the problem simplifies to:



Example – 3 (contd.)

- **Finally the simplified problem is:**



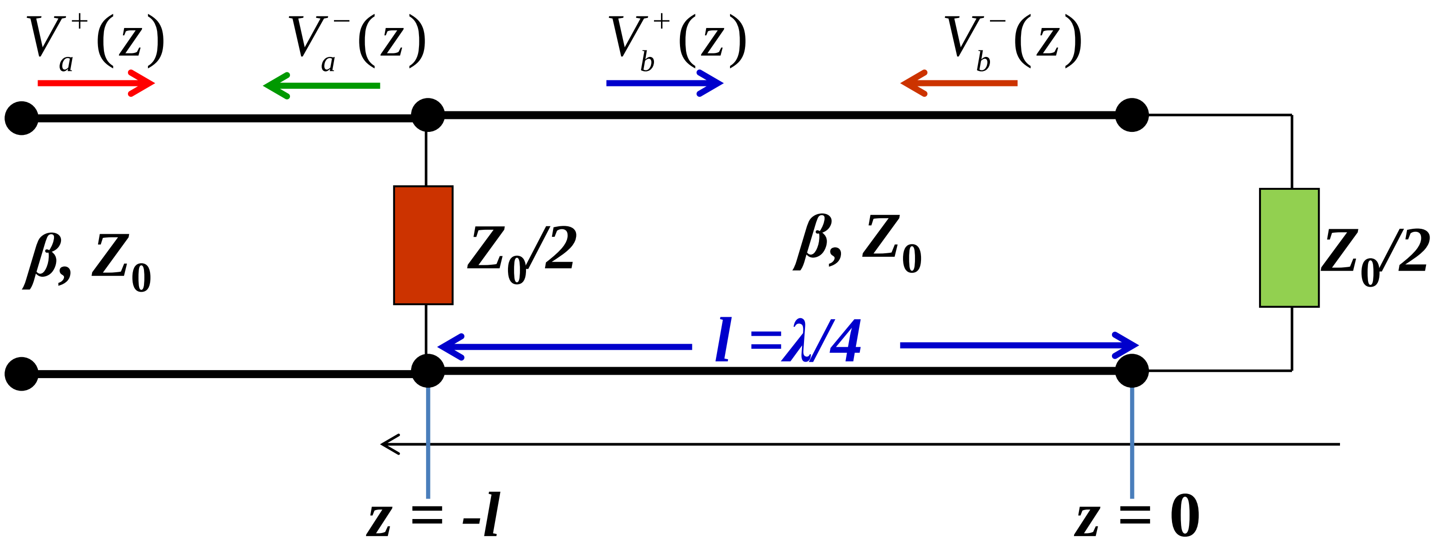
TL is a half wavelength → special case we discussed earlier → input impedance equals the load impedance

$$\therefore Z_{in} = Z_4 = 0.22 + j0.028$$

Example – 4

For the following circuit determine:

$$\frac{V_a^-}{V_a^+} \qquad \frac{V_b^+}{V_a^+} \qquad \frac{V_b^-}{V_a^+}$$



Given:

$$V(z) = V_a^+(z) + V_a^-(z) = V_a^+ e^{-j\beta z} + V_a^- e^{+j\beta z}$$ **For z < -l**

$$V(z) = V_b^+(z) + V_b^-(z) = V_b^+ e^{-j\beta z} + V_b^- e^{+j\beta z}$$ **For –l < z < 0**

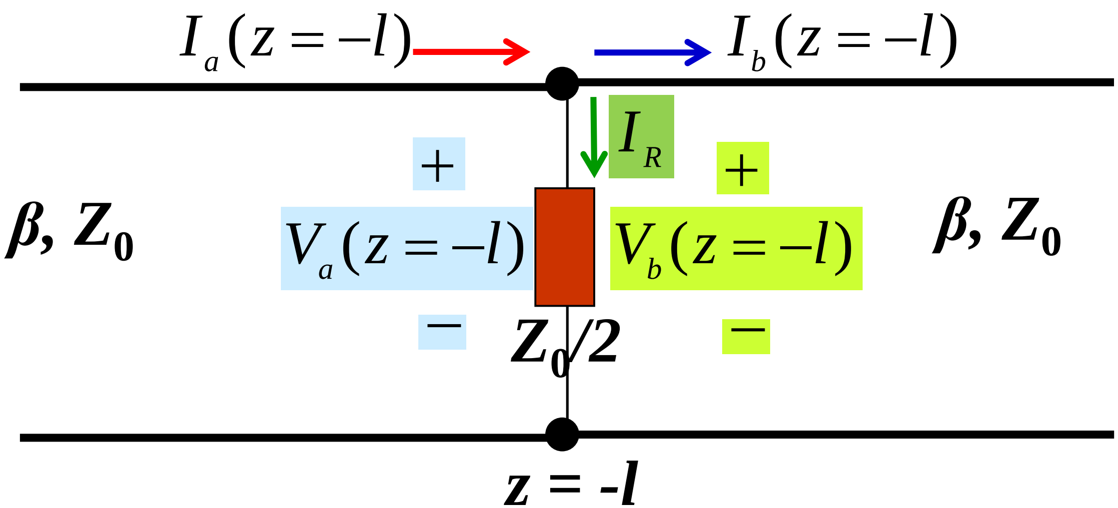
Example – 4 (contd.)

- We can write current equations as:

$$I(z) = \frac{V_a^+(z)}{Z_0} - \frac{V_a^-(z)}{Z_0} = \frac{V_a^+}{Z_0} e^{-j\beta z} - \frac{V_a^-}{Z_0} e^{+j\beta z} \qquad \text{For } z < -l$$

$$I(z) = \frac{V_b^+(z)}{Z_0} - \frac{V_b^-(z)}{Z_0} = \frac{V_b^+}{Z_0} e^{-j\beta z} - \frac{V_b^-}{Z_0} e^{+j\beta z} \qquad \text{For } -l < z < 0$$

- At $z = -l$:



KVL gives:

$$V_a(z=-l) = V_b(z=-l)$$

KCL gives:

$$I_a(z=-l) = I_b(z=-l) + I_R$$

Ohm's Law gives: $I_R = \dfrac{V_a(z=-l)}{Z_0/2} = \dfrac{2V_a(z=-l)}{Z_0} = \dfrac{2V_b(z=-l)}{Z_0}$

Example – 4 (contd.)

- **At z = -l:**

$$V_a(z=-l) = V_a^+(z=-l) + V_a^-(z=-l) = V_a^+ e^{-j\beta(-l)} + V_a^- e^{+j\beta(-l)} = V_a^+ e^{+j\beta l} + V_a^- e^{-j\beta l}$$

It is given: $l = \dfrac{\lambda}{4}$ → $\beta l = \dfrac{2\pi}{\lambda}\dfrac{\lambda}{4} = \dfrac{\pi}{2}$

$$\therefore V_a(z=-l) = V_a^+ e^{+j(\pi/2)} + V_a^- e^{-j(\pi/2)} = j\left(V_a^+ - V_a^-\right)$$

Similarly:

$$V_b(z=-l) = j\left(V_b^+ - V_b^-\right)$$

$$I_a(z=-l) = j\left(\frac{V_a^+ + V_a^-}{Z_0}\right)$$

$$I_b(z=-l) = j\left(\frac{V_b^+ + V_b^-}{Z_0}\right)$$

Example – 4 (contd.)

- Now let us revisit the expressions achieved from KVL, KCL and Ohm's Law

KVL

$$V_a(z=-l)=V_b(z=-l)$$

$$\Rightarrow j\left(V_a^+ - V_a^-\right) = j\left(V_b^+ - V_b^-\right)$$

$$\therefore 1-\frac{V_a^-}{V_a^+}=\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+}$$

Ohm's Law

$$I_R=\frac{2V_a(z=-l)}{Z_0}=\frac{2j\left(V_a^+ - V_a^-\right)}{Z_0}$$

$$I_R=\frac{2V_b(z=-l)}{Z_0}=\frac{2j\left(V_b^+ - V_b^-\right)}{Z_0}$$

KCL

$$I_a(z=-l)=I_b(z=-l)+I_R$$

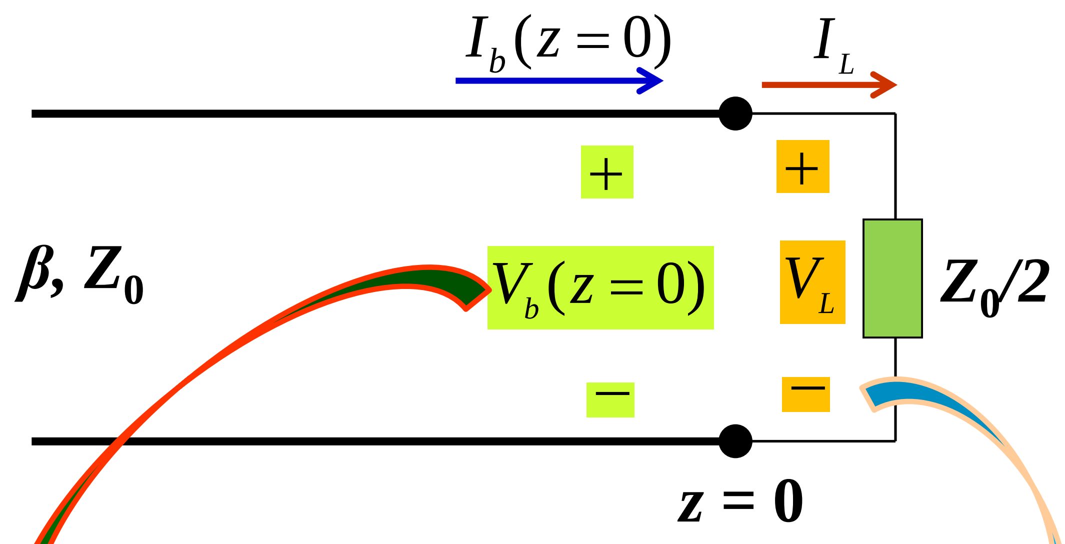
$$\Rightarrow j\left(\frac{V_a^+ + V_a^-}{Z_0}\right)=j\left(\frac{V_b^+ + V_b^-}{Z_0}\right)+I_R$$

$$V_a^+ + V_a^- = V_b^+ + V_b^- - jI_R Z_0$$

$$\therefore 1+\frac{V_a^-}{V_a^+}=3\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+}$$

Example – 4 (contd.)

- **At $z = 0$:**



$$V_b(z=0) = V_b^+ e^{-j\beta(0)} + V_b^- e^{+j\beta(0)} = V_b^+ + V_b^-$$

$$I_b(z=0) = \frac{V_b^+}{Z_0} e^{-j\beta(0)} - \frac{V_b^-}{Z_0} e^{+j\beta(0)} = \frac{V_b^+ - V_b^-}{Z_0}$$

KVL: $V_b(z=0) = V_L$

KCL: $I_b(z=0) = I_L$

Ohm's Law: $I_L = \dfrac{V_L}{Z_0/2} = \dfrac{2V_L}{Z_0}$

Example – 4 (contd.)

At $z = 0$:

$$I_L = \frac{V_L}{Z_0/2} = \frac{2V_L}{Z_0}$$

$$= I_b(z=0) = \frac{V_b^+ - V_b^-}{Z_0}$$

$$= V_b(z=0) = V_b^+ + V_b^-$$

$$\frac{V_b^+ - V_b^-}{Z_0} = \frac{2\left(V_b^+ + V_b^-\right)}{Z_0}$$

simplify

$$V_b^- = -\frac{1}{3}V_b^+$$

You can also achieve this result by writing the expression for reflection coefficient

Example – 4 (contd.)

Let us bring all the three simplified equations together

$$1-\frac{V_a^-}{V_a^+}=\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+} \quad (1)$$

$$1+\frac{V_a^-}{V_a^+}=3\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+} \quad (2)$$

$$V_b^-=-\frac{1}{3}V_b^+ \quad (3)$$

Simplification of (1) and (3) results in: $$1-\frac{V_a^-}{V_a^+}=\frac{4}{3}\frac{V_b^+}{V_a^+} \quad (4)$$

Simplification of (2) and (3) results in: $$1+\frac{V_a^-}{V_a^+}=\frac{10}{3}\frac{V_b^+}{V_a^+} \quad (5)$$

Simplify all of these to obtain the values of

$$\frac{V_a^-}{V_a^+} \qquad \frac{V_b^+}{V_a^+} \qquad \frac{V_b^-}{V_a^+}$$

Example – 4 (contd.)

Let us now summarize the fruits of our effort

$$\frac{V_a^-}{V_a^+} = \frac{3}{7}$$

$$\frac{V_b^+}{V_a^+} = \frac{3}{7}$$

$$\frac{V_b^-}{V_a^+} = -\frac{1}{7}$$