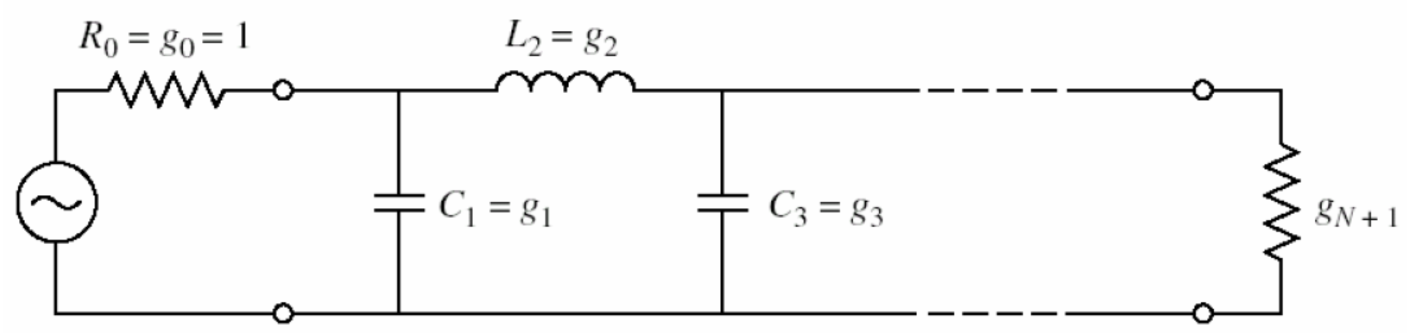
Lecture – 20 Date: 30.03.2017

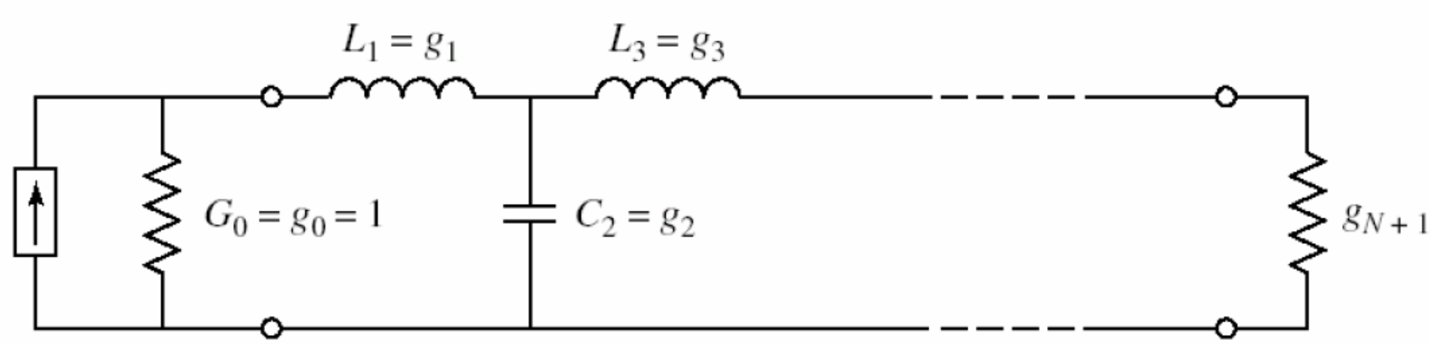
- Filter Realization using Lumped Components
- Richard’s Transformation
- Kuroda Identities
- Stepped-Impedance Method

Filter Realizations Using Lumped Elements

- Our **first** filter circuit will be “**realized**” with lumped elements.
- **Lumped elements**—we mean inductors L and capacitors C !
- Since each of these elements are (ideally) perfectly **reactive**, the resulting filter will be **lossless** (ideally).

- Let us first consider two configurations of a **ladder circuit**:





Filter Realizations Using Lumped Elements (contd.)

Note that these two structures provide a **low-pass** filter response (evaluate the circuits at ω = 0 and ω = ∞!).

Moreover, these structures have N different **reactive elements** (i.e., N degrees of design freedom) and thus can be used to realize an **N-order** power loss ratio.

- For example, consider the **Butterworth** power loss ratio function:

$$P_{LR}(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2N}$$

- Recall this is a **low-pass** function, as $P_{LR} = 1$ at ω = 0, and $P_{LR} = \infty$ at $\omega = \infty$. Note also that at $\omega_c = \omega$:

$$P_{LR}(\omega = \omega_c) = 1 + \left(\frac{\omega_c}{\omega_c}\right)^{2N} = 2$$

Thus

$$\Gamma(\omega = \omega_c) = T(\omega = \omega_c) = 1/2$$

In other words, ω_c defines the 3dB bandwidth of the low-pass filter.

Filter Realizations Using Lumped Elements (contd.)

- Likewise, we find that this Butterworth function is **maximally flat** at ω = 0:

$$P_{LR}(\omega=0) = 1 + \left(\frac{0}{\omega_c}\right)^{2N} = 1$$

and:

$$\frac{d^n P_{LR}(\omega)}{d\omega^n}\Big|_{\omega=0} = 0$$

For all n

- Now, we can determine the function $P_{LR}(\omega)$ for a lumped element ladder circuit of N elements using our knowledge of **complex circuit theory**.
- Then, we can **equate** the resulting polynomial to the **maximally flat** function above. In this manner, we can determine the appropriate **values** of all inductors L and capacitors C!
- Finding these L an C requires little bit of complex algebra.
- Pozar provides tables of complete Butterworth and Chebychev low-pass solutions.

Filter Realizations Using Lumped Elements (contd.)

TABLE 8.3 Element Values for Maximally Flat Low-Pass Filter Prototypes ($g_0 = 1$, $\omega_c = 1$, $N = 1$ to 10)

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.0000	1.0000									
2	1.4142	1.4142	1.0000								
3	1.0000	2.0000	1.0000	1.0000							
4	0.7654	1.8478	1.8478	0.7654	1.0000						
5	0.6180	1.6180	2.0000	1.6180	0.6180	1.0000					
6	0.5176	1.4142	1.9318	1.9318	1.4142	0.5176	1.0000				
7	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450	1.0000			
8	0.3902	1.1111	1.6629	1.9615	1.9615	1.6629	1.1111	0.3902	1.0000		
9	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473	1.0000	
10	0.3129	0.9080	1.4142	1.7820	1.9754	1.9754	1.7820	1.4142	0.9080	0.3129	1.0000

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Filter Realizations Using Lumped Elements (contd.)

TABLE 8.4 **Element Values for Equal-Ripple Low-Pass Filter Prototypes ($g_0 = 1, \omega_c = 1, N = 1$ to 10, 0.5 dB and 3.0 dB ripple)**

					0.5 dB Ripple						
N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	0.6986	1.0000									
2	1.4029	0.7071	1.9841								
3	1.5963	1.0967	1.5963	1.0000							
4	1.6703	1.1926	2.3661	0.8419	1.9841						
5	1.7058	1.2296	2.5408	1.2296	1.7058	1.0000					
6	1.7254	1.2479	2.6064	1.3137	2.4758	0.8696	1.9841				
7	1.7372	1.2583	2.6381	1.3444	2.6381	1.2583	1.7372	1.0000			
8	1.7451	1.2647	2.6564	1.3590	2.6964	1.3389	2.5093	0.8796	1.9841		
9	1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504	1.0000	
10	1.7543	1.2721	2.6754	1.3725	2.7392	1.3806	2.7231	1.3485	2.5239	0.8842	1.9841
					3.0 dB Ripple						
N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	1.9953	1.0000									
2	3.1013	0.5339	5.8095								
3	3.3487	0.7117	3.3487	1.0000							
4	3.4389	0.7483	4.3471	0.5920	5.8095						
5	3.4817	0.7618	4.5381	0.7618	3.4817	1.0000					
6	3.5045	0.7685	4.6061	0.7929	4.4641	0.6033	5.8095				
7	3.5182	0.7723	4.6386	0.8039	4.6386	0.7723	3.5182	1.0000			
8	3.5277	0.7745	4.6575	0.8089	4.6990	0.8018	4.4990	0.6073	5.8095		
9	3.5340	0.7760	4.6692	0.8118	4.7272	0.8118	4.6692	0.7760	3.5340	1.0000	
10	3.5384	0.7771	4.6768	0.8136	4.7425	0.8164	4.7260	0.8051	4.5142	0.6091	5.8095

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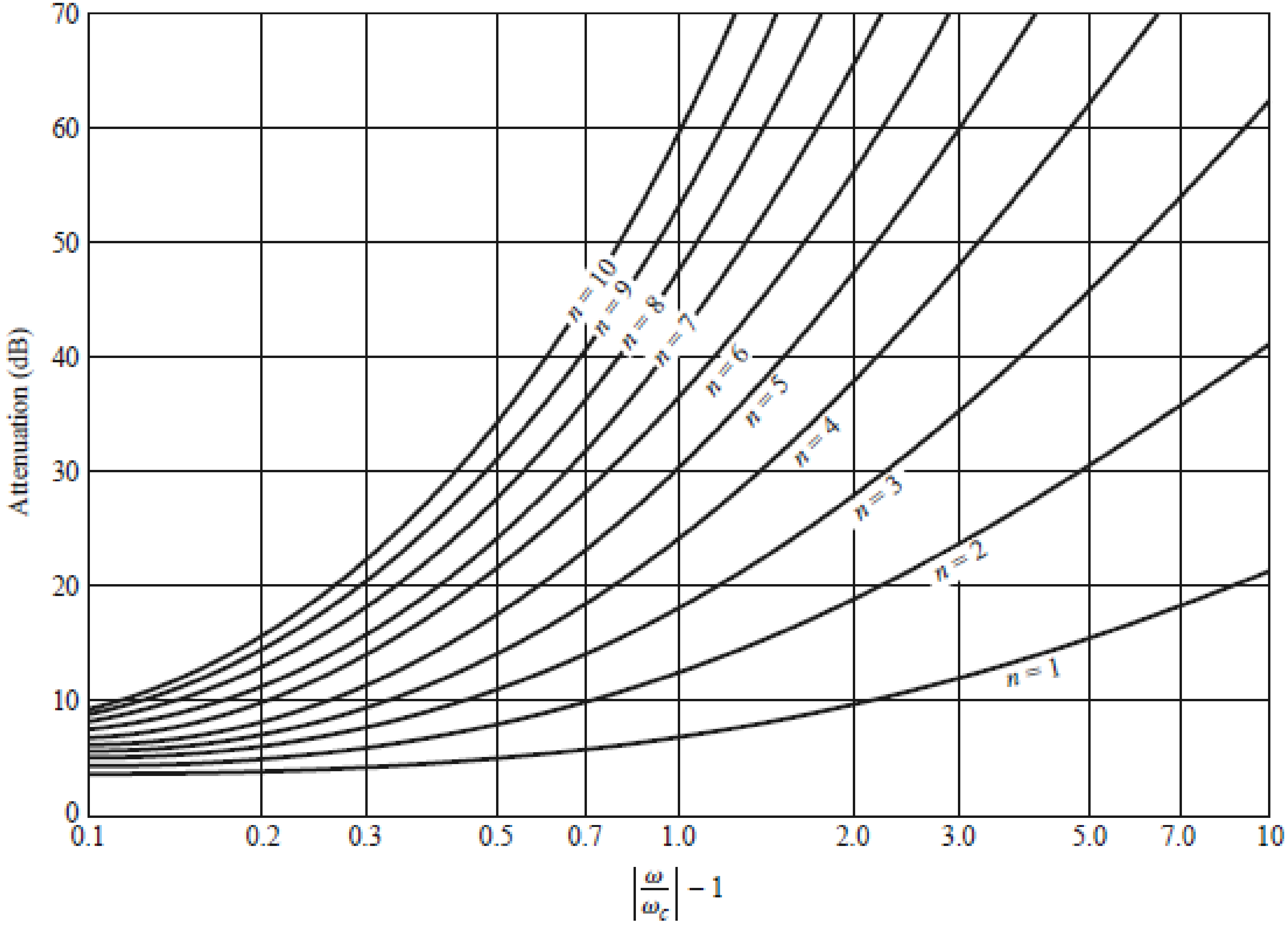
Insertion Loss Method



Attenuation versus Normalized Frequency

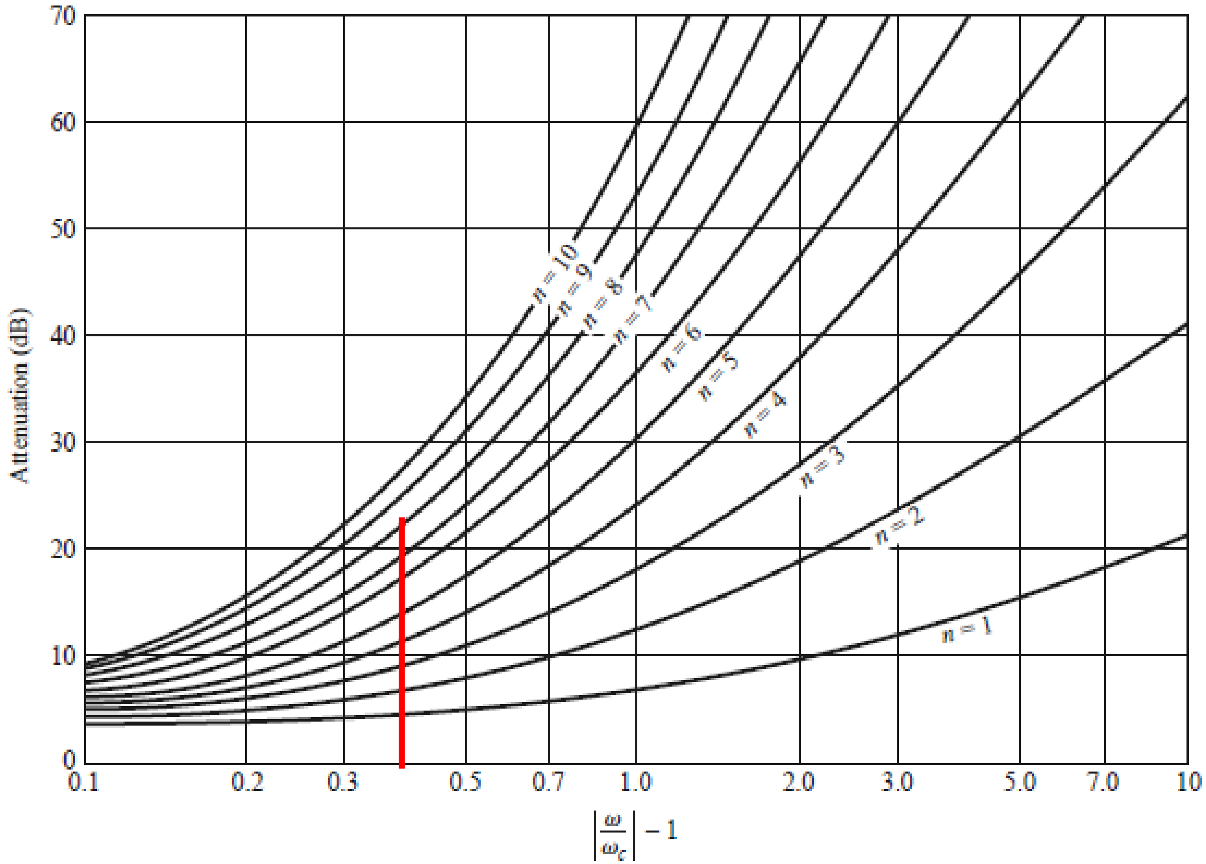
Example – 1

A maximally flat low-pass filter is to be designed with a cut-off frequency of 8GHz and a minimum attenuation of 20dB at 11GHz. How many filter elements are required?

We have:

$$\frac{\omega/2\pi}{\omega_c/2\pi}-1=\frac{11}{8}-1=0.375$$

N=8



Example – 2

Design a maximally flat low-pass filter with a cut-off frequency of 2GHz, impedance of 50Ω and at least 15dB insertion loss at 3GHz.

- First, find the required order of the maximally flat filter to satisfy the insertion loss specification at 3GHz.
- We have:

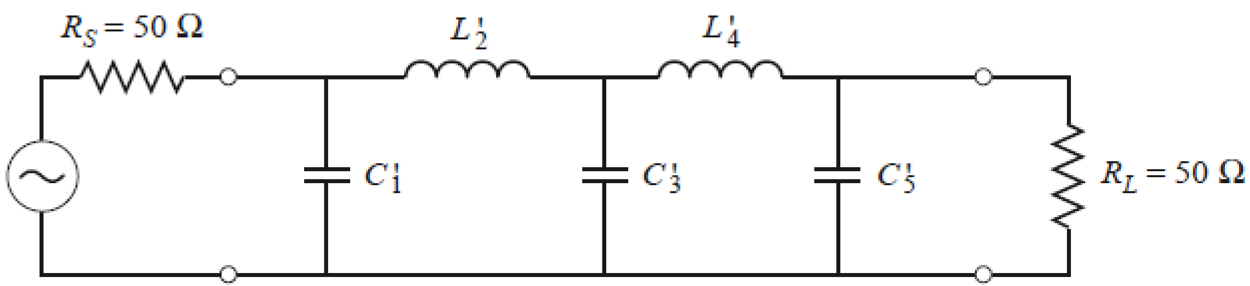
$$\frac{\omega/2\pi}{\omega_c/2\pi}-1=\frac{3}{2}-1=0.5$$

- It is apparent that N =5 will be sufficient.
- From the table we get: $g_1 = 0.618$, $g_2 = 1.618$, $g_3 = 2.000$, $g_4 = 1.618$, $g_5 = 0.618$.

Example – 2 (contd.)

- The Analysis of N-element filters give: $L_n = g_n\left(\frac{R_s}{\omega_c}\right)$ $C_n = g_n\left(\frac{1}{R_s\omega_c}\right)$
- The elements are therefore:

$$C_1 = 0.984\,pF \quad L_2 = 6.438\,nH \quad C_3 = 3.183\,pF \quad L_4 = 6.438\,nH \quad C_5 = 0.984\,pF$$



Filter Realizations Using Lumped Elements (contd.)

Q: What?! What the heck do these values g_n mean?

A: We can use the values g_n to find the values of inductors and capacitors required for a given **cut-off frequency** ω_c and source resistance R_s (Z_0).

- Specifically, we use the values of g_n to find ladder circuit **inductor** and **capacitor** values as:

$$L_n = g_n \left(\frac{R_s}{\omega_c}\right)$$

$$C_n = g_n \left(\frac{1}{R_s \omega_c}\right)$$

where *n* =*1,2,…, N*

- Likewise, the value g_{N+1} describes the **load impedance**. Specifically, we find that **if** the **last** reactive element (i.e., g_N) of the ladder circuit is a **shunt capacitor**, then:

$$R_L = g_{N+1} R_s$$

- Whereas, **if** the **last** reactive element (i.e., g_N) of the ladder circuit is a **series inductor**, then:

$$R_L = \frac{R_s}{g_{N+1}}$$

- Note, however, for the **Butterworth** solutions **(in Table 8.3)** we find that g_{N+1}=1 **always**, and therefore:

$$R_L = R_s$$

← Regardless of the last element

Filter Realizations Using Lumped Elements (contd.)

- Moreover, we note (in Table 8.4) that this (i.e., $g_{N+1}=1$) is likewise true for the Chebyshev solutions – provided that **N is odd**.
- Thus, we typically desire a filter where:

$$R_L = R_s = Z_0$$

We can use **any** order of **Butterworth** filter, or an **odd** order of **Chebyshev.**

In other words, avoid even order Chebyshev filters!

Q: OK, so we now have the solutions for Chebyshev and Butterworth **low-pass** filters. But what about high-pass, band-pass, or band-stop filters?

A: Surprisingly, the low-pass filter solutions **likewise** provide us with the solutions for **any** and **all** high-pass, band-pass and band-stop filters! All we need to do is apply **filter transformations**.

We can use the concept of **filter transformations** to determine the **new** filter designs from a low-pass design. As a result, we can construct a 3rd-order Butterworth **high-pass** filter or a 5th-order Chebyshev **band-pass** filter!

Filter Transformations

It will be apparent that the mathematics for each filter design will be very **similar**. Eg, the difference between a low-pass and high-pass filter is essentially an **inverse**—the frequencies below ω_c are mapped into frequencies above ω_c —and vice versa.

It is evident that:

$$T_{lp}(\omega = 0) = \mathrm{T}_{hp}(\omega = \infty) = 1$$

$$\Gamma_{lp}(\omega = \infty) = \Gamma_{hp}(\omega = 0) = 0$$

- However:

$$T_{lp}(\omega = \omega_c) = \mathrm{T}_{hp}(\omega = \omega_c) = 0.5$$

- Therefore, we can express:

$$T_{lp}(\omega = \alpha\omega_c) = \mathrm{T}_{hp}(\omega = \frac{1}{\alpha}\omega_c)$$

where α is some positive, real value (i.e., 0 ≤α < ∞).

- For example, if α = 0.5, then: $T_{lp}(\omega = 0.5\omega_c) = \mathrm{T}_{hp}(\omega = 2\omega_c)$

Filter Transformations (contd.)

In other words, the transmission through a low-pass filter at one half the cut-off frequency will be equal to the transmission through a (mathematically similar) high-pass filter at twice the cut-off frequency.

- Now, recall the loss-ratio functions for Butterworth and Chebyshev low-pass filters:

$$P_{LR}^{lp}(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2N}$$

$$P_{LR}^{lp}(\omega) = 1 + k^2 T_N^2\left(\frac{\omega}{\omega_c}\right)$$

- Note in each case that the argument of the function has the form: $\frac{\omega}{\omega_c}$

In other words, the frequency is **normalized** by the cut-off frequency.

- Consider now **this** mapping: $\frac{\omega}{\omega_c} \Rightarrow -\frac{\omega_c}{\omega}$

- This mapping **transforms** the lpf response into a corresponding high pass filter response! i.e.:

$$P_{LR}^{hp}(\omega) = 1 + \left(\frac{\omega_c}{\omega}\right)^{2N}$$

$$P_{LR}^{hp}(\omega) = 1 - k^2 T_N^2\left(\frac{\omega_c}{\omega}\right)$$

Filter Transformations (contd.)

Q: Yikes! Where did this mapping come from? Are sure this works?

Consider again the case where $\omega = \alpha\omega_c$; the low pass responses are:

$$P_{LR}^{lp}(\omega) = 1 + (\alpha)^{2N}$$

$$P_{LR}^{lp}(\omega) = 1 + k^2 T_N^2(\alpha)$$

Now consider the high-pass responses where $\omega = -\omega_c/\alpha$:

$$P_{LR}^{hp}(\omega) = 1 + (\alpha)^{2N}$$

$$P_{LR}^{hp}(\omega) = 1 - k^2 T_N^2(\alpha)$$

- Thus, we can conclude from this mapping that:

$$P_{LR}^{lp}(\omega = \alpha\omega_c) = P_{LR}^{hp}(\omega = -\omega_c / \alpha)$$

- And since $T = P_{LR}^{-1}$:

$$T_{lp}(\omega = \alpha\omega_c) = T_{hp}(\omega = -\frac{1}{\alpha}\omega_c)$$

Exactly the result that we expected! Our mapping provides a method for transforming a low-pass filter into a high-pass filter!

Filter Transformations (contd.)

Q: OK Poindexter, you have succeeded in providing another one of your "fascinating" mathematical insights, but does this "mapping" provide anything useful for us engineers?

A: Absolutely! We can apply this mapping one component element (capacitor or inductor) at a time to our low-pass schematic design, and the result will be a direct transformation into a high-pass filter schematic.

- Recall the reactance of an inductor element in a low-pass filter design is:

$$jX_n^{lp} = j\omega L_n^{lp} = j\omega g_n\left(\frac{R_s}{\omega_c}\right) = jg_n R_s\left(\frac{\omega}{\omega_c}\right)$$

- while that of a capacitor is:

$$jX_n^{lp} = \frac{1}{j\omega C_n^{lp}} = -j\frac{R_s}{g_n}\left(\frac{\omega_c}{\omega}\right)$$

- Now apply the mapping:

$$\frac{\omega}{\omega_c} \Rightarrow -\frac{\omega_c}{\omega}$$

Filter Transformations (contd.)

- The inductor becomes:

$$jX_n^{hp} = jg_nR_s\left(-\frac{\omega_c}{\omega}\right) = -j\frac{g_nR_s\omega_c}{\omega} = \frac{1}{j\left(g_nR_s\omega_c\right)^{-1}\omega}$$

- and the capacitor:

$$jX_n^{hp} = -j\frac{R_s}{g_n}\left(-\frac{\omega_c}{\omega}\right) = j\omega\left(\frac{R_s}{g_n\omega_c}\right)$$

It is clear (do you see why?) that the transformation has converted a positive (i.e., inductive) reactance into a negative (i.e., capacitive) reactance—and vice versa.

- As a result, to transform a low-pass filter schematic into a high-pass filter schematic, we:

1. Replace each inductor with a capacitor of value:

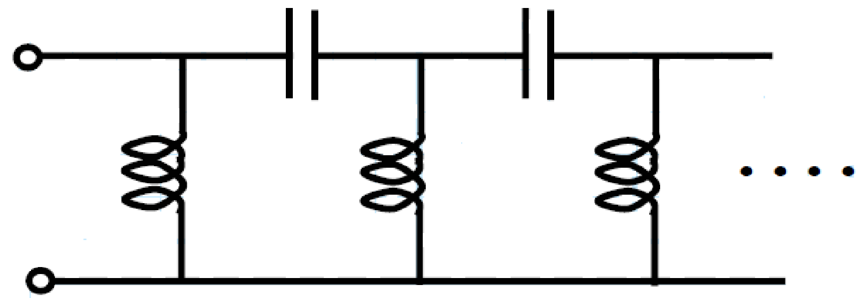
$$C_n^{hp} = \frac{1}{g_nR_s\omega_c} = \frac{1}{\omega_c^2 L_n^{lp}}$$

2. Replace each capacitor with an inductor of value:

$$L_n^{hp} = \frac{R_s}{g_n\omega_c} = \frac{1}{\omega_c^2 C_n^{lp}}$$

Filter Transformations (contd.)

- Thus, a **high-pass ladder circuit** consists of **series capacitors** and **shunt inductors** (compare this to the low-pass) ladder circuit!).



Q: What about band-pass filters?

A: The difference between a low-pass and band-pass filter is simply a **shift** in the "center" frequency of the filter, where the center frequency of a low-pass filter is essentially ω = 0.

- For this case, we find the **mapping**:

$$\frac{\omega}{\omega_c} \Rightarrow \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

transforms a low-pass function into a **band-pass function**, where Δ is the **normalized bandwidth**:

$$\Delta = \frac{\omega_2 - \omega_1}{\omega_0}$$

ω_1 and ω_2 define the two **3dB frequencies** of the bandpass filter.

Filter Transformations (contd.)

- For example, the Butterworth **low-pass** function:

$$P_{LR}^{lp}(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2N}$$

- becomes a Butterworth **band-pass** function:

$$P_{LR}^{bp}(\omega) = 1 + \frac{1}{\Delta^{2N}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^{2N}$$

- Applying this transform to the reactance of a low-pass inductive element:

$$jX_n^{bp} = jg_nR_s\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = j\omega\left(\frac{g_nR_s}{\omega_0\Delta}\right) + \frac{1}{j\omega\left(\Delta / g_n\omega_0R_s\right)}$$

- Look what happened! The transformation turned the inductive reactance into an inductive reactance in series with a capacitive reactance.
- A similar analysis of the transformation of the low-pass capacitive reactance shows that it is transformed into an inductive reactance in parallel with an capacitive reactance.

Filter Transformations (contd.)

- As a result, to transform a low-pass filter schematic into a band-pass filter schematic, we:

1. Replace each series inductor with a capacitor and inductor in series, with values:

$$L_n^{bp} = g_n \frac{R_s}{\omega_0 \Delta}$$

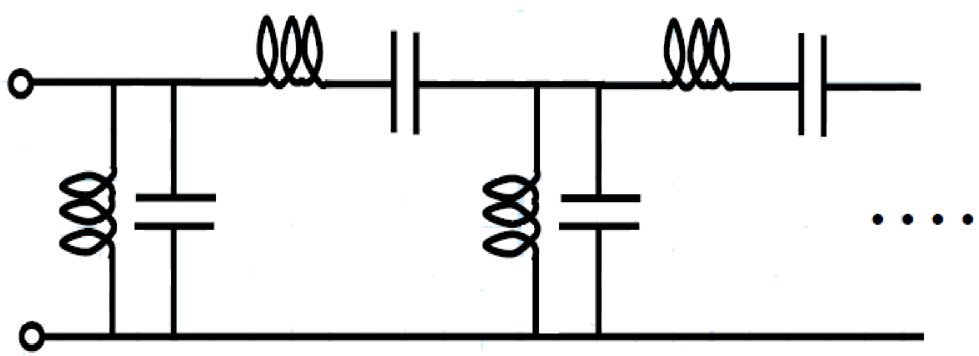
$$C_n^{bp} = \frac{1}{g_n} \frac{\Delta}{\omega_0 R_s}$$

2. Replace each shunt capacitor with an inductor and capacitor in parallel, with values:

$$L_n^{bp} = \frac{1}{g_n} \frac{\Delta R_s}{\omega_0}$$

$$C_n^{bp} = g_n \frac{1}{\omega_0 \Delta R_s}$$

- Thus, the ladder circuit for **band-pass circuit** is simply a ladder network of LC resonators, both series and parallel:



Filter Implementations

Q: So, we now know how to make any and all filters with **lumped** elements—but this is a **RF/microwave** engineering course!

You said that lumped elements were difficult to make and implement at high frequencies. **You** said that distributed elements were used to make microwave components. So **how** do we make a filter with **distributed elements**!?!

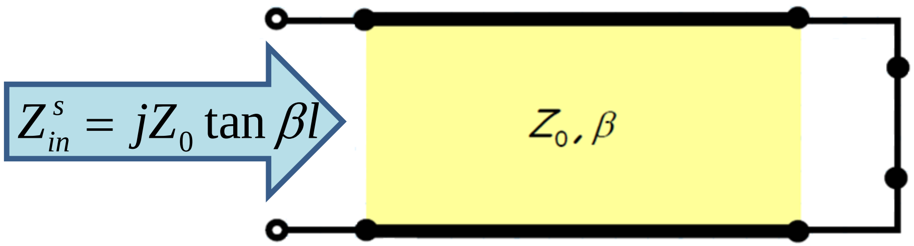
A: There are **many** ways to make RF/microwave filters with distributed elements. Perhaps the most straightforward is to "**realize**" each individual lumped element with transmission line sections, and then insert these **approximations** in our lumped element solutions.

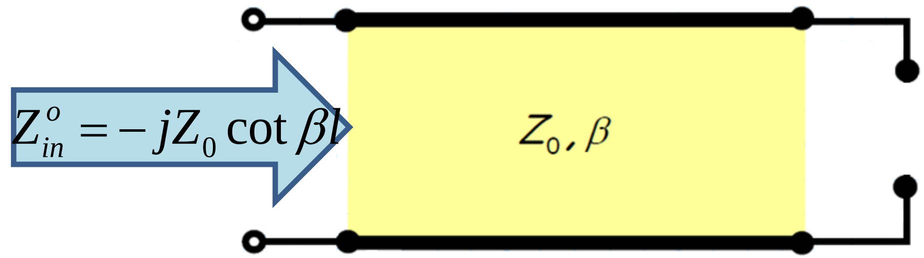
The **first** of these realizations is: Richard's Transformations

To easily **implement** Richard's Transforms in a microstrip or stripline circuit, we must apply one of **Kuroda's Identities**.

Richard's Transformations

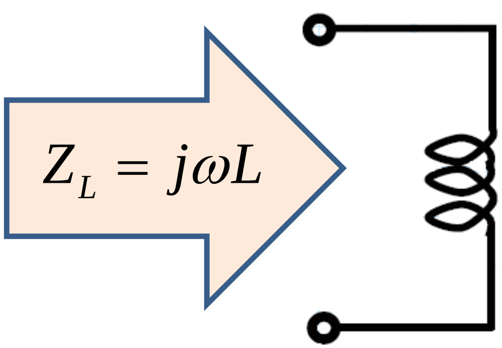
- Recall the input impedances of short-circuited and open-circuited transmission line **stubs**.

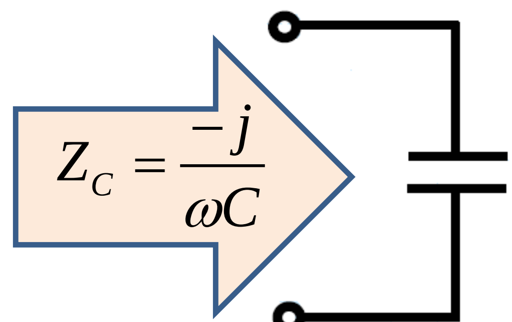




Note that the input impedances are purely **reactive**—just like **lumped** elements!

- However, the reactance of lumped inductors and capacitors have a **much** different mathematical form to that of transmission line stubs:





Richard's Transformations (contd.)

- In other words, the impedance of transmission line stubs and lumped elements (capacitors and inductors) are different functions with respect to **frequency**. Therefore, we can say in general that, for example:

$$Z_{in}^{s} \neq Z_{L}$$

$$Z_{in}^{o} \neq Z_{C}$$

However, for a given lumped element (L or C) and a given stub (with a given Z_0 and length l) the functions **will** be equal at precisely **one frequency**!

- For example, there is one frequency—let's call it ω_c —that satisfies **this** equation for a given L, Z_0, and l:

$$j\omega_c L = jZ_0 \tan \beta_c l = jZ_0 \tan\left[\frac{\omega_c}{v_p} l\right]$$

- Similarly:

$$\frac{-j}{\omega_c C} = -jZ_0 \cot \beta_c l = -jZ_0 \cot\left[\frac{\omega_c}{v_p} l\right]$$

- To make things easier, let's set the **length** of our transmission line stub to $\lambda_c/8$, where:

$$\lambda_c = \frac{v_p}{\omega_c} = \frac{2\pi}{\beta_c}$$

Richard's Transformations (contd.)

Q: Why $l = \lambda_c/8$?

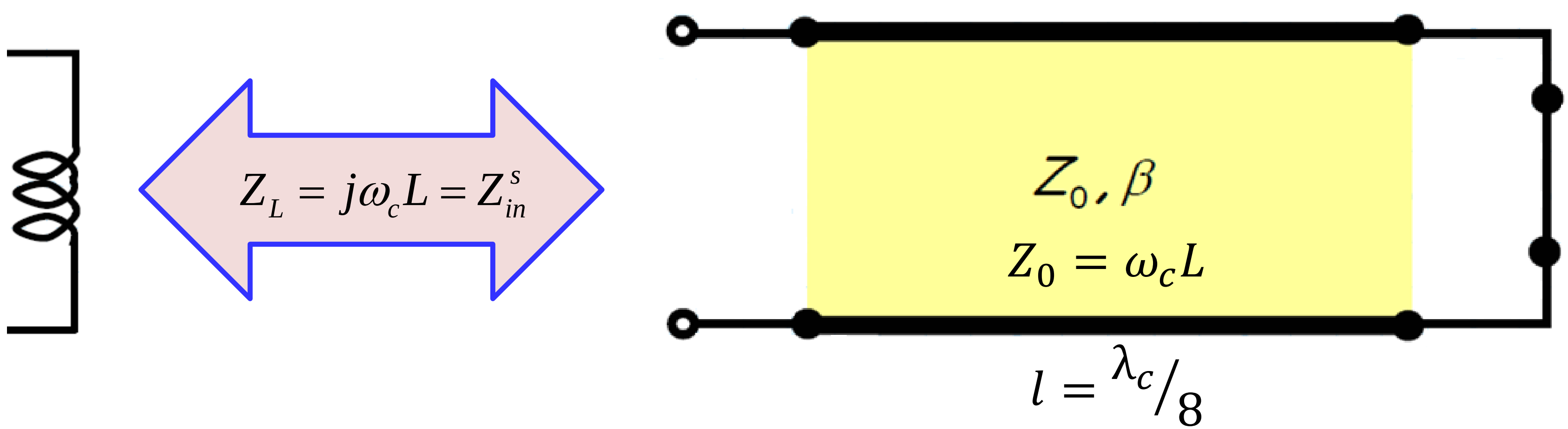
A: Well, for **one** reason, $\beta_c l = \pi/4$ and therefore $\tan(\pi/4) = 1.0$!

- This greatly **simplifies** our earlier results:

$$j\omega_c L = jZ_0 \tan\left(\frac{\pi}{4}\right) = jZ_0$$

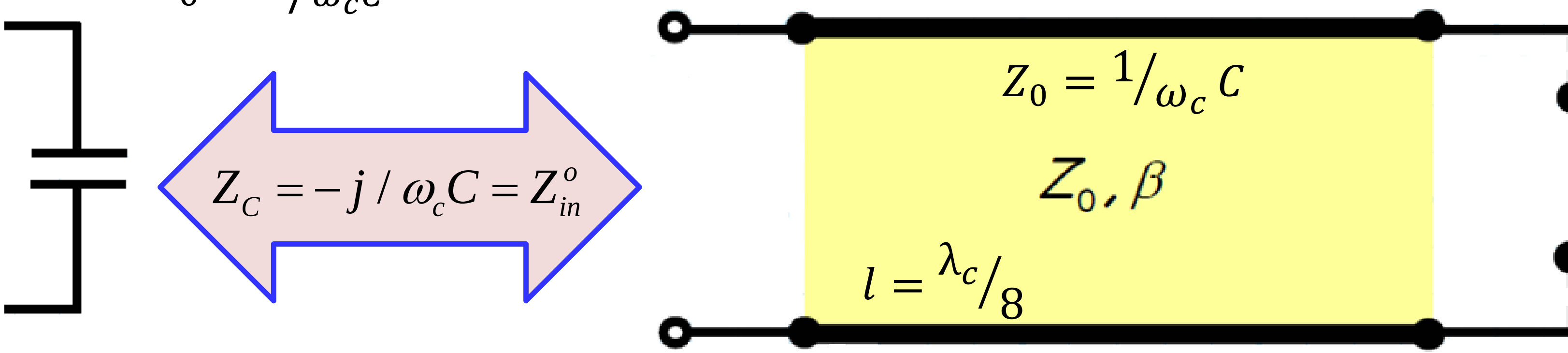
$$\frac{-j}{\omega_c C} = -jZ_0 \cot\left(\frac{\pi}{4}\right) = -jZ_0$$

- Therefore, if we wish to build a **short-circuited** stub with the **same** impedance as an **inductor** L at frequency ω_c, we set the **characteristic impedance** of the stub transmission line to be $Z_0 = \omega_c L$:



Richard's Transformations (contd.)

- Similarly, if we wish to build **open-circuited** stub with the **same** impedance as a **capacitor** C at ω_c, we set the **characteristic impedance** of the stub transmission line to be $Z_0 = {}^{1}/_{\omega_c C}$:



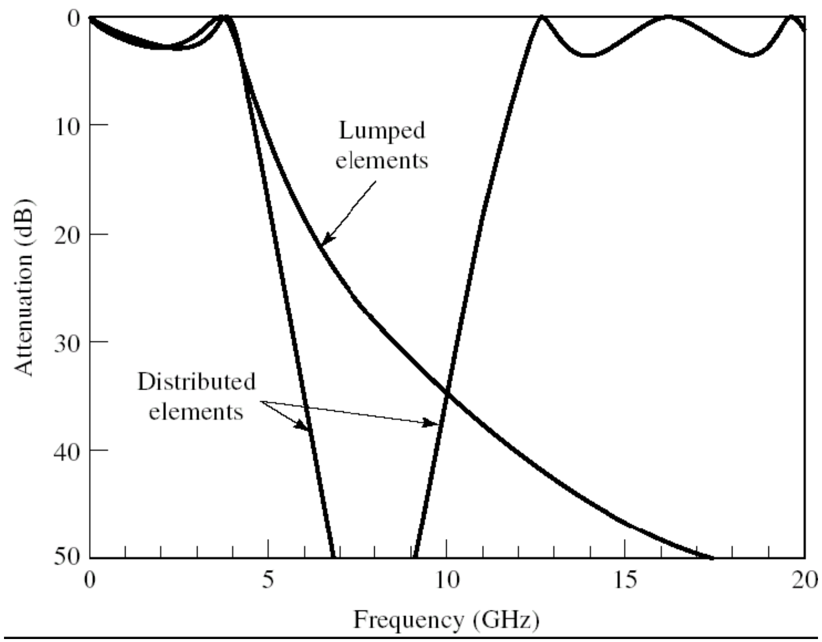
We call these two results as **Richard's Transformation.**

However, remember that Richard's Transformations do **not** result in **perfect** replacements for lumped elements—the stubs **do not** behave like C and L!

- Instead, the transformation is **perfect**—the impedances are **equal**—at **only one frequency** (ω_c).
- We can use Richard's transformations to replace the inductors and capacitors of a lumped element filter design. In fact, for **low-pass filter design**, the frequency ω_c is the filter's **cut-off frequency**.

Richard's Transformations (contd.)

- Using these stubs to **replace** inductors and capacitors will result in a filter response **similar** to that of the lumped element design—a low pass filter with cut-off frequency ω_c.
- However, the behavior of the filter in the **stop-band** will be **very** different from the lumped element design. For example, at the (high) frequencies where the stub length becomes a **multiple** of $\lambda/2$, the filter response will be that of $\omega = 0$—near perfect **transmission**!



Q: So **why** does the filter response match the lumped element response so **well** in the **pass-band**?

A: To see why, we first note that the **Taylor Series approximation** for $tan\varphi$ and $cot\varphi$ when φ is small (i.e., $\varphi \ll 1$) is:

$tan\varphi \approx \varphi$ **and** $cot\varphi \approx \dfrac{1}{\varphi}$ **for** $\varphi \ll 1$

φ is expressed in **radians.**

Richard's Transformations (contd.)

- The **impedance** of Richard's transformation shorted stub at some **arbitrary frequency** ω is therefore:

$$Z_{in}^{s}(\omega) = jZ_0 \tan\left(\beta \frac{\lambda_c}{8}\right) = j(\omega_c L)\tan\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right)$$

- Therefore, when $\omega \ll \omega_c$(i.e., frequencies in the **pass-band** of a low-pass filter!), we can **approximate** this impedance as:

$$Z_{in}^{s}(\omega) = j(\omega_c L)\tan\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right) \approx j(\omega_c L)\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right) = j\omega L\left(\frac{\pi}{4}\right)$$

Compare this to a **lumped inductor** impedance

$$Z_L = j\omega L$$

Since the value π/4 is **relatively** close to one, we find that the Richard's Transformation shorted stub has an input impedance **very** close to the lumped element inductor for **all** frequencies **less than** ω_c (i.e., all frequencies of the low-pass filter pass-band)!

Richard's Transformations (contd.)

- Similarly, we find that the Richard's transformation **open-circuit** stub, when $\omega \ll \omega_c$, has an input impedance of **approximately**:

$$Z_{in}^{o}(\omega) = \frac{-j}{\omega_c C}\cot\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right) \approx \frac{-j}{\omega_c C}\left(\frac{\omega_c}{\omega}\frac{4}{\pi}\right) = \frac{1}{j\omega C}\left(\frac{4}{\pi}\right)$$

$$Z_C = \frac{1}{j\omega C}$$

Compare this to a **lumped capacitor** impedance

we find that results are approximately the **same** for all pass-band frequencies (i.e., when $\omega \ll \omega_c$).

Kuroda's Identities

- We will find that **Kuroda's Identities** can be very useful in making the implementation of Richard's transformations more **practicable.**
- Kuroda's Identities essentially provide a list of **equivalent** two port networks. By equivalent, we mean that they have **precisely** the same scattering/impedance/admittance/transmission matrices.
- In other words, we can **replace** one two-port network with its equivalent in a circuit, and the behavior and characteristics (e.g., its scattering matrix) of the circuit will **not** change!

Q: Why would we want to do this?

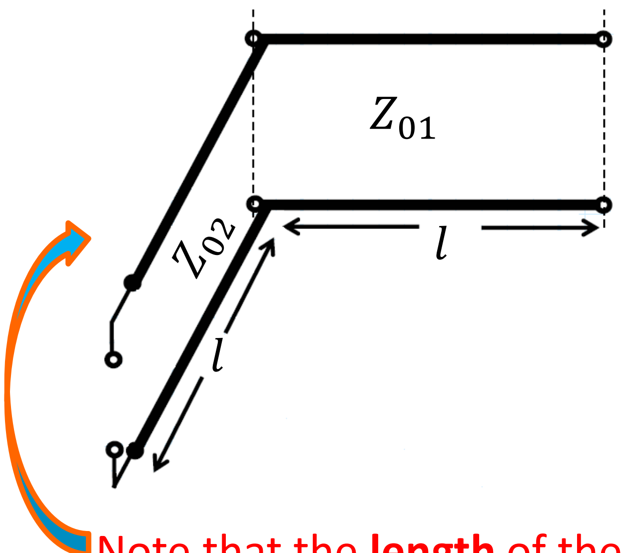
A: Because one of the equivalent may be more **practical** to implement!

For example, we can use Kuroda's Identities to:

1. Physically **separate** transmission line stubs.
2. Transform series stubs into **shunt** stubs.
3. Change impractical **characteristic impedances** into more realizable ones.

Kuroda's Identities (contd.)

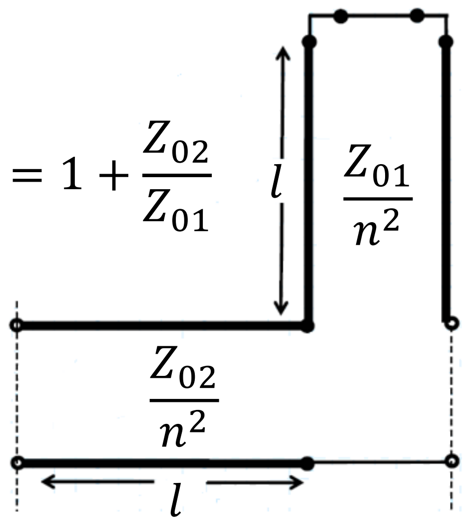
- **Four** Kuroda's identities are provided in a very **ambiguous** and confusing table (Table 8.7) in your **book**. We will find the **first two** identities to be the most useful.
- Consider the following two-port network, constructed with a length of transmission line, and an **open**-circuit **shunt** stub:



Note that the **length** of the stub and the transmission line are **identical**, but the characteristic **impedance** of each are **different**.

- The **first Kuroda identity** states that this two-port network is **precisely** the same two-port network as **the following**:

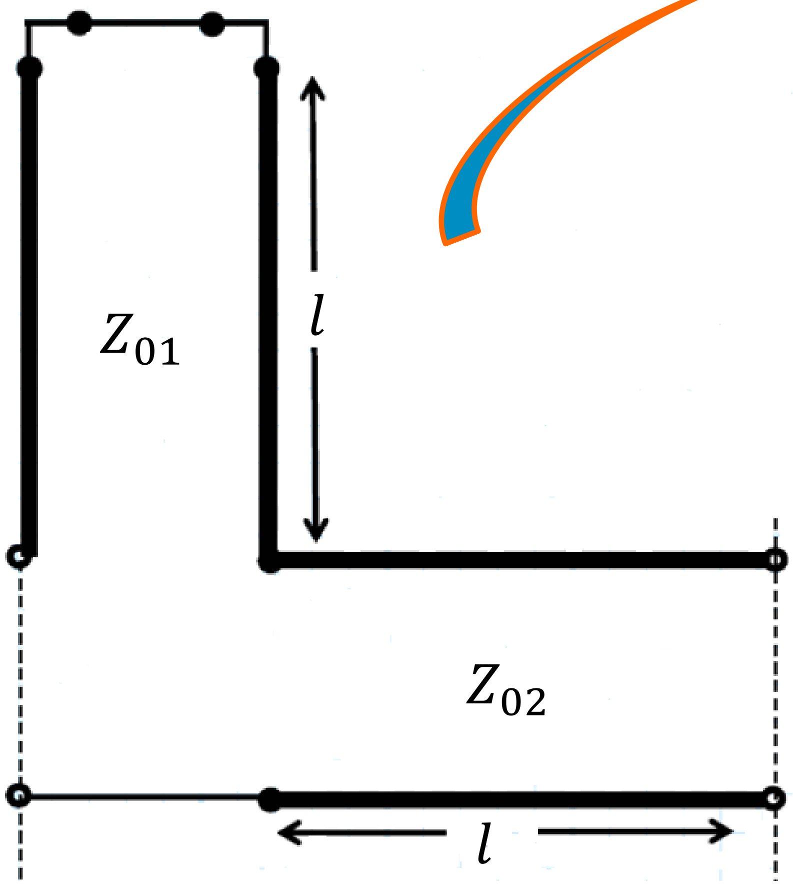
$$n^2 = 1 + \frac{Z_{02}}{Z_{01}}$$



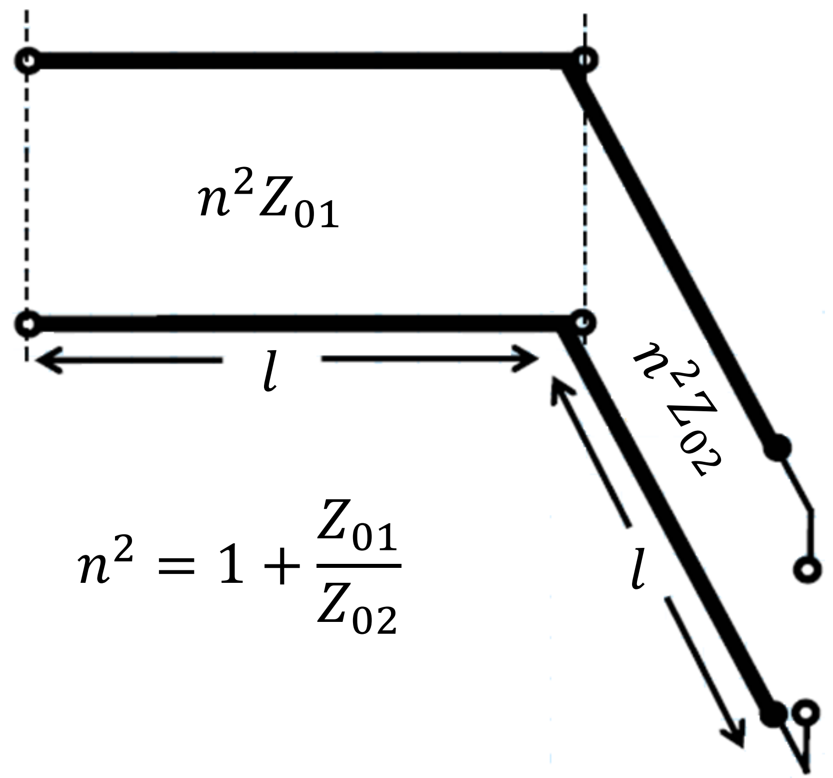
- Thus, we can **replace** the first structure in some circuit with this, and the behavior of that circuit will **not change** in the least!
- Note this equivalent circuit uses a **short-circuited series** stub.

Kuroda's Identities (contd.)

- The **second** of Kuroda's Identities states that this two port network:



Is **precisely identical** to this two-port network:

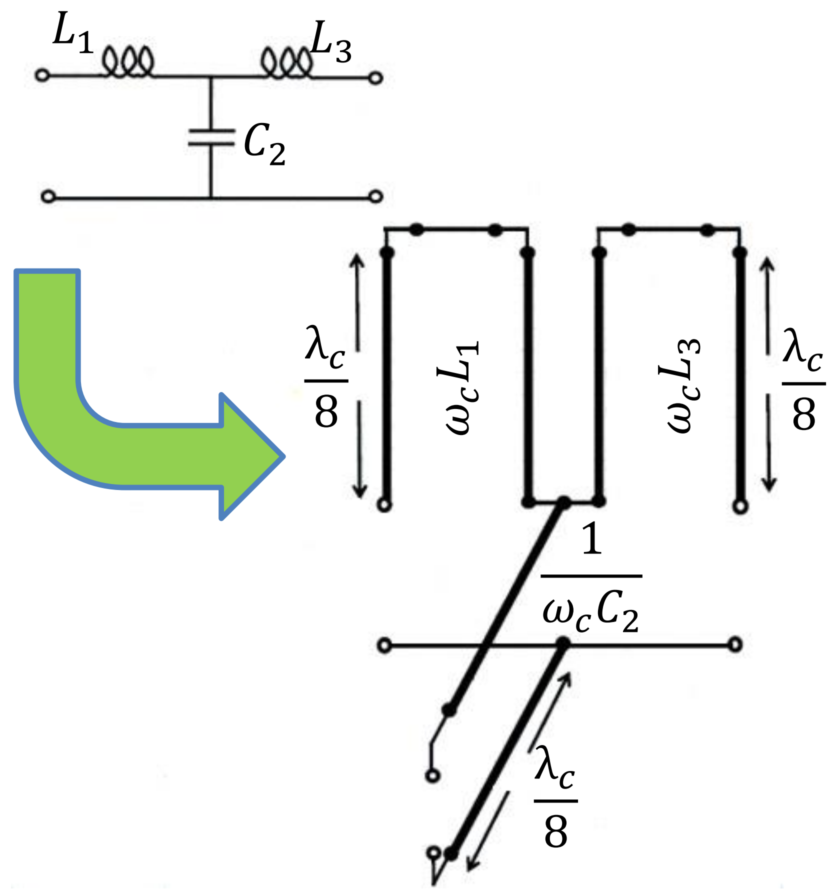


$$n^2 = 1 + \frac{Z_{01}}{Z_{02}}$$

Kuroda's Identities (contd.)

- With regard to **Richard's Transformation**, these identities are useful when we replace the series **inductors** with **shorted stubs**.

- To see **why** this is useful when implementing a **lowpass filter** with distributed elements, consider this third order filter example, realized using Richard's Transformations:



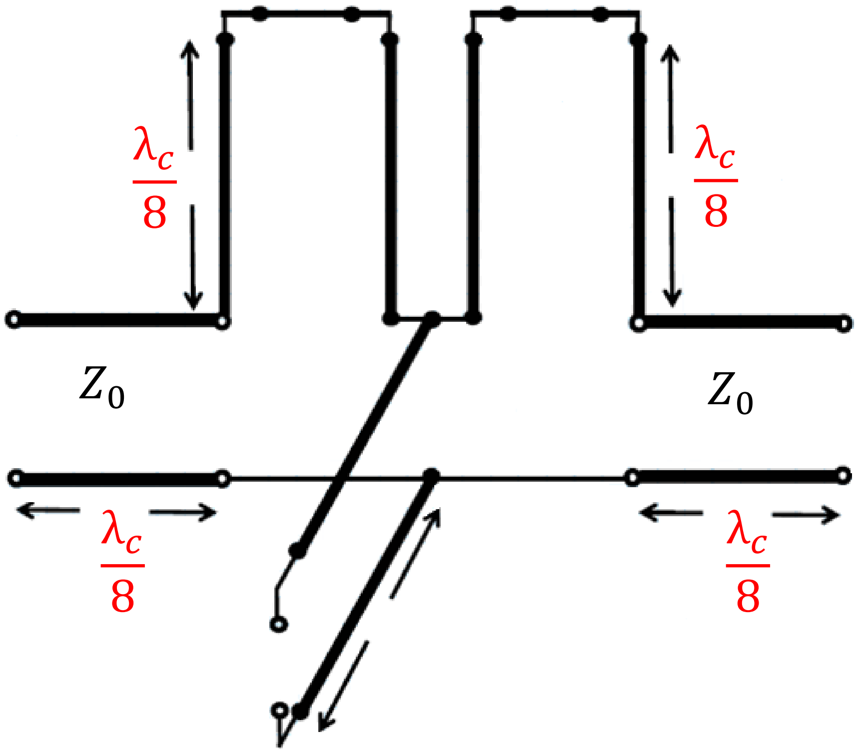
Note that we have a few **problems** in terms of implementing this design!

Kuroda's Identities (contd.)

- First of all the stubs are ideally **infinitely close** to each other— how do we build that? We could physically **separate** them, but this would introduce some transmission **line length** between them that would **mess up** our filter response!

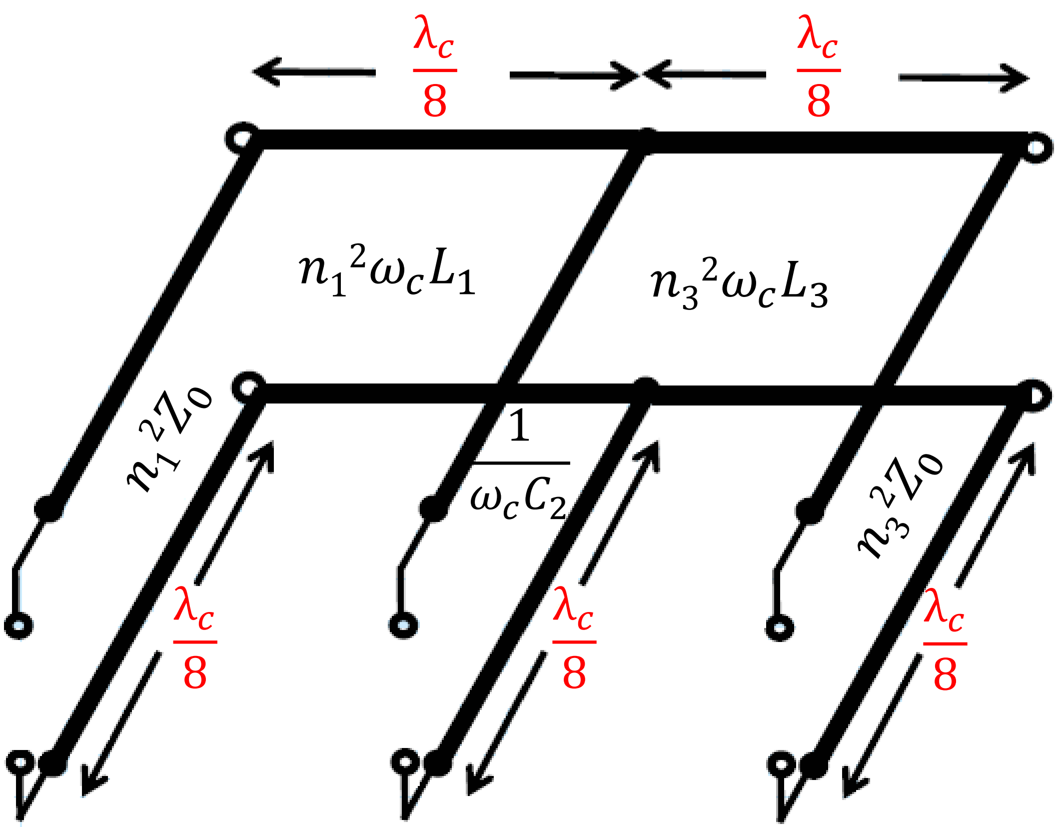
- Secondly, **series** stubs are difficult to construct in microstrip/ stripline—we like **shunt** stubs **much** better!

- To solve these problems, we first **add** a short length of transmission line (Z_0 and $l = \lambda_c/8$) to the **beginning** and **end** of the filter:



Kuroda's Identities (contd.)

- Note adding these lengths only results in a **phase shift** in the filter response—the transmission and reflection functions will remain **unchanged**.
- Then we can use the second of **Kuroda's Identities** to replace the **series** stubs with **shunts**:



Where:

$$n_1^2 = 1 + \frac{Z_0}{\omega_c L_1}$$

$$n_3^2 = 1 + \frac{Z_0}{\omega_c L_3}$$

Now **this** is a realizable filter! Note the **three stubs** are separated, and they are all **shunt** stubs.

Stepped-Impedance Low-Pass Filters

- Another distributed element realization of a lumped element low-pass filter designs is the **stepped-impedance** low-pass filter.
- These filters are also known as **“hi-Z, low-Z”** filters, and we’re about to find out why!

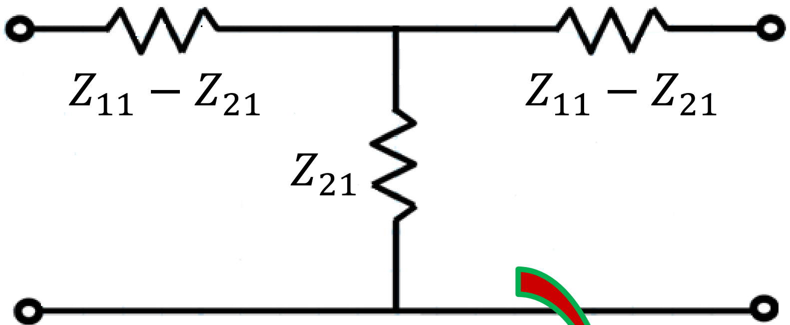
All distributed elements (e.g., transmission lines, coupled lines, resonators, stubs) exhibit **some** frequency dependency. If we are clever, we can construct these structures in a way that their frequency dependency (i.e., $S_{21}(\omega)$) conforms to a desirable function of ω.

Stepped-Impedance Low-Pass Filters (contd.)

- Say we know the impedance matrix of a **symmetric** two-port device:

$$Z = \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{21} & Z_{11} \end{bmatrix}$$

- **Regardless** of the construction of this two port device, we can **model** it as a simple "T-circuit", consisting of three impedances:

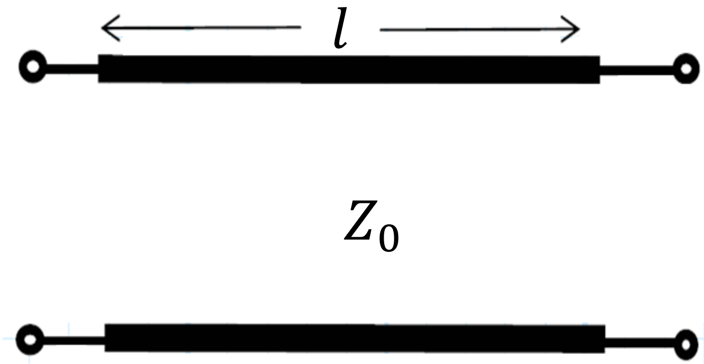


- In other words, if the two **series impedances** have an impedance value equal to $Z_{11} - Z_{21}$, and the **shunt impedance** has a value equal to Z_{21}, the impedance matrix of this "T-circuit" is:

$$Z = \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{21} & Z_{11} \end{bmatrix}$$

Thus, **any** symmetric two-port network can be modeled by this "T-circuit"!

Stepped-Impedance Low-Pass Filters (contd.)



- For example, consider a length l of **transmission line** (a symmetric two-port network!):

- Recall (or determine for yourself!) that the **impedance parameters** of this two port network are:

$$Z_{11} = Z_{22} = -jZ_0 \cot \beta l$$

$$Z_{12} = Z_{21} = -jZ_0 \operatorname{cosec} \beta l$$

- With a little **trigonometry:**

$$Z_{11} - Z_{12} = jZ_0 \tan\left(\frac{\beta l}{2}\right)$$

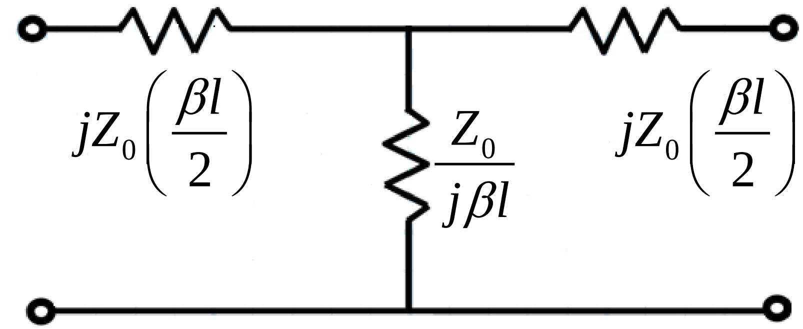
- For small βl:

$$Z_{11} - Z_{12} \approx jZ_0\left(\frac{\beta l}{2}\right)$$

$$Z_{12} = Z_{21} = -jZ_0 \operatorname{cosec} \beta l \approx \frac{Z_0}{j\beta l}$$

Stepped-Impedance Low-Pass Filters (contd.)

- Thus, an **electrically short** ($\beta l \ll 1$) transmission line can be **approximately modeled** with a "T-circuit" as:



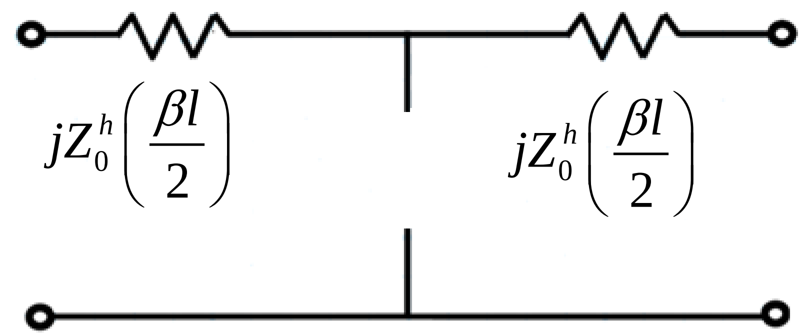
- Now, consider also the case where the **characteristic impedance** of the transmission line is **relatively large**. We'll **denote** this large characteristic impedance as ${Z_0}^h$.
- Note the **shunt** impedance value $\frac{{Z_0}^h}{j\beta l}$. Since the **numerator** (${Z_0}^h$) is relatively **large**, and the **denominator** ($j\beta l$) is **small**, the impedance of shunt device is **very large**.
- So large, in fact, that we can approximate it as an **open circuit**!

$$\boxed{\frac{Z_0^h}{j\beta l} \approx \infty}$$

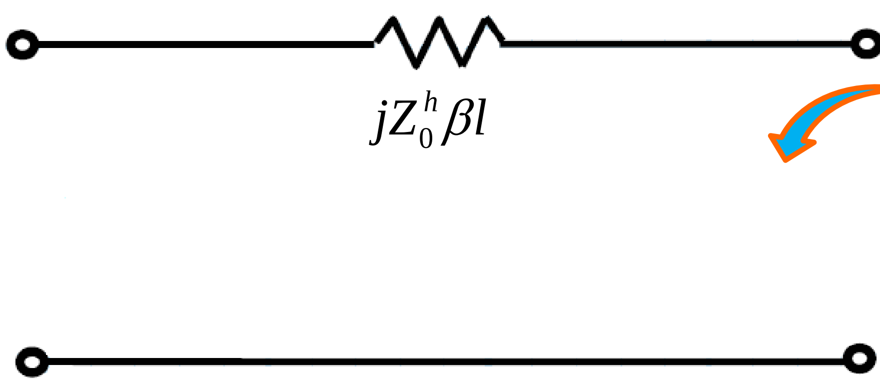
For $\beta l \ll 1$ and ${Z_0}^h \gg Z_0$

Stepped-Impedance Low-Pass Filters (contd.)

- So now we have a further **simplification** of our **model**:



- The remaining impedances are now in **series**, so the circuit can be further simplified to:

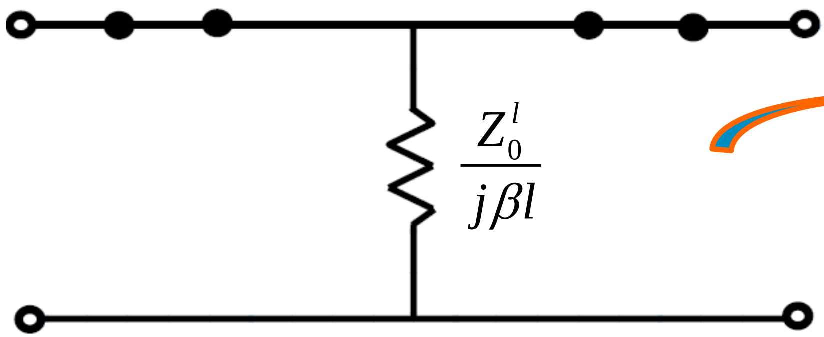


The equivalent circuit for transmission line with short electrical length ($\beta l \ll 1$) and large characteristic impedance ($Z_0{}^h \gg Z_0$)

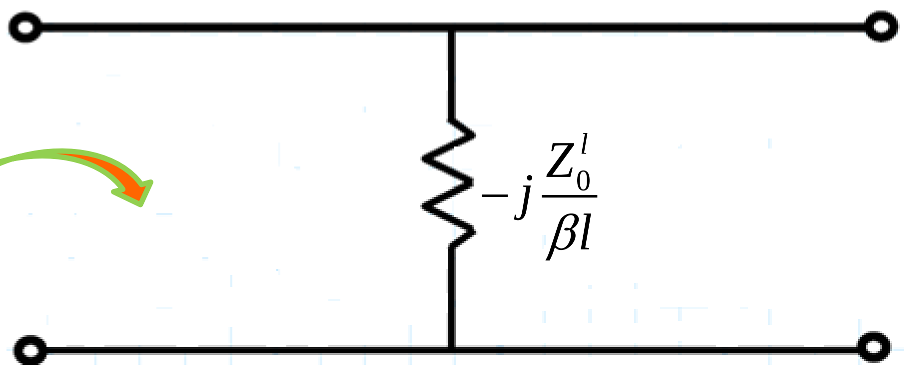
- Now, consider the case where the **characteristic impedance** of the transmission line has a relatively **low value**, denoted as $Z_0{}^l$, where $Z_0{}^l \ll Z_0$.

Stepped-Impedance Low-Pass Filters (contd.)

- In such a case: $Z_{11} - Z_{12} \approx jZ_0^l\left(\frac{\beta l}{2}\right) \approx 0$ For $\beta l \ll 1$ and $Z_0{}^l \ll Z_0$
- So now we have **another simplification** of our model:



Which **of course** further simplifies to



The equivalent circuit for transmission line with short electrical length ($\beta l \ll 1$) and small characteristic impedance (($Z_0{}^l \ll Z_0$).

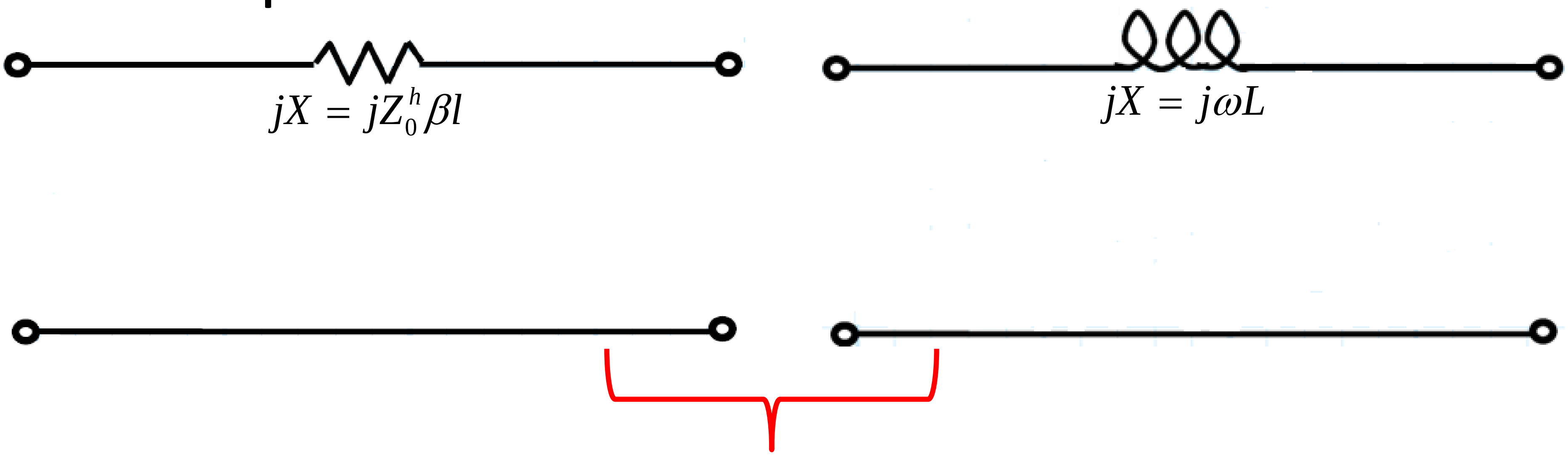
Q: But, **what** does all this have to do with constructing a **low-pass filter**?

A: **Plenty!** Recall that a lossless low-pass filter constructed with **lumped** elements consists of a "circuit ladder" of **series inductors** and **shunt capacitors**!

Stepped-Impedance Low-Pass Filters (contd.)

Q: So?

A: Look at the two **equivalent circuits** for an electrically short transmission line. The one with **large** characteristic impedance $Z_0{}^h$ has the form of a **series inductor**, and the one with **small** characteristic impedance $Z_0{}^l$ has the form of a **shunt capacitor**!



are identical **if**:

$$jZ_0^h \beta l = j\omega L \quad \Rightarrow \quad Z_0^h \beta l = \omega L$$

Stepped-Impedance Low-Pass Filters (contd.)

- Thus, the "**series inductance**" of our **transmission line** length is:

$$L = \frac{Z_0^h \beta l}{\omega}$$

Q: Yikes! Our inductance appears to be a function of **frequency** ω. I **assume** we simply **set** this value to **cut-off frequency** ω_c, just like we did for Richard's transformation?

A: Nope! We can **simplify** the result a bit more. Recall that $\beta = {}^{\omega}/_{v_p}$, so that:

$$L = \frac{Z_0^h \beta l}{\omega} = \frac{Z_0^h l}{v_p}$$

- In other words, the **series impedance** resulting from our short transmission line is:

$$Z = j\omega L = j\omega\left(\frac{Z_0^h l}{v_p}\right)$$

Q: Wow! This realization seems to give us a result that **precisely** matches an inductor at **all** frequencies—**right**?

A: Not quite! Recall this result was obtained from applying a few **approximations**—the result is **not** exact!

Stepped-Impedance Low-Pass Filters (contd.)

Moreover, one of the approximations was that the **electrical length** of the transmission line be **small**. This obviously **cannot** be true at **all** frequencies. As the signal frequency **increases**, so does the **electrical length**—our **approximate** solution will **no longer be valid**.

Thus, this realization is accurate **only** for "**low** frequencies"— recall that was **likewise** true for **Richard's transformations**!

Q: Low-frequencies? How **low** is **low**?

A: Well, for our filter to provide a response that **accurately** follows the **lumped element** design, our approximation should be valid for frequencies **up to** (and including!) the **filter cut-off frequency** ω_c.

- A general "**rule-of-thumb**" is that a **small electrical length** is defined as being **less than** π/4 radians. Thus, to maintain this small electrical length at frequency ω_c, our realization **must** satisfy the relationship:

$$\beta_c l = \frac{\omega L}{Z_0^h} < \frac{\pi}{4}$$

Stepped-Impedance Low-Pass Filters (contd.)

- Note that this criterion is **difficult** to satisfy if the **filter cut-off frequency** and/or the **inductance value L** that we are trying to realize is **large**.
- Our **only** recourse for these challenging conditions is to **increase** the value of **characteristic impedance** Z_0^h .

Q: Is there some particular difficulty with increasing Z_0^h ?

A: Could be! There is always a **practical** limit to how large (or small) we can make the **characteristic impedance** of a transmission line.

For example, a **large** characteristic impedance implemented in **microstip/stripline** requires a **very narrow** conductor **width W.** But manufacturing tolerances, power handling capability and/or line loss (line resistance R increases as W decreases) place a **lower bound** on how **narrow** we can make these conductors!

- However, assuming that we **can** satisfy the above constraint, we can approximately **"realize"** a **lumped inductor** of inductance **value L** by selecting the correct **characteristic impedance** Z_0^h and **line length** l of our short transmission line.

$$L = \frac{Z_0^h l}{v_p}$$

Stepped-Impedance Low-Pass Filters (contd.)

Q: For **Richard's Transformation**, we **first** set the stub length to a **fixed** value (i.e., $l = \lambda_c/8$), and **then** determined the **specific characteristic impedance** necessary to realize a **specific inductor value** L. I **assume** we follow the same procedure here?

A: **Nope!** When constructing stepped-impedance low-pass filters, we typically do the **opposite**!

1. **First**, we select the value of Z_0^h, making sure that the short electrical length inequality is **satisfied** for the **largest inductance value** L in our lumped element filter:

$$Z_0^h > \frac{4\omega_c L}{\pi}$$

This characteristic impedance value is typically used to realize **all** inductor values L in our low-pass filter, **regardless** of the actual value of inductance L.

2. Then, we determine the **specific lengths** l_n of the transmission line required to realize **specific** filter **inductors values** L_n:

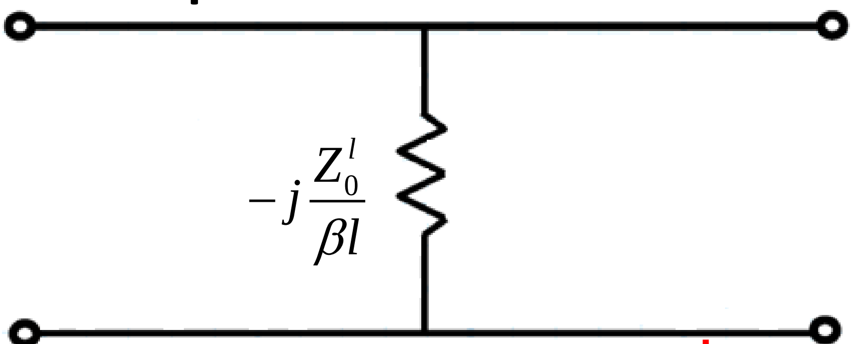
$$l_n = \left(\frac{v_p}{Z_0^h}\right) L_n$$

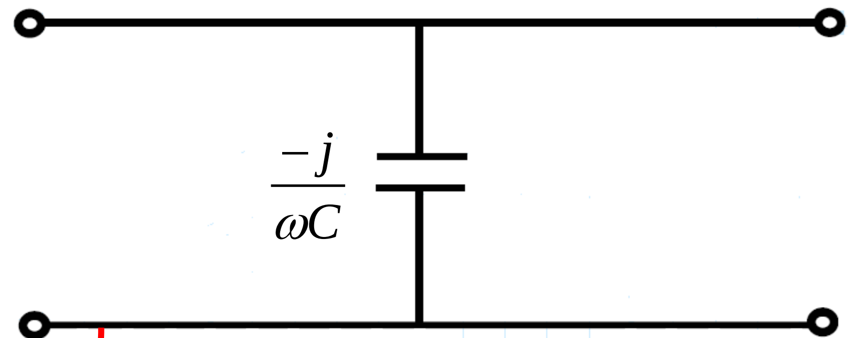
Stepped-Impedance Low-Pass Filters (contd.)

Q: What about the **shunt capacitors**?

A: Almost forgot!

- Recall the **low-impedance** transmission line provided a **shunt impedance** that matched a shunt capacitor:





$$-j\frac{Z_0^l}{\beta l} = \frac{-j}{\omega C}$$

are identical **if**:

$$\frac{\beta l}{Z_0^l} = \omega C$$

- Thus, the "**shunt capacitance**" of our transmission line length is:

$$C = \frac{\beta l}{\omega Z_0^l} \quad \Rightarrow \quad C = \frac{l}{v_p Z_0^l}$$

Stepped-Impedance Low-Pass Filters (contd.)

- And thus the **shunt reactance** of our transmission line realization is:

$$Z = \frac{-j}{\omega}\left(\frac{v_p Z_0^l}{l}\right)$$

Although this again **appears** to provide **exactly** the same behavior as a **capacitor** (as a function of frequency), it is likewise accurate **only** for **low frequencies**, where $\beta l < \frac{\pi}{4}$.

- Thus from our realization **equality**: $\frac{\beta l}{Z_0^l} = \omega C$

- We can conclude that for our approximations to be valid at all frequencies **up to** the filter **cut-off frequency**, the following inequality **must** be valid:

$$\beta_c l = \omega_c C Z_0^l < \frac{\pi}{4}$$

Note that for **difficult** design cases where ω_c and/or C is **very large**, the line **characteristic impedance** $Z_0{}^l$ must be made **very small**.

Stepped-Impedance Low-Pass Filters (contd.)

Q: I suppose there is **likewise** a problem with making $Z_0{}^l$ **very small**?

A: Yes! In microstrip and stripline, making $Z_0{}^l$ **small** means making conductor width W **very large**. In other words, it will take up **lots of space** on our substrate. For most applications the **surface area** of the substrate is both **limited** and precious, and thus there is generally a **practical limit** on how wide we can make width W (i.e., how **low** we can make $Z_0{}^l$).

- However, assuming that we **can** satisfy the above constraint, we can approximately **"realize"** a **lumped capacitor** of inductance value C by selecting the correct **characteristic impedance** $Z_0{}^l$ and **line length** l of our short transmission line:

$$C = \frac{l}{v_p Z_0^l}$$

Stepped-Impedance Low-Pass Filters (contd.)

- The **design rules** for **shunt capacitor realization** are:

1. **First**, we select the value of $Z_0{}^l$, making sure that the short electrical length inequality is **satisfied** for the **largest** capacitance value C in our lumped element filter:

$$Z_0^l < \frac{\pi}{4\omega C}$$

This characteristic impedance value is typically used to realize **all** capacitor values C in our low-pass filter, **regardless** of the actual value of capacitance C.

2. Then, we determine the **specific lengths** l_n of the transmission line required to realize **specific** filter capacitor values C_n:

$$l_n = \left(v_p Z_0^l\right) C_n$$

Stepped-Impedance Low-Pass Filters (contd.)

- An **example** of a low-pass, stepped-impedance filter design is provided on page 414-416 of **your** book