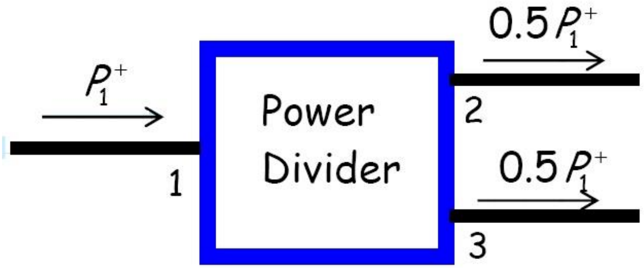
Lecture – 16

Date: 06.03.2017

- Wilkinson Power Divider
- Wilkinson Power Divider Analysis

The (Nearly) Ideal T- Junction Power Divider

- Recall that we **cannot build a matched, lossless** reciprocal **three-port device.**
- So, let's **mathematically try and determine the scattering** matrix of the best possible T-junction 3 dB **power divider.**



- To **efficiently divide the power incident on the input port, the** port (port 1) must first be **matched (i.e., all incident power** should be delivered to port 1): $S_{11} = 0$
- Likewise, this delivered power to port 1 must be divided efficiently (i.e., **without loss) between ports 2 and 3.**
- Mathematically, this means that the first column of the scattering matrix must have **magnitude of 1.0:**

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1 \quad \xrightarrow{S_{11}=0} \quad |S_{21}|^2 + |S_{31}|^2 = 1$$

- Provided that we wish to evenly divide the input power, we can conclude from the expression above that:

$$|S_{21}|^2 = |S_{31}|^2 = 1/2 \quad \Longrightarrow \quad |S_{21}| = |S_{31}| = 1/\sqrt{2}$$

The (Nearly) Ideal T- Junction Power Divider (contd.)

- Note that **this device would take the power into port 1 and** divide into **two equal parts—half exiting port 2, and half** exiting **port3 (provided ports 2 and 3 are terminated in** matched loads!).

$$P_2^- = |S_{21}|^2 P_1^+ = 0.5 P_1^+$$

$$P_3^- = |S_{31}|^2 P_1^+ = 0.5 P_1^+$$

- In addition, it is **desirable that ports 2 and 3 be matched** (the whole device is thus matched):

$$S_{22} = S_{33} = 0$$

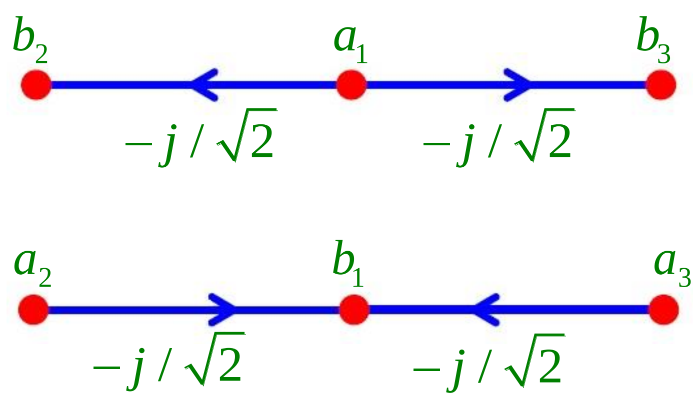
- And also **desirable that ports 2 and 3 be isolated:**

$$S_{23} = S_{32} = 0$$

This last requirement ensures that no signal incident on port 2 (e.g., reflected from a load) will **"leak" into port 3—and vice** versa.

- The ideal 3 dB power divider **could therefore have the form:**

$$S = \begin{bmatrix} 0 & -j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 0 & 0 \\ -j/\sqrt{2} & 0 & 0 \end{bmatrix}$$



Since we can describe this ideal power divider **mathematically, we can potentially build it physically!**

The (Nearly) Ideal T- Junction Power Divider (contd.)

Q: Huh!? I thought you said that a matched, lossless, reciprocal three-port device is **impossible?**

A: It is! This divider is clearly a lossy device. The magnitudes of both column 2 and 3 are less than one:

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = \left|-j/\sqrt{2}\right|^2 + 0 + 0 = 0.5 < 1 \qquad |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = \left|-j/\sqrt{2}\right|^2 + 0 + 0 = 0.5 < 1$$

Note then that **half the power incident on port 2 (or port 3)** of this device would **exit port 1 (i.e., reciprocity), but no power** would exit port 3 (port2), since ports 2 and 3 are **isolated. i.e.,**

$$P_1^- = |S_{12}|^2 P_2^+ = 0.5 P_2^+ \qquad P_3^- = |S_{32}|^2 P_2^+ = 0 * P_2^+ = 0 \qquad P_1^- = |S_{13}|^2 P_3^+ = 0.5 P_3^+ \qquad P_2^- = |S_{23}|^2 P_3^+ = 0 * P_3^+ = 0$$

Q: Any ideas on how to build this thing?

A: Note that the first column of the scattering matrix is precisely the same as that of the **lossless 3 dB divider.**

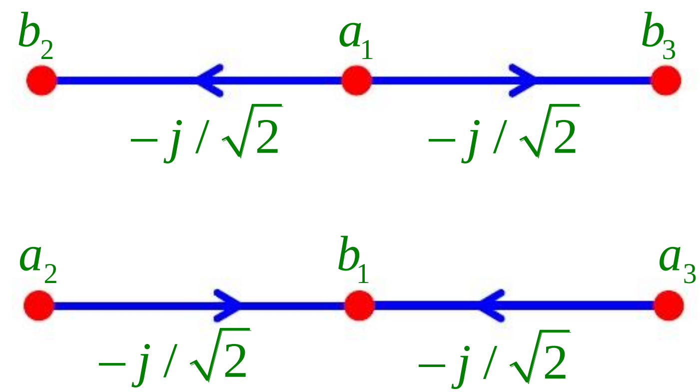
Also note that since the device is **lossy, the design must** include some **resistors.**

Lossless Divider + resistors = The Wilkinson Power Divider

Wilkinson Power Divider

- Wilkinson power divider is the **nearly** ideal T-junction power divider → It is **lossy, matched** and **reciprocal.**
- Therefore, the scattering matrix and SFG of Wilkinson power divider is of the form:

$$S = \begin{bmatrix} 0 & -j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 0 & 0 \\ -j/\sqrt{2} & 0 & 0 \end{bmatrix}$$



- Note this device is **matched at port 1 (S_{11} = 0), and we find that** magnitude of column 1 is: $|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1$
- Just like the **lossless divider, the incident power on port 1** is **evenly and efficiently divided between the outputs of port 2** and port 3

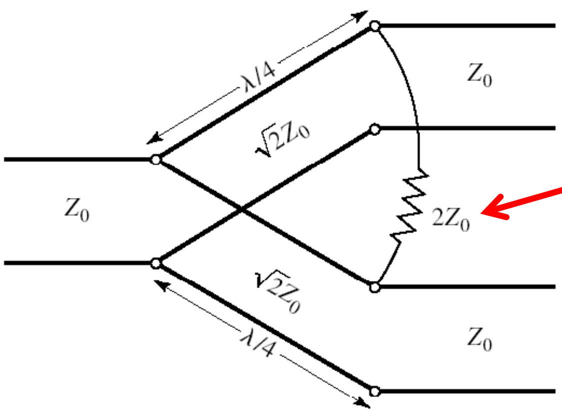
$$P_2^- = |S_{21}|^2 P_1^+ = 0.5P_1^+ \qquad P_3^- = |S_{31}|^2 P_1^+ = 0.5P_1^+$$

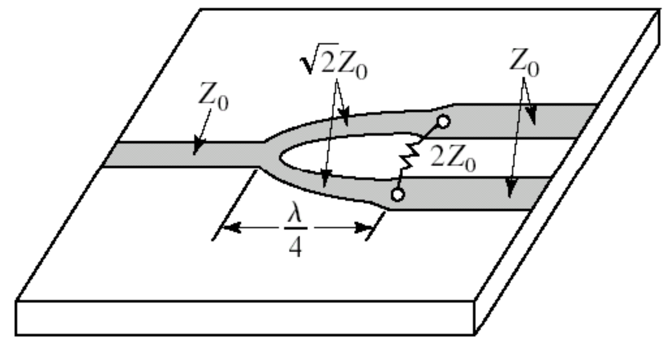
- It is also apparent that the ports 2 and 3 of this device are **matched** ! $S_{22} = S_{33} = 0$
- We also note that ports 2 and ports 3 are **isolated**: $S_{23} = S_{32} = 0$

Wilkinson Power Divider (contd.)

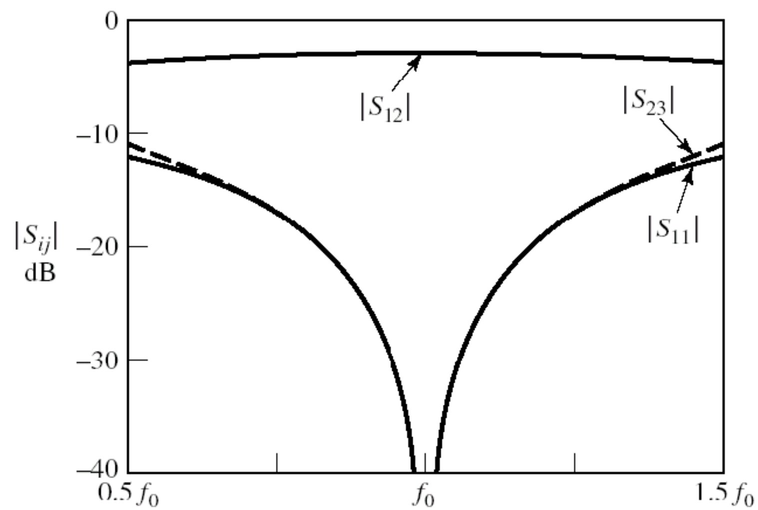
Q: Ok, so it is a (nearly) ideal divider → **but** how do we **make** this Wilkinson power divider?

A: It looks a lot like a **lossless 3dB divider**, only with an additional **resistor** of value $2Z_0$ between ports 2 and 3:



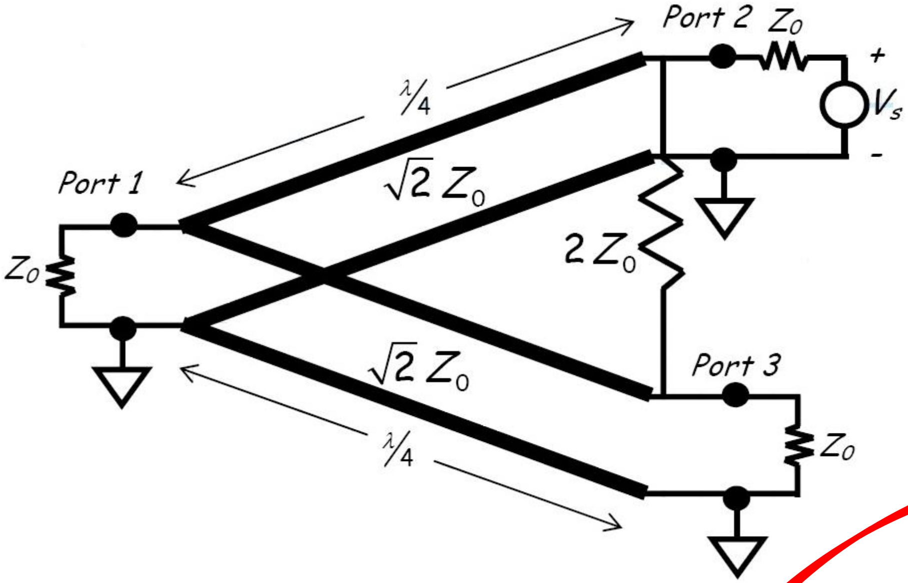


- This resistor is the **secret** to the Wilkinson power divider, and is the reason that it is **matched** at ports 2 and 3, and the reason that ports 2 and 3 are **isolated**.
- Note however, that the **quarter-wave** transmission line sections make the Wilkinson power divider a **narrow-band** device.

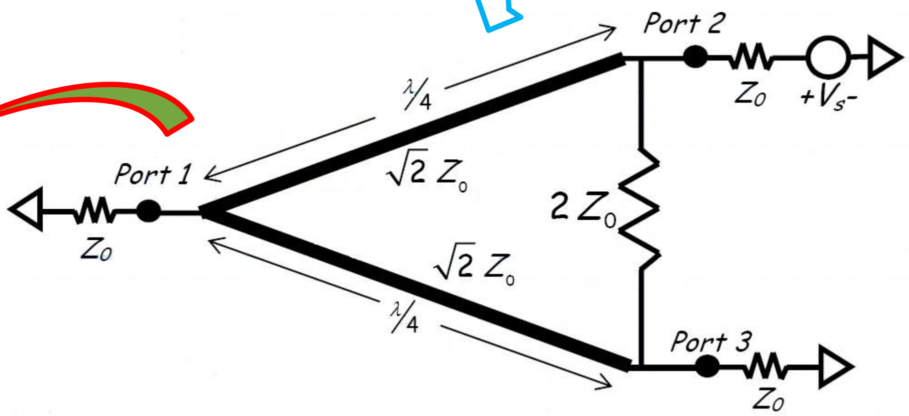


Analysis of Wilkinson Power Divider

- Consider a matched **Wilkinson power divider**, with a **source at port 2**:



To **simplify** this schematic, let us **remove** the ground plane, which includes the **bottom conductor** of the transmission lines:



Q: How do we **analyze** this circuit ?

A: Use **Even-Odd mode** analysis!

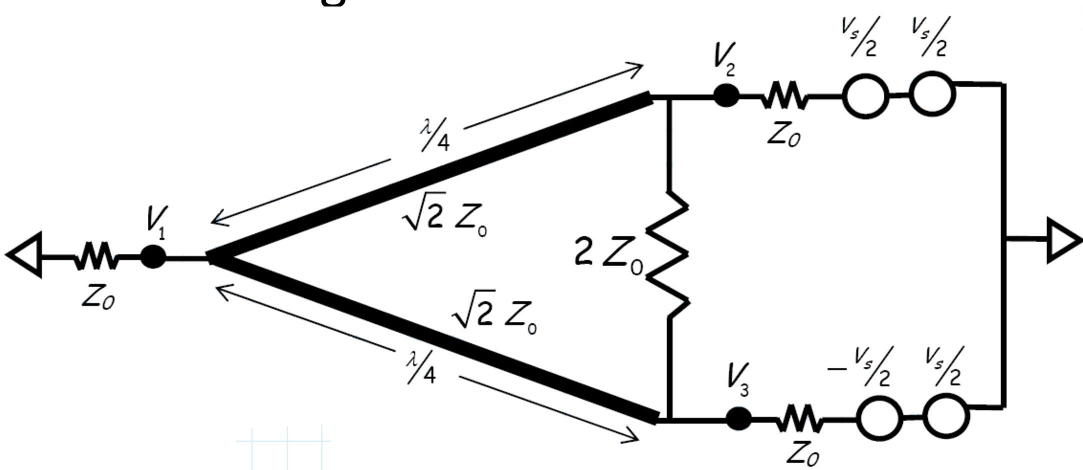
Analysis of Wilkinson Power Divider (contd.)

Remember, even-odd mode analysis uses **two** important principles:

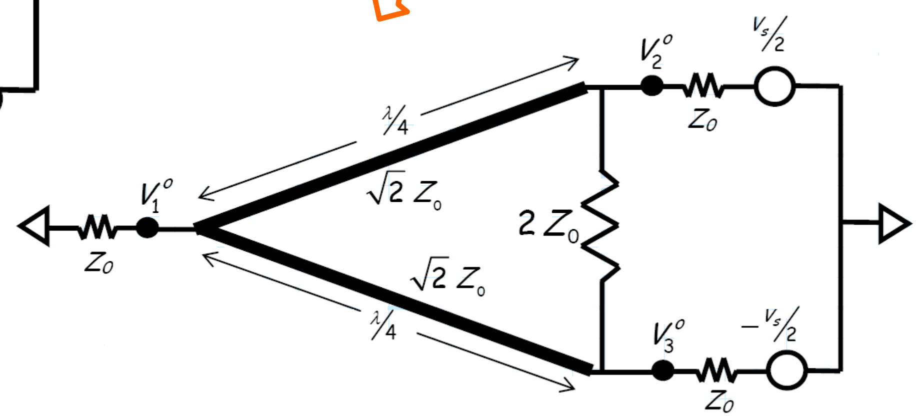
a) **superposition**

b) circuit **symmetry**

- To see how we apply these principles, let's first rewrite the circuit with **four** voltage sources:

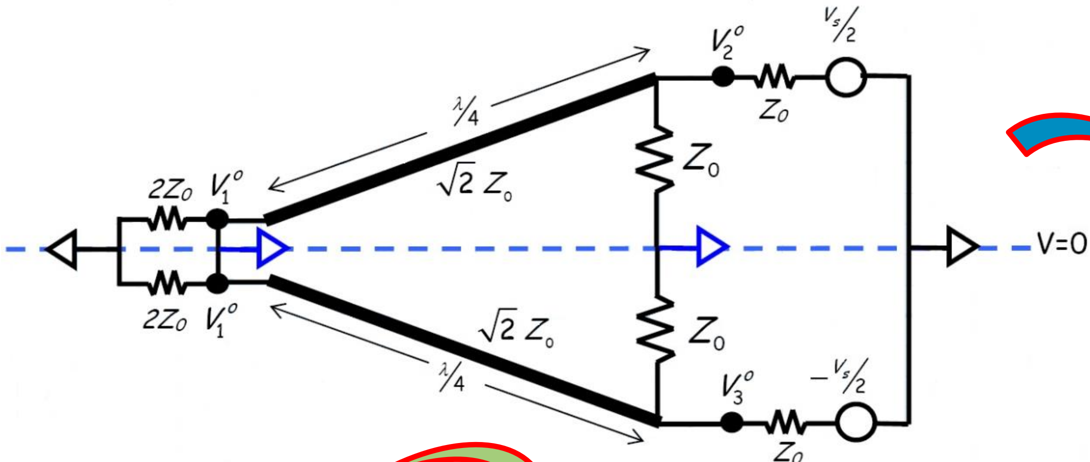


Turning off **one positive source** at each port, we are left with an **odd mode** circuit

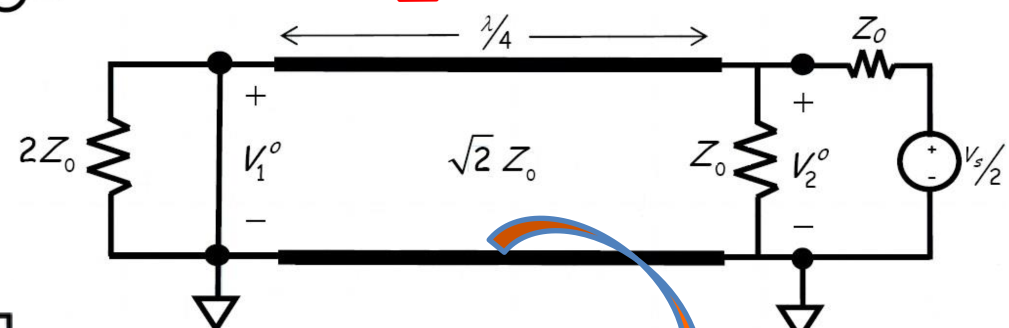


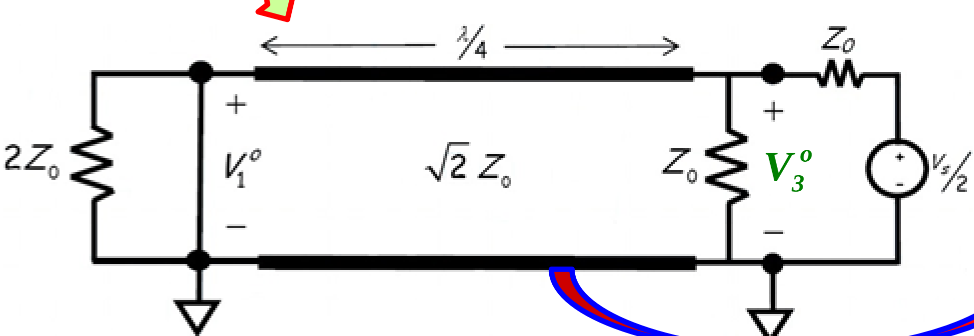
Analysis of Wilkinson Power Divider (contd.)

- Note the circuit has **odd symmetry**, and thus the plane of symmetry becomes a **virtual short**, and in this case, a virtual **ground**!



Dividing the circuit into two **half-circuits**, we get:

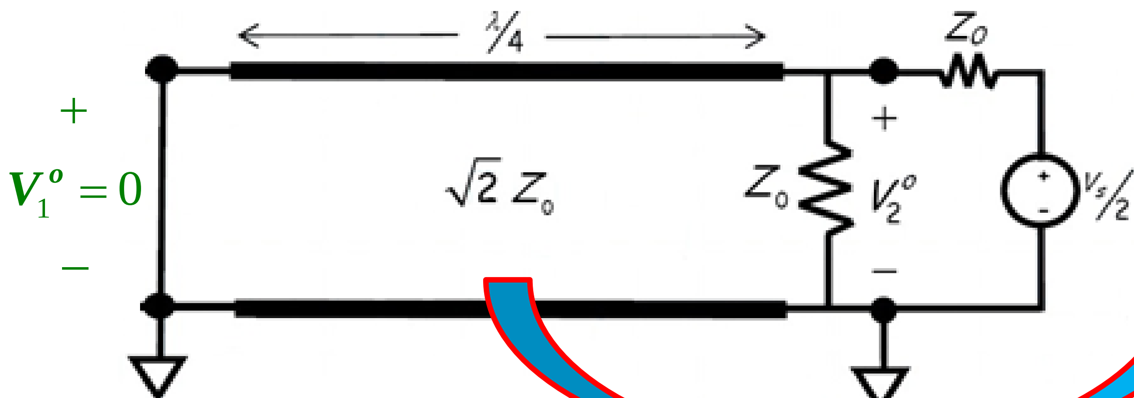




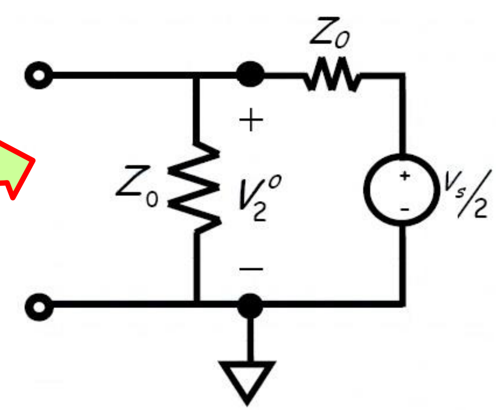
Note the **bottom conductor** of the transmission line (a **ground plane**) has been redrawn to enhance **clarity** (I **hope**!)

Analysis of Wilkinson Power Divider (contd.)

- Analyzing the first half-circuit, we find that the transmission line is terminated in a **short** circuit in **parallel** with a **resistor** of value $2Z_0$. Thus, the transmission line is terminated in a **short circuit**!



Now, since the transmission line is a **quarter wavelength**, this **short** circuit at the **end** of the transmission line transforms to an **open** circuit at the **beginning**!



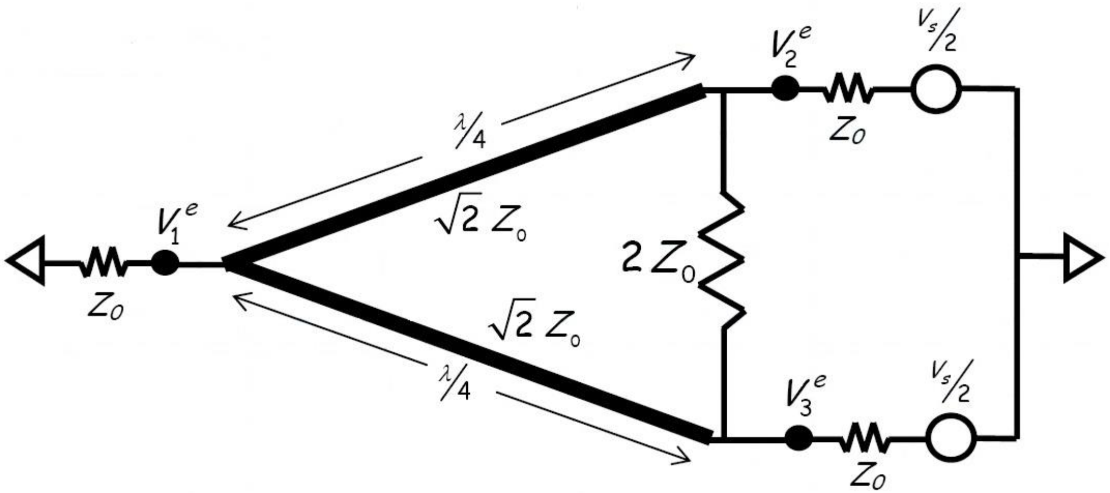
$$V_2^o = \frac{V_s}{2}\frac{Z_0}{Z_0+Z_0} = \frac{V_s}{4}$$

- From the **odd symmetry** of the circuit, we can similarly determine:

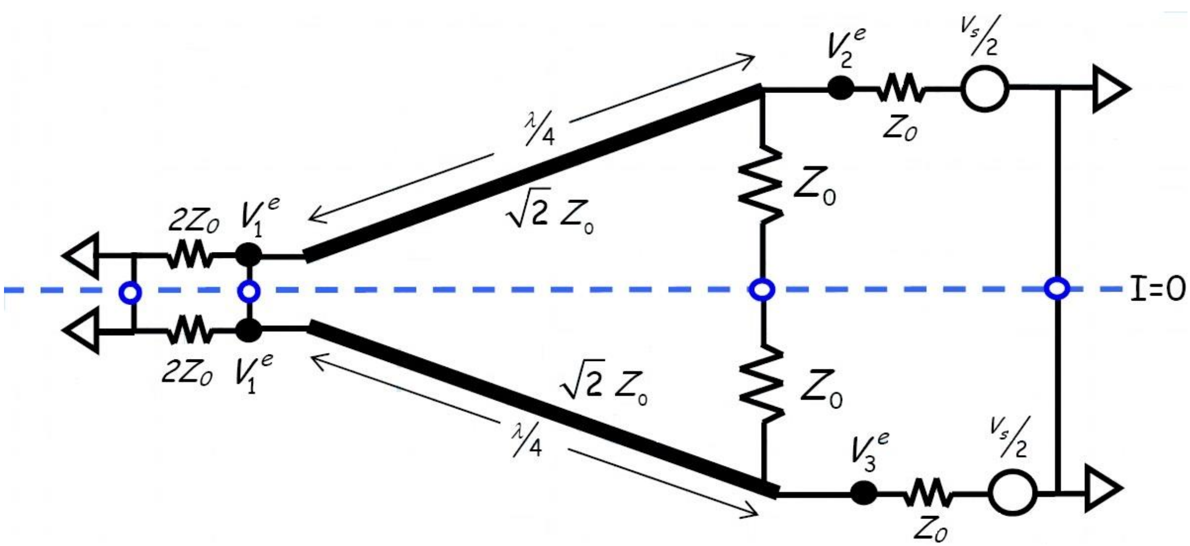
$$V_3^o = -\frac{V_s}{4}$$

Analysis of Wilkinson Power Divider (contd.)

- Now, let's turn **off** the **odd mode sources**, and turn back **on** the **even mode sources.**

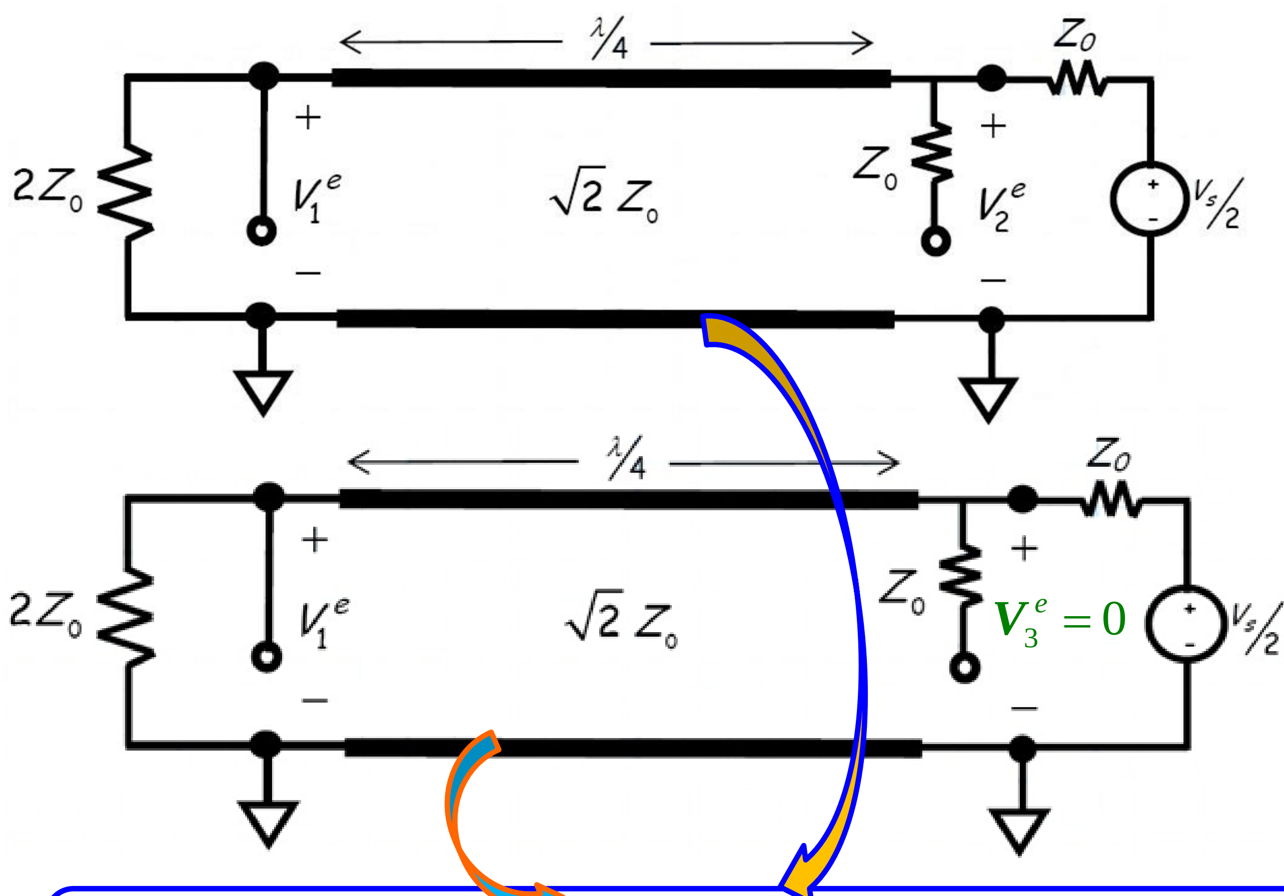


Note the circuit has **even symmetry**, and thus the plane of symmetry becomes a **virtual open**.



Analysis of Wilkinson Power Divider (contd.)

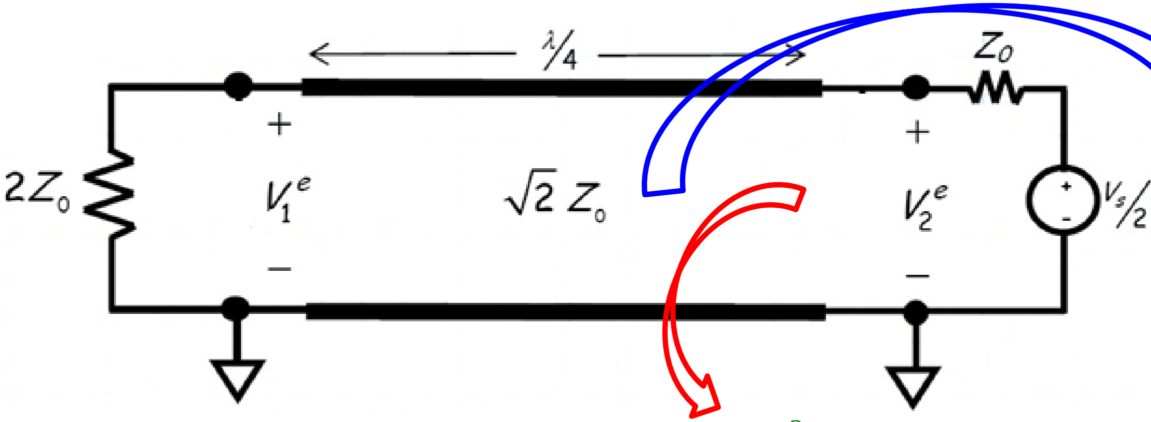
- Dividing the circuit into two **half-circuits**, we get:



Note we have **again** drawn the **bottom conductor** of the transmission line (a **ground plane**).

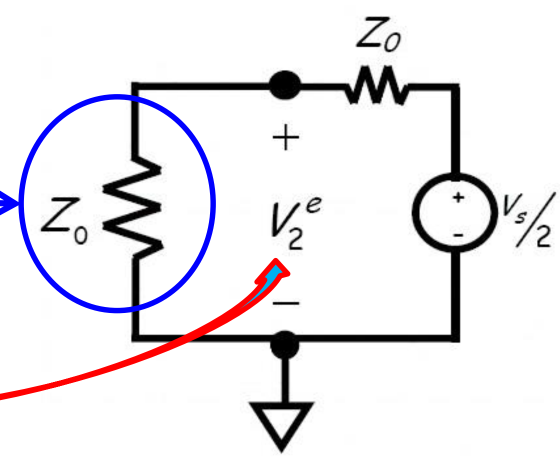
Analysis of Wilkinson Power Divider (contd.)

- Analyzing the first circuit, we find that the transmission line is terminated in an **open** circuit in **parallel** with a **resistor** of value $2Z_0$. Thus, the transmission line is terminated in a **resistor** valued $2Z_0$.



Now, since the transmission line is a **quarter wavelength**, the $2Z_0$ resistor at the **end** of the TL transforms the circuit as:

$$Z_{in} = \frac{\left(\sqrt{2}Z_0\right)^2}{2Z_0} = Z_0$$



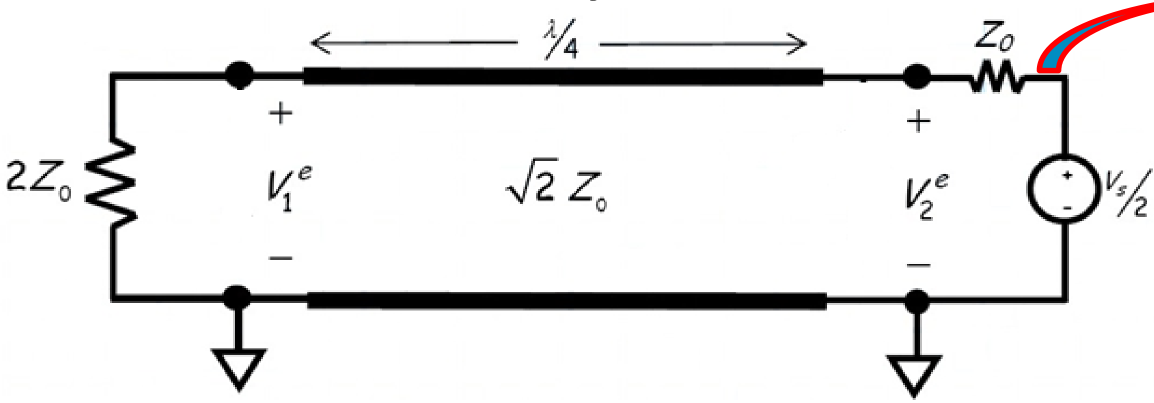
$$V_2^e = \frac{V_s}{2}\frac{Z_0}{Z_0+Z_0} = \frac{V_s}{4}$$

- Then due to the **even symmetry** of the circuit, we can say:

$$V_3^e = \frac{V_s}{4}$$

Analysis of Wilkinson Power Divider (contd.)

- there's **no** direct or easy way to find V_1^e. We must apply TL theory (i.e., the solution to the **telegrapher's equations** + **boundary conditions**) to find this value. This means **applying** the knowledge and skills acquired during our **scholarly** examination of **TL Theory**!



If we **carefully** and **patiently** analyze the above TL circuit, we find that (see if **you** can verify this!):

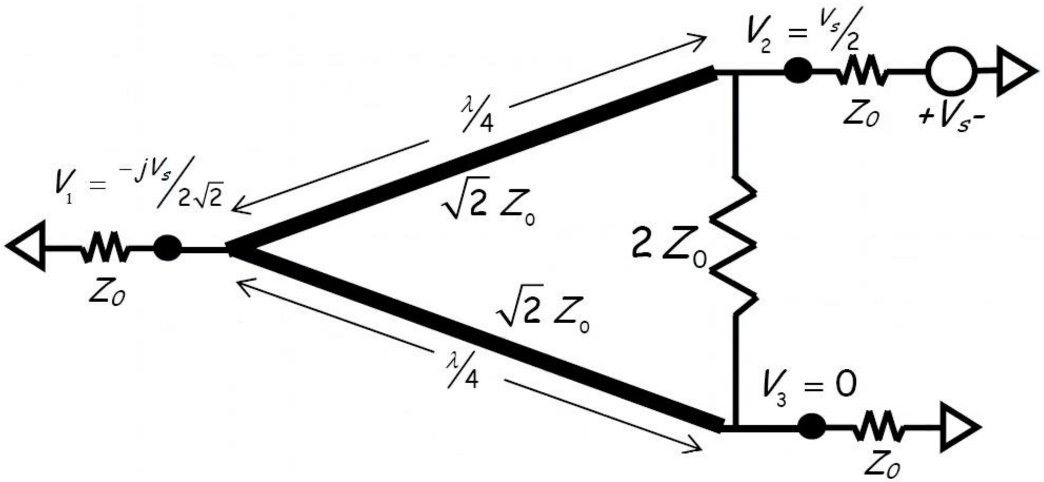
$$V_1^e = \frac{-jV_s}{2\sqrt{2}} \qquad V_1^o = 0$$

- This completes our symmetry analysis and then from **superposition**, the voltages within the circuit is simply found from the **sum** of the solutions of each mode:

$$V_1 = V_1^o + V_1^e = 0 + \frac{(-jV_s)}{2\sqrt{2}} = -\frac{jV_s}{2\sqrt{2}}$$

$$V_2 = V_2^o + V_2^e = \frac{V_s}{4} + \frac{V_s}{4} = \frac{V_s}{2} \qquad V_3 = V_3^o + V_3^e = -\frac{V_s}{4} + \frac{V_s}{4} = 0$$

Analysis of Wilkinson Power Divider (contd.)



- Note that the voltages we calculated are **total voltages**—the **sum** of the **incident** and **exiting** waves at each port:

$$V_1 \doteq V_1(z_1 = z_{1p}) = V_1^+(z_1 = z_{1p}) + V_1^-(z_1 = z_{1p})$$

$$V_2 \doteq V_2(z_2 = z_{2p}) = V_2^+(z_2 = z_{2p}) + V_2^-(z_2 = z_{2p})$$

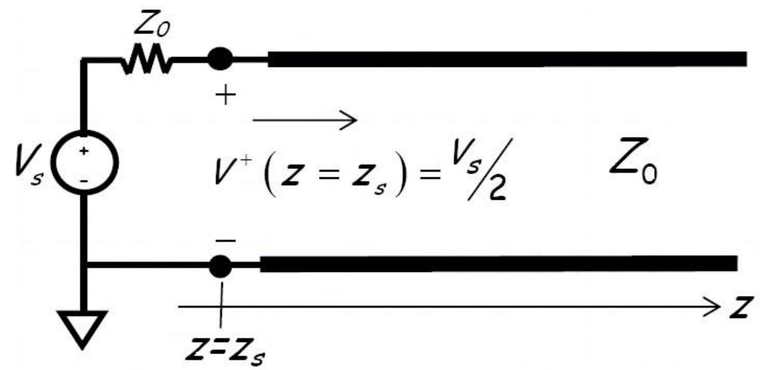
$$V_3 \doteq V_3(z_3 = z_{3p}) = V_3^+(z_3 = z_{3p}) + V_3^-(z_3 = z_{3p})$$

- Since ports 1 and 3 are terminated in **matched loads**, and we also know that the **incident** wave on those ports are **zero**. As a result, the **total** voltage is equal to the value of the exiting waves at those ports.

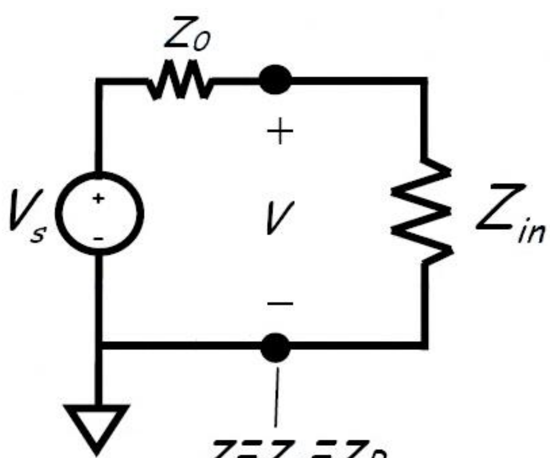
$$V_1^+(z_1 = z_{1p}) = 0 \qquad V_1^-(z_1 = z_{1p}) = \frac{-jV_s}{2\sqrt{2}} \qquad V_3^+(z_3 = z_{3p}) = 0 \qquad V_3^-(z_3 = z_{3p}) = 0$$

Analysis of Wilkinson Power Divider (contd.)

- The problem now is to determine the values of the **incident** and **exiting** waves at port 2.
- For this purpose, let us consider this circuit where the **source impedance** is **matched** to TL characteristic impedance (i.e., $Z_s = Z_0$). We can find, the incident wave "launched" by the source **always** has the value **$V_s/2$** at the start of the line.



- Now, if the length of the transmission line connecting the source to a port (or load) is **electrically very small** (i.e., $\beta l \ll 1$), then the source is effectively **connected directly** to the source (i.e, $\beta z_s = \beta z_p$):



Thus the **total** voltage is:

$$V = V^+(z = z_p) + V^-(z = z_p)$$

$$= V^+(z = z_s) + V^-(z = z_p)$$

$$\therefore V = \frac{V_s}{2} + V^-(z = z_p) \Rightarrow V^-(z = z_p) = V - \frac{V_s}{2}$$

Analysis of Wilkinson Power Divider (contd.)

- **Therefore**, **for port 2** of the Wilkinson power divider we can write:

$$V_2^+\left(z_2=z_{2p}\right)=\frac{V_s}{2} \qquad V_2^-\left(z_2=z_{2p}\right)=V_2-\frac{V_s}{2}=\frac{V_s}{2}-\frac{V_s}{2}=0$$

$$S_{12}=\frac{V_1^-\left(z_1=z_{1p}\right)}{V_2^+\left(z_2=z_{2p}\right)}=\left(\frac{-jV_s}{2\sqrt{2}}\right)\frac{2}{V_s}=\frac{-j}{\sqrt{2}}$$

- Now, we can **finally** determine the following scattering parameters:

$$S_{22}=\frac{V_2^-\left(z_2=z_{2p}\right)}{V_2^+\left(z_2=z_{2p}\right)}=(0)\frac{2}{V_s}=0$$

$$S_{32}=\frac{V_3^-\left(z_3=z_{3p}\right)}{V_2^+\left(z_2=z_{2p}\right)}=(0)\frac{2}{V_s}=0$$

Q: Wow! That seemed like a **lot** of hard work, and we're only 1/3 of the way done. Do we **have** to move the source to port 1 and then port 3 and perform similar analyses?

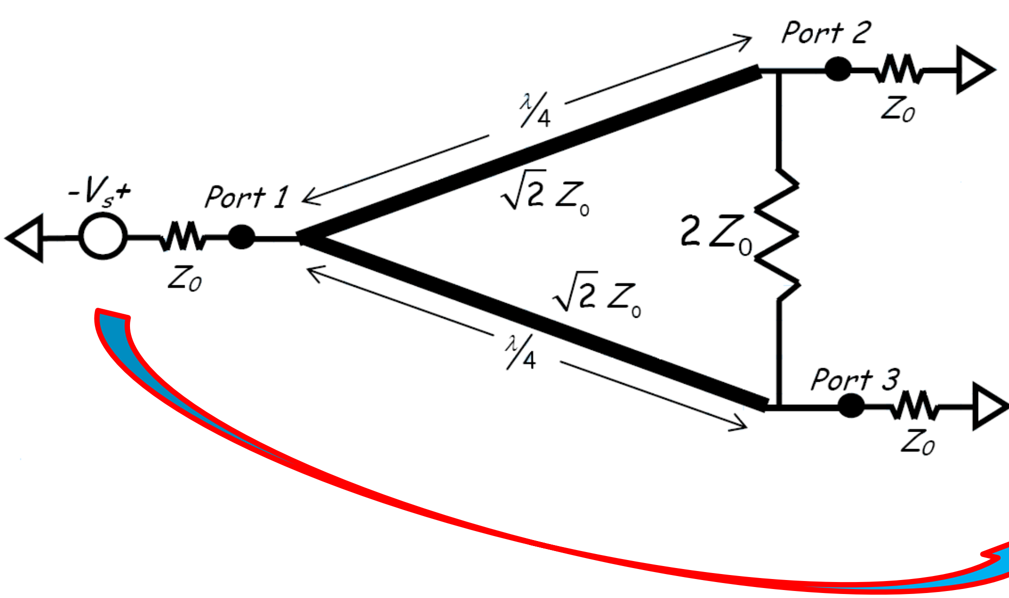
A: **Nope!** Using the bilateral **symmetry** of the circuit (1→1, 2→3, 3→2), we can conclude:

$$S_{13}=S_{12}=\frac{-j}{\sqrt{2}} \qquad S_{33}=S_{22}=0 \qquad S_{23}=S_{32}=0$$

Analysis of Wilkinson Power Divider (contd.)

- and from **reciprocity** we can say: $S_{21} = S_{12} = \frac{-j}{\sqrt{2}}$ $S_{31} = S_{13} = \frac{-j}{\sqrt{2}}$

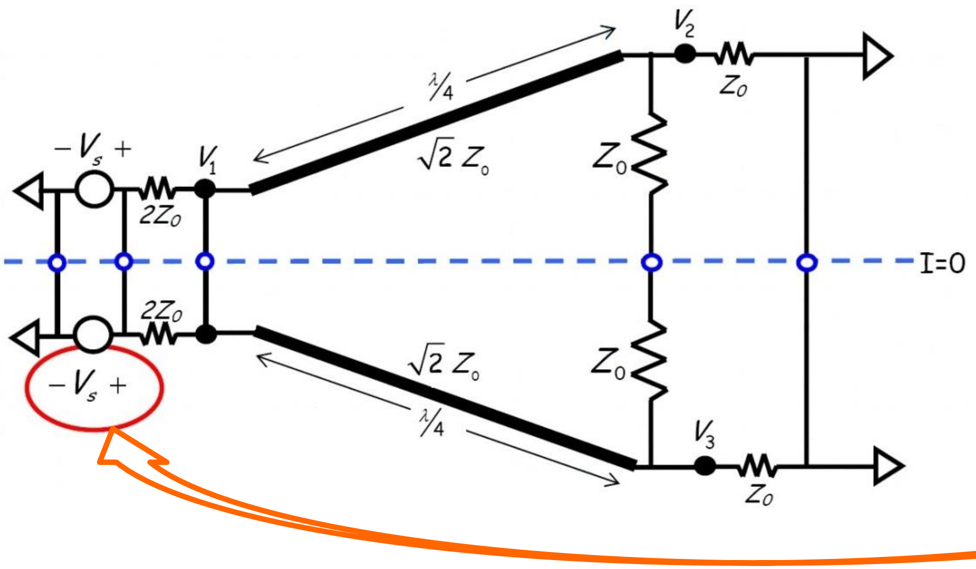
- We thus have determined **8** of the **9** scattering parameters needed to characterize this 3-port device. The **remaining** is the scattering parameter $\mathbf{S_{11}}$. To find this value, we must move the **source to port 1** and analyze.



This source does **not** alter the bilateral symmetry of the circuit. We can thus use this symmetry to **help analyze** the circuit, **without** having to specifically define odd and even mode sources.

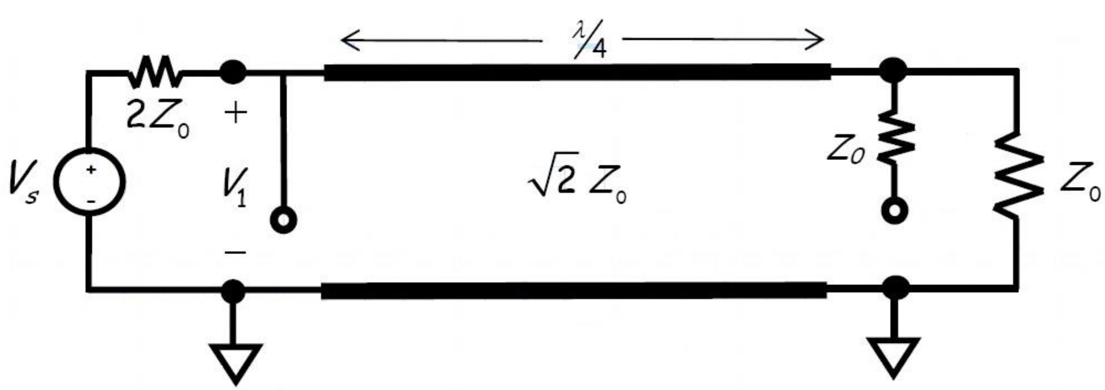
Analysis of Wilkinson Power Divider (contd.)

- Since the circuit has bilateral symmetry, we know that the symmetry plane forms a **virtual open**.



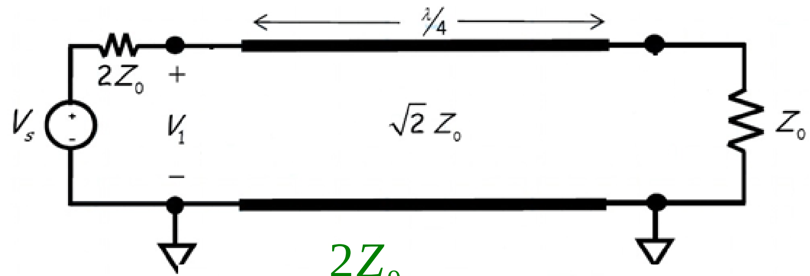
Note the **value** of the voltage sources. They have a value of V_s (as **opposed** to, say, $2V_s$ or $V_s/2$) because two equal voltage sources in **parallel** is equivalent to one voltage source of the **same value**.

- **Splitting** the circuit into **two** half-circuits, we find the **top** half-circuit to be:

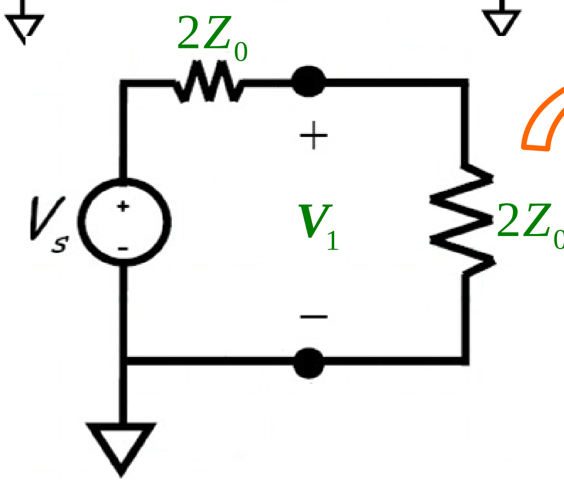


Analysis of Wilkinson Power Divider (contd.)

- Which simplifies to:



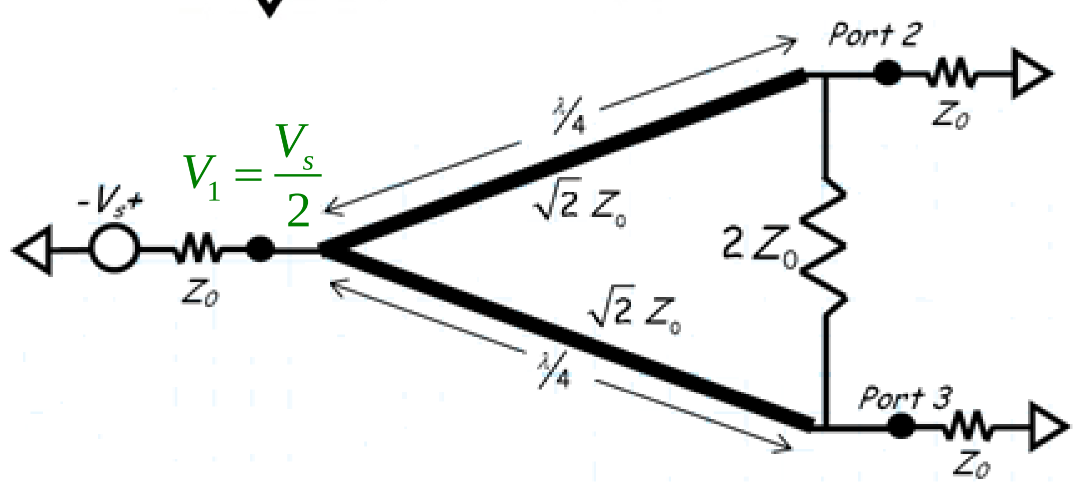
- **Transforming** the load resistor at the end of the λ/4 line back to the start:



It Gives

$$V_1 = V_s\left(\frac{2Z_0}{2Z_0 + 2Z_0}\right) = \frac{V_s}{2}$$

Therefore



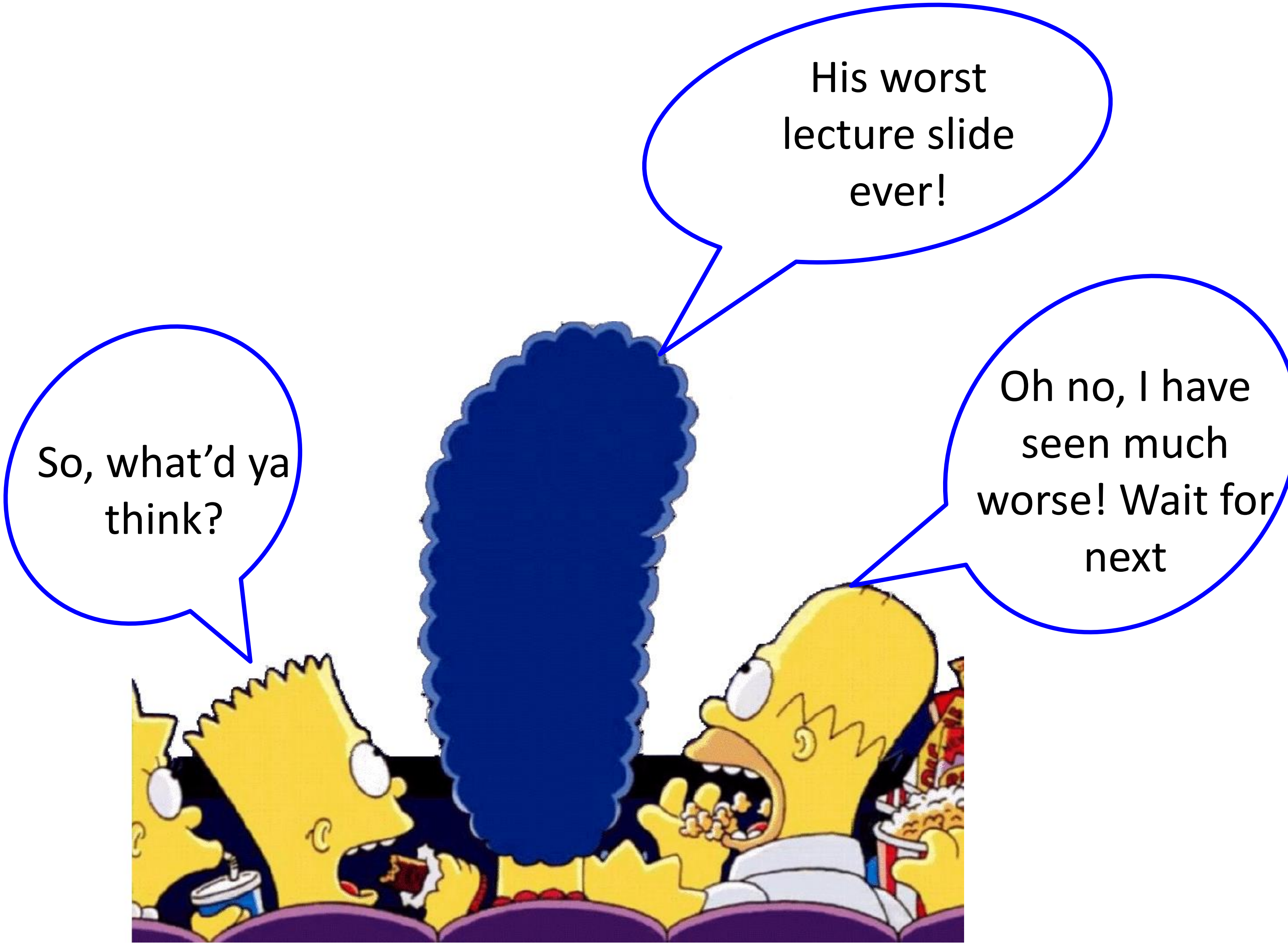
Analysis of Wilkinson Power Divider (contd.)

- And since the **source is matched**: $V_1^+(z_1 = z_{1p}) = \frac{V_s}{2}$ $\quad V_1^-(z_1 = z_{1p}) = V_1 - \frac{V_s}{2} = \frac{V_s}{2} - \frac{V_s}{2} = 0$
- So our **final** scattering element is revealed!

$$S_{11} = \frac{V_1^-(z_1 = z_{1p})}{V_1^+(z_1 = z_{1p})} = (0)\frac{2}{V_s} = 0$$

- So the scattering matrix of a **Wilkinson power divider** has been **confirmed**:

$$S = \begin{bmatrix} 0 & -j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 0 & 0 \\ -j/\sqrt{2} & 0 & 0 \end{bmatrix}$$


So, what’d ya think?
His worst lecture slide ever!
Oh no, I have seen much worse! Wait for next