Lecture – 13 Date: 13.02.2017

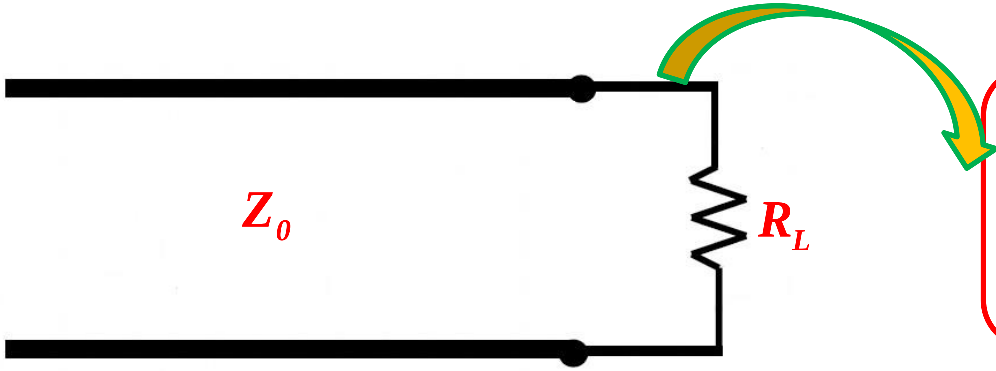
- Quarter Wave Impedance Transformer
- Multiple Reflection Viewpoints
- Theory of Small Reflections

The Quarter Wave Transformer

- By now you must have noticed that a **quarter-wave length** of transmission line ($l = \lambda/4$, $2\beta l = \pi$) appears **often** in RF/microwave engineering problems.
- Another application of the $l = \lambda/4$ transmission line is as an **impedance matching network.**

Q: Why does the quarter-wave matching network work — after all, the quarter-wave line is **mismatched** at both ends?

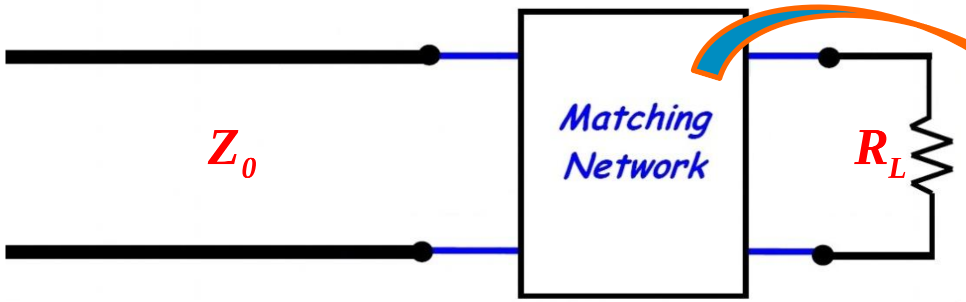
- Let us consider a TL (with characteristic impedance Z_0) where the end is terminated with a **resistive** (i.e., real) load:



Unless $R_L = Z_0$, the resistor is **mismatched** to the line, and thus some of the incident power will be **reflected**.

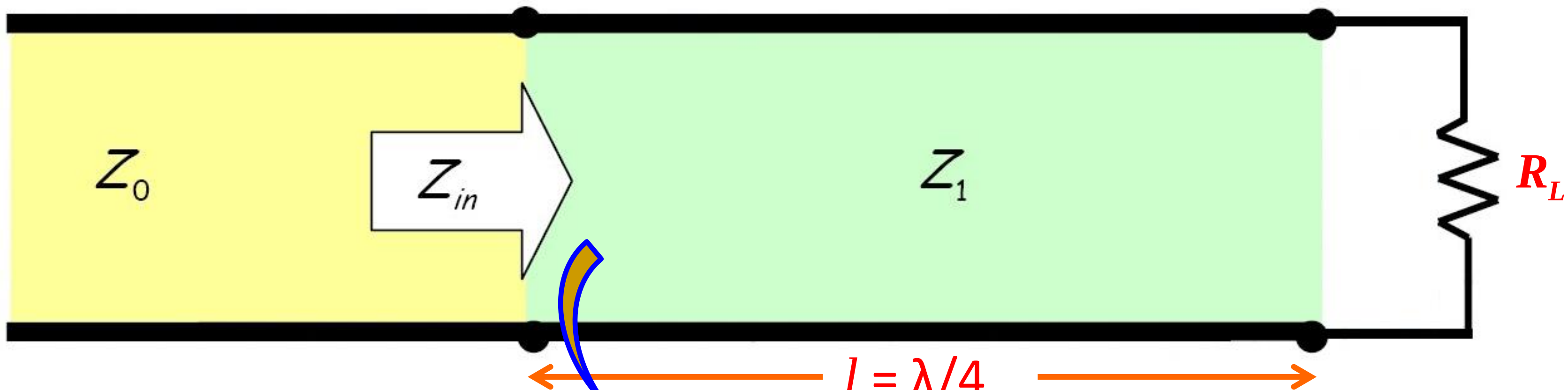
The Quarter Wave Transformer (contd.)

- We can correct this situation by placing a matching network between the line and the load:



In addition to the designs we have just studied (e.g., L-networks, stub tuners), one of the simplest matching network designs is the **quarter-wave transformer**.

- The quarter-wave transformer is simply a transmission line with characteristic impedance Z_1 and length $l = \lambda/4$ (i.e., a quarter-wave line).



This λ/4 line **is** the **matching network**!

Q: But what about the characteristic impedance Z_1; what **should** its value be??

The Quarter Wave Transformer (contd.)

A: Remember, the λ/4 case is one of the **special** cases that we studied. In such a situation the **input** impedance of the line is:

$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

- Thus, if we wish for Z_{in} to be numerically equal to Z_0, we find:

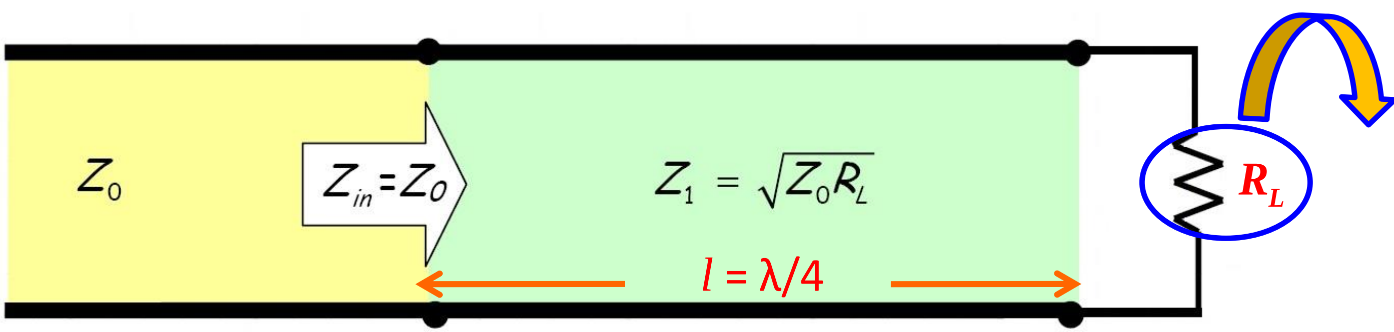
$$Z_{in} = \frac{(Z_1)^2}{R_L} = Z_0$$

- we find the **required** value of Z_1 be:

$$Z_1 = \sqrt{Z_0 R_L}$$

In other words, the characteristic impedance of the quarter wave line is the **geometric average** of Z_0 and R_L!

Therefore, a λ/4 line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will **match** a transmission line with characteristic impedance Z_0 **to** a resistive load R_L



This ensures that **all power** is delivered to load R_L!

Alas, the quarter-wave transformer (like all our designs) have a few problems!

The Quarter Wave Transformer (contd.)

Problem #1

- The matching **bandwidth** is **narrow** !
- In other words, we obtain a **perfect** match at precisely the frequency where the length of the matching transmission line is a **quarter**-wavelength.

remember, this length can be a quarter-wavelength at just **one** frequency!

- **Wavelength** is related to **frequency** as:

$$\lambda = \frac{v_p}{f} = \frac{1}{f\sqrt{LC}}$$

v_p is propagation velocity of wave

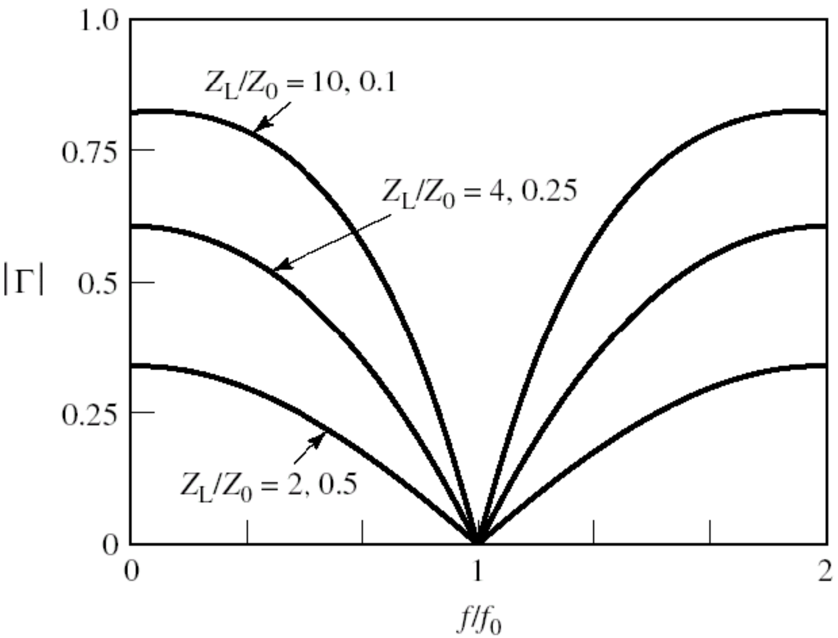
- For **example**, assuming that v_p = c (the speed of light in vacuum), one wavelength at 1 GHz is 30 cm (λ = 0.3m), while one wavelength at 3 GHz is 10 cm (λ = 0.1m). As a result, a TL length l = 7.5cm is a quarter wavelength for a signal at 1GHz **only**.

Thus, a quarter-wave transformer provides a perfect match (Γ_{in} = 0) at one and only one signal frequency!

In other words, as the signal frequency (i.e., wavelength) changes, the **electrical** length of the matching TL segment changes. It will **no longer** be a **quarter** wavelength, and thus we **no longer** will have a **perfect** match

The Quarter Wave Transformer (contd.)

It can be observed that the **closer** R_L (or R_{in}) is to characteristic impedance Z_0, the **wider** the bandwidth of the quarter wavelength transformer



In principle, the bandwidth can be **increased** by adding **multiple** λ/4 sections!

Problem #2

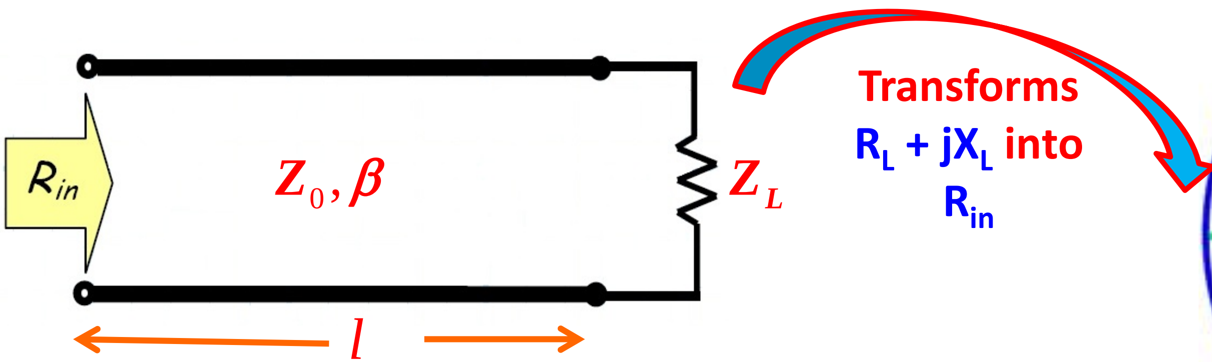
Recall the matching solution was limited to loads that were **purely real**! i.e.:

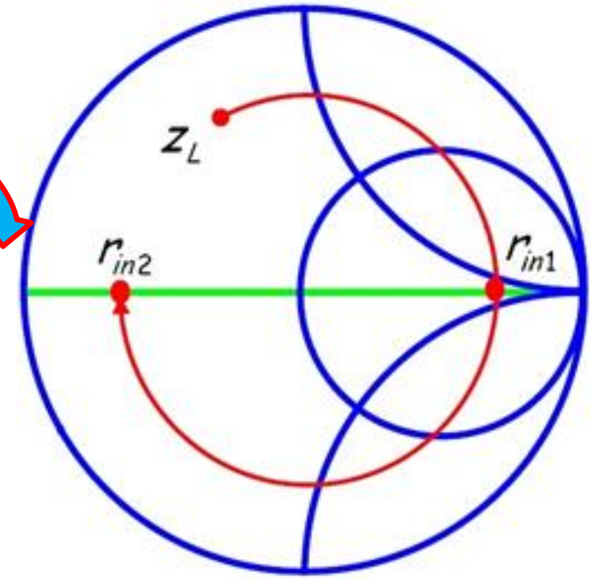
$$Z_L = R_L + j0$$

Obviously, this is a BIG problem, as most loads will have a **reactive** component!

The Quarter Wave Transformer (contd.)

- Fortunately, we have a relatively easy **solution** to this problem, as we can always add some **length** l of TL to the load to make the impedance completely **real**:



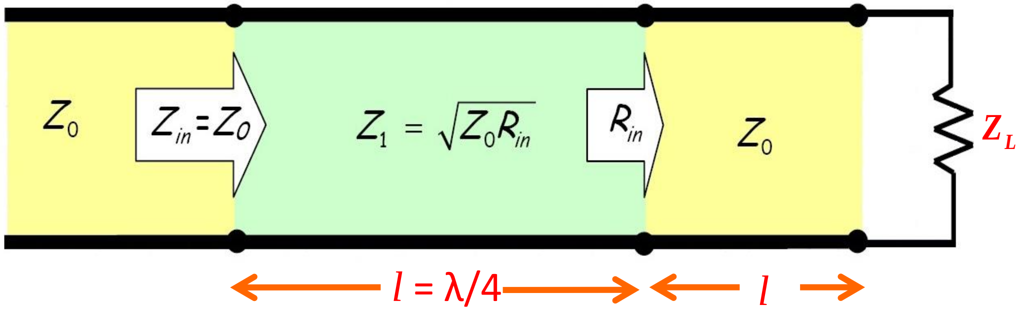


Clearly two possible solutions

However, it should be understood that the input impedance will be purely real at only **one** frequency!

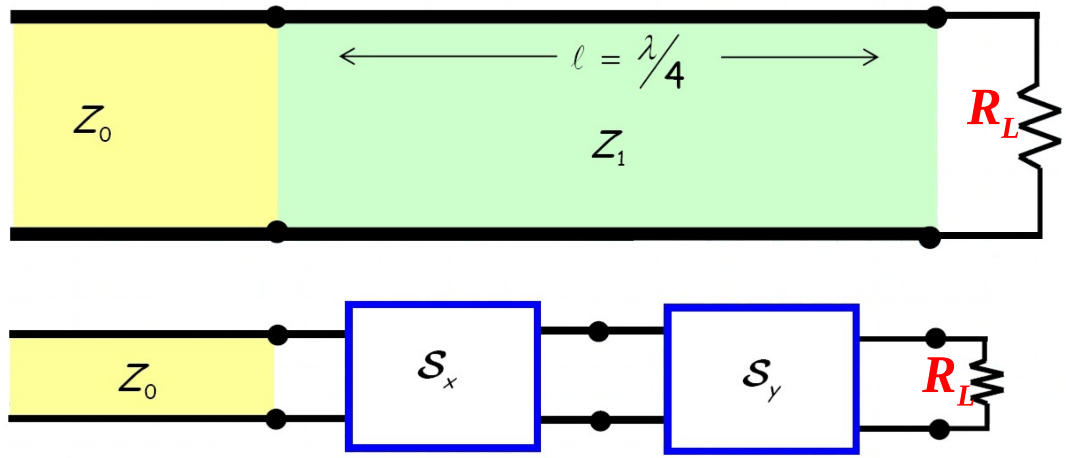
Once the output impedance has been converted to purely real, one can then build a quarter-wave transformer to **match** the line Z_0 to resistance R_{in}

The Quarter Wave Transformer (contd.)



Again, since the transmission lines are lossless, **all** of the incident power is delivered to the **load** Z_L .

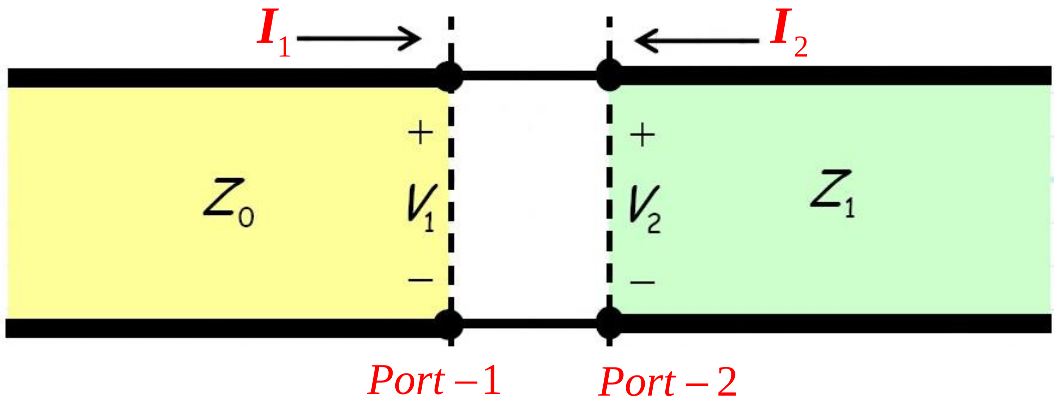
- A quarter wave transformer can be thought of as a cascaded series of **two** two-port devices, terminated with a load R_L:



Q: **Two** two-port devices? It appears to me that a quarter-wave transformer is **not** that complex. What **are** the **two** two–port devices?

A: The **first** is a "**connector**". Note a connector is the interface between one transmission line (characteristic impedance Z_0) to a second transmission line (characteristic impedance Z_1).

The Quarter Wave Transformer (contd.)



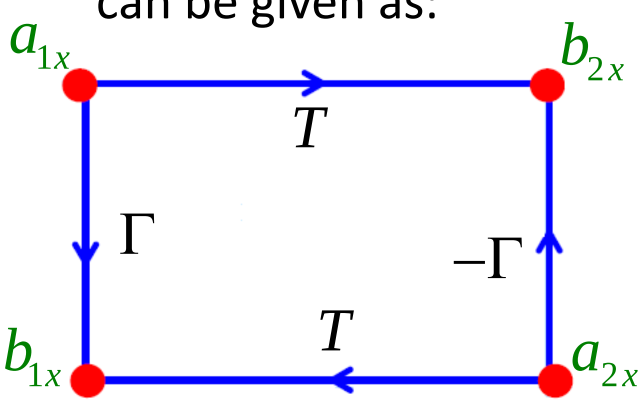
- we **earlier** determined the scattering matrix of this two-port device as:

$$S_x = \begin{bmatrix} \frac{Z_1 - Z_0}{Z_1 + Z_0} & \frac{2\sqrt{Z_0 Z_1}}{Z_1 + Z_0} \\ \frac{2\sqrt{Z_0 Z_1}}{Z_1 + Z_0} & \frac{Z_0 - Z_1}{Z_1 + Z_0} \end{bmatrix}$$

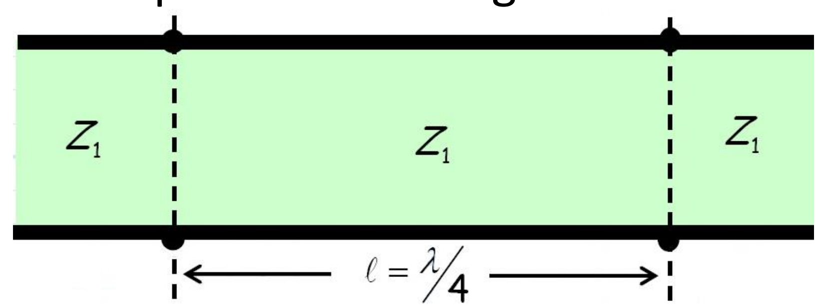
Compact Form

$$S_x = \begin{bmatrix} \Gamma & T \\ T & -\Gamma \end{bmatrix}$$

- Therefore signal flow graph of the connector can be given as:



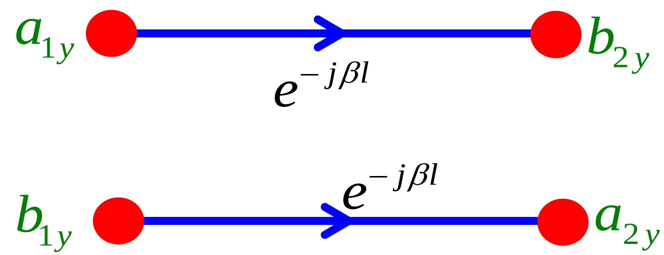
- Now, the **second** two-port device is a quarter wavelength of **TL:**



The Quarter Wave Transformer (contd.)

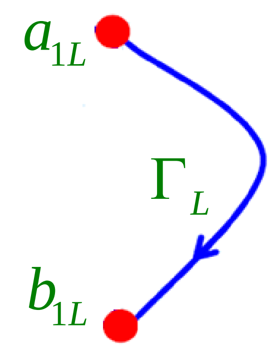
- The second device has the scattering matrix and SFG as:

$$S_y = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix}$$

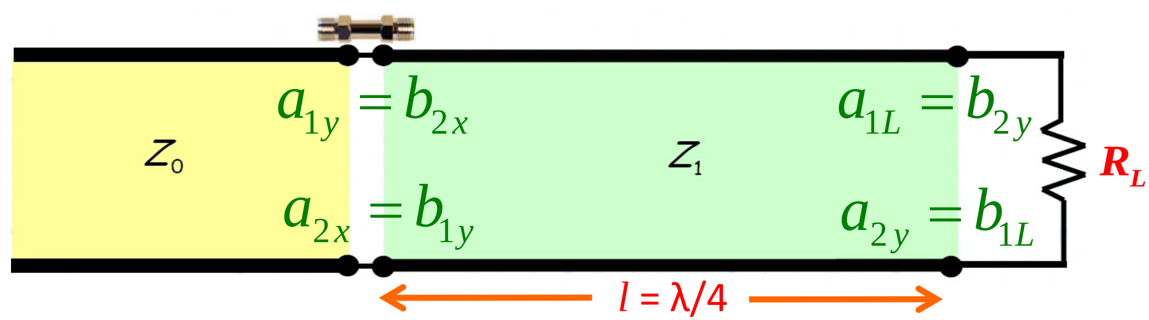


- Finally, a **load** has a “scattering matrix” and SFG as:

$$S = \begin{bmatrix} \frac{R_L - Z_1}{R_L + Z_1} \end{bmatrix} = \Gamma_L$$



- if we connect the ideal connector to a λ/4 of transmission line, and terminate the whole thing with load R_L, we have formed a **λ/4 matching network**!

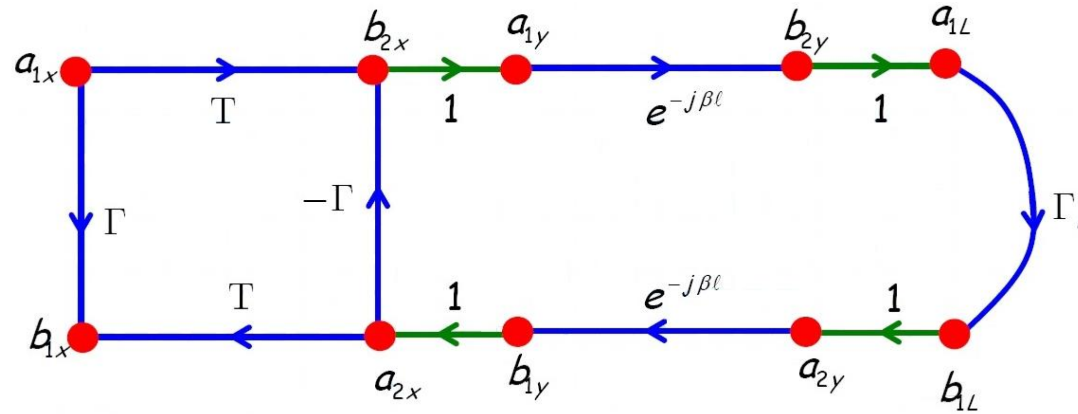


- The boundary conditions associated with these connections are likewise:

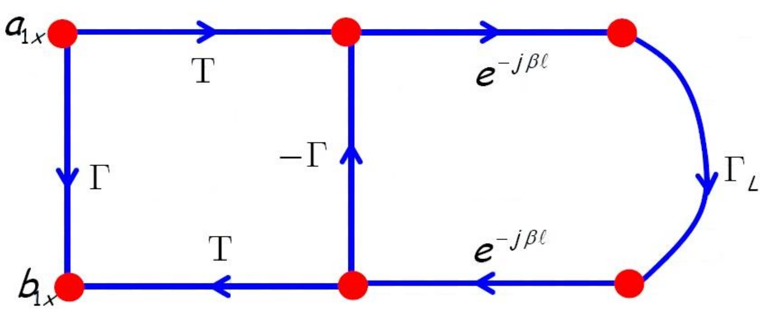
$$a_{1y} = b_{2x} \qquad a_{2x} = b_{1y} \qquad a_{1L} = b_{2y} \qquad a_{2y} = b_{1L}$$

The Quarter Wave Transformer (contd.)

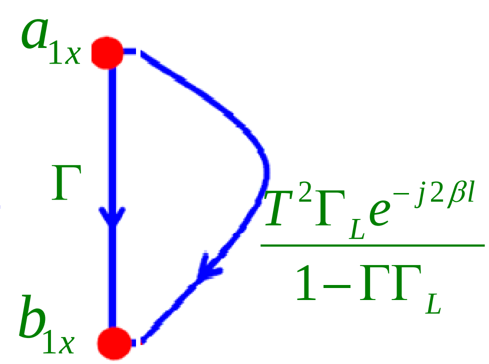
- Therefore, we can put the signal-flow graph pieces together to form the **signal-flow graph** of the quarter wave network:



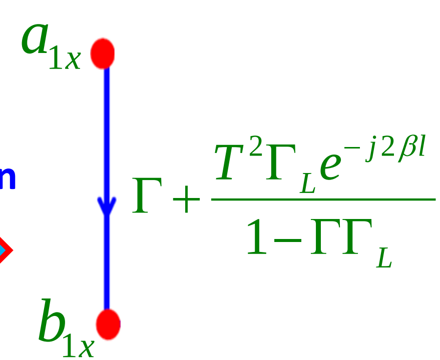
- **Simplification gives:**



Simplification:



Final Simplification



The Quarter Wave Transformer (contd.)

Therefore:

$$\Gamma_{in} \doteq \frac{b_{1x}}{a_{1x}} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta l}}{1 - \Gamma \Gamma_L}$$

Q: Hey wait! If the quarter-wave transformer is a **matching network**, shouldn't $\Gamma_{in} = 0$??

A: Who says it isn't! Consider now **three important facts**.

- For a **quarter wave transformer**, we set Z_1 such that:

$$Z_1^2 = Z_0 R_L \quad \Rightarrow \quad Z_0 = \frac{Z_1^2}{R_L}$$

- **Inserting** this into the scattering parameter S_{11} of the connector, we find:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{Z_1 - Z_1^2 / R_L}{Z_1 + Z_1^2 / R_L} = \frac{R_L - Z_1}{R_L + Z_1}$$

- For the quarter-wave transformer, the **connector** S_{11} value (i.e., Γ) is the **same** as the **load** reflection coefficient Γ_L :

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L$$

Fact 1

- Since the connector is **lossless** (unitary scattering matrix!), we can conclude (and likewise show) that:

$$1 = |S_{11}|^2 + |S_{21}|^2 = |\Gamma|^2 + |T|^2$$

The Quarter Wave Transformer (contd.)

- Since Z_0 , Z_1 , and R_L are all real, the values Γ and T are also **real valued**. As a result, $|\Gamma|^2 = \Gamma^2$ and $|T|^2 = T^2$, and we can likewise conclude:

$$|\Gamma|^2 + |T|^2 = \Gamma^2 + T^2 = 1$$

Fact 2

- Likewise, the Z_1 transmission line has $l = \lambda/4$, so that:

$$2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi \Rightarrow e^{-j\beta l} = e^{-j\pi} = -1$$

Fact 3

- As a result:

$$\Gamma_{in} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta l}}{1 - \Gamma \Gamma_L} = \Gamma - \frac{T^2 \Gamma_L}{1 - \Gamma \Gamma_L}$$

- And using the **newly discovered** fact that (for a correctly designed transformer) $\Gamma_L = \Gamma$:

$$\Gamma_{in} = \Gamma - \frac{T^2 \Gamma}{1 - \Gamma^2}$$

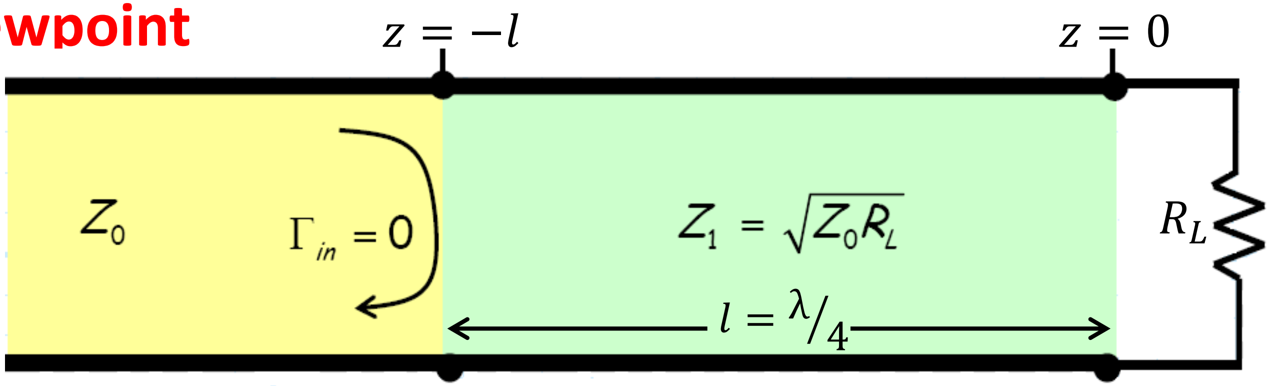
- We also have a **recent** discovery that says $T^2 = 1 - \Gamma^2$, therefore:

$$\Gamma_{in} = \Gamma - \frac{T^2 \Gamma}{1 - \Gamma^2} = \Gamma - \frac{T^2 \Gamma}{T^2} = 0$$

A **perfect match**! The quarter-wave transformer does indeed work!

Multiple Reflection Viewpoint

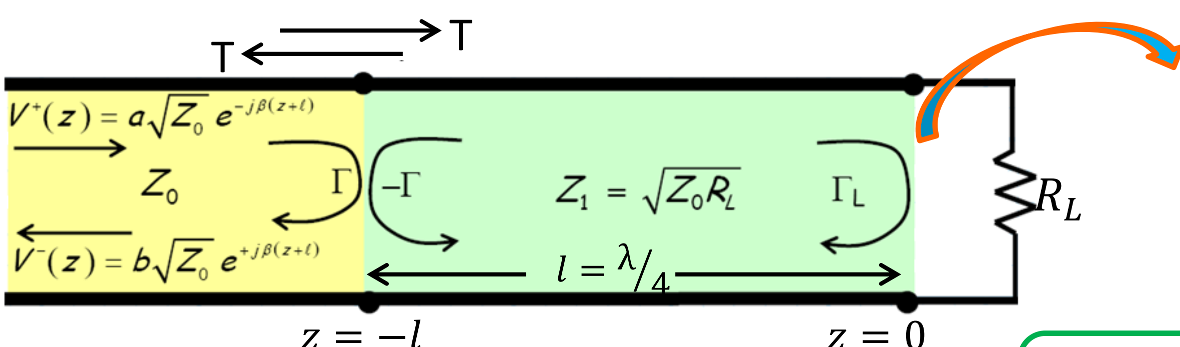
- The **λ/4 - transformer** brings up an interesting question in μ-wave engineering.

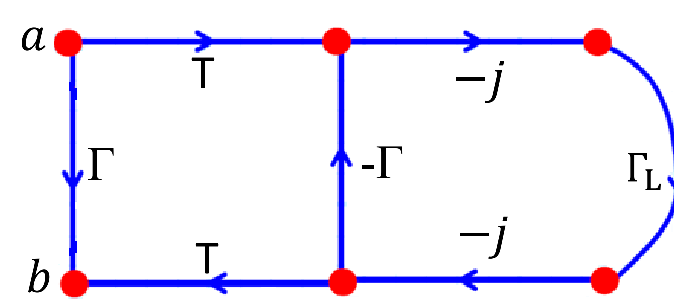


Q: Why is there **no** reflection at $z = -l$? It appears that the line is **mismatched** at both $z = 0$ and $z = -l$.

A: there **are** reflections at the mismatched interfaces—an **infinite** number of them!

We can use **signal flow graph** to determine the propagation series, once we determine all the **propagation paths** through the quarter-wave transformer.



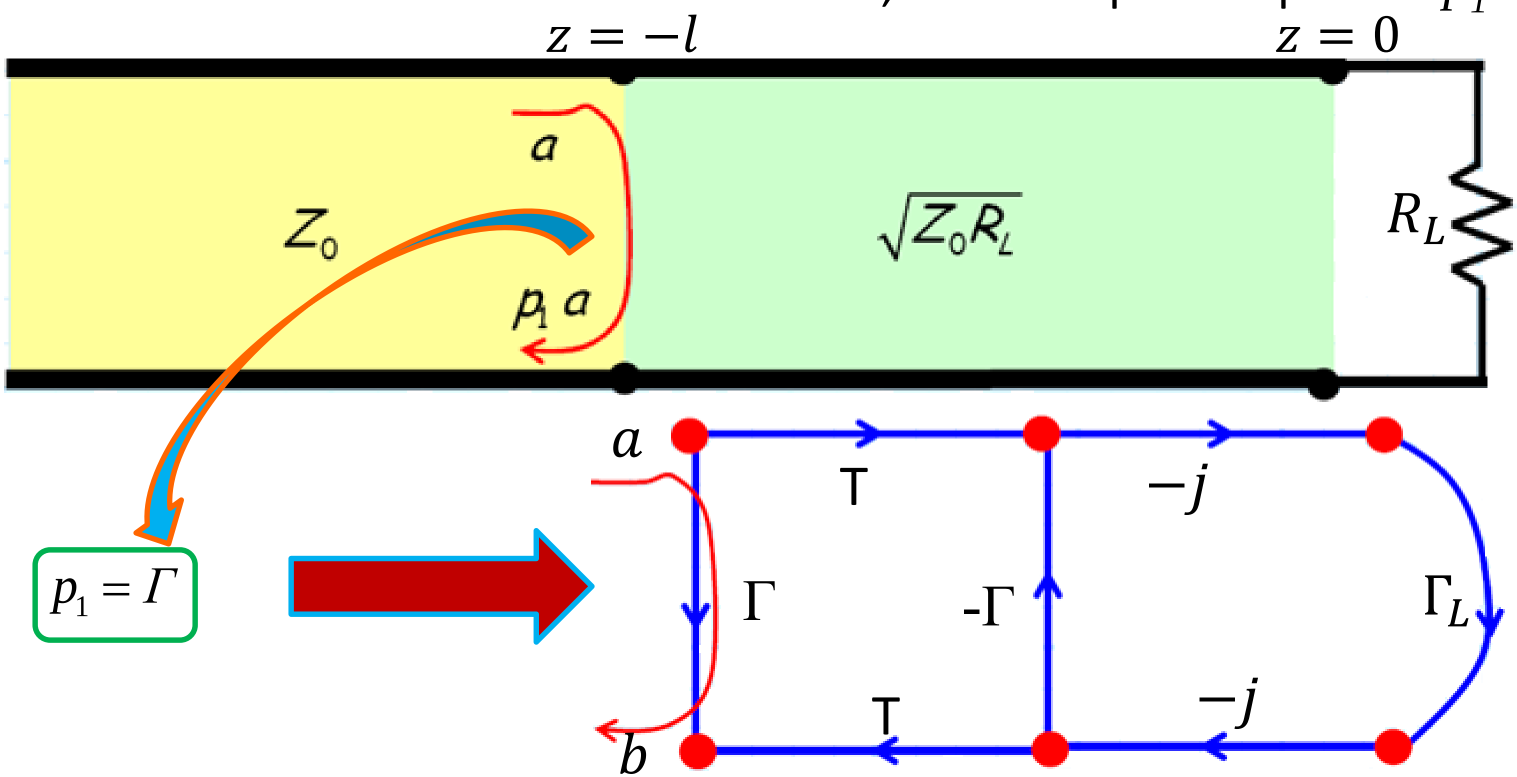


$$b = a\sum_{n=1}^{\infty} p_n$$

Multiple Reflection Viewpoint (contd.)

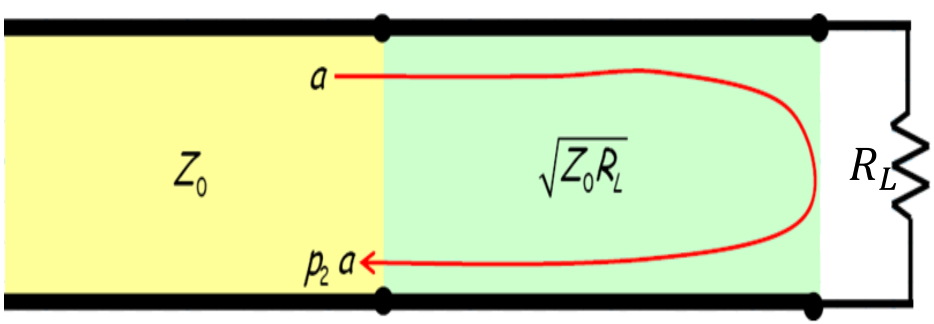
- Now, let's try to interpret what **physically** happens when the **incident** voltage wave reaches the interface at $z = -l$.
- We find that there are **two forward paths** through the quarter-wave transformer signal flow graph.

Path 1. At $z = -l$, the characteristic impedance of the transmission line changes from Z_0 to Z_1. This mismatch creates a **reflected** wave, with complex amplitude $p_1 a$:



Multiple Reflection Viewpoint (contd.)

Path 2. However, a **portion** of the incident wave is transmitted (T) across the interface at $z = -l$, this wave travels a distance of $\beta l = 90^{\circ}$ to the load at $z = 0$, where a portion of it is reflected (Γ_L). This wave travels back $\beta l = 90^{\circ}$ to the interface at $z = -l$, where a portion is again transmitted (T) across into the Z_0 transmission line—**another** reflected wave !

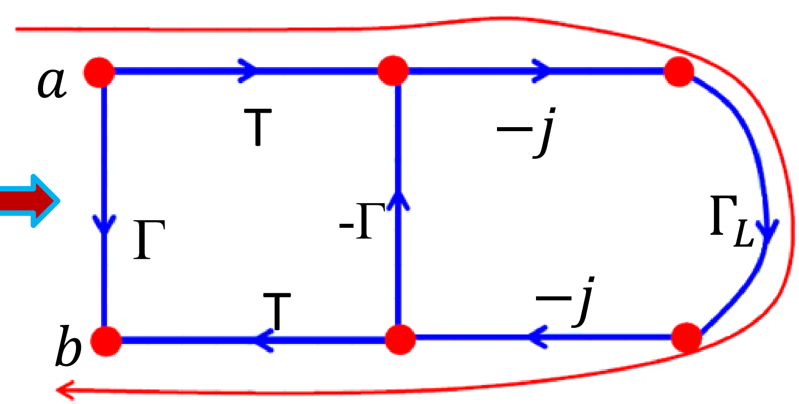


- So the **second direct path** is:

$$p_2 = T e^{-j90^{\circ}} \Gamma_L e^{-j90^{\circ}} T = -T^2 \Gamma_L$$

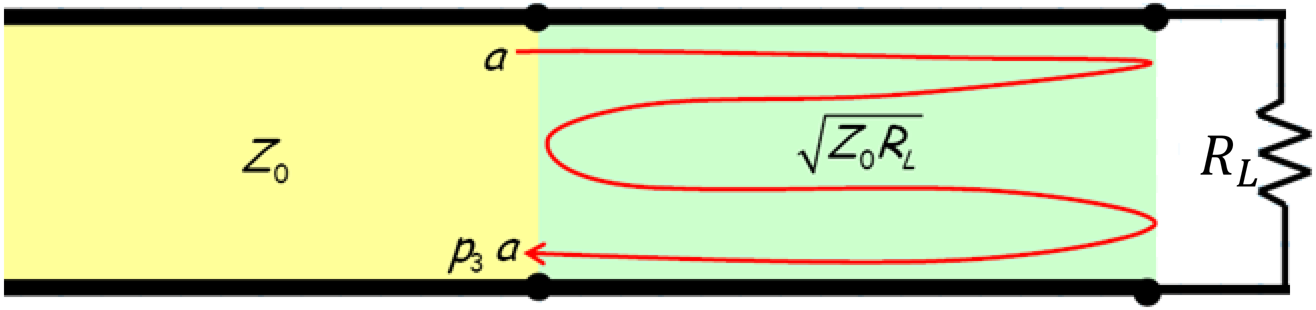
note that traveling $2\beta l = 180^{\circ}$ has produced a **minus** sign in the result.

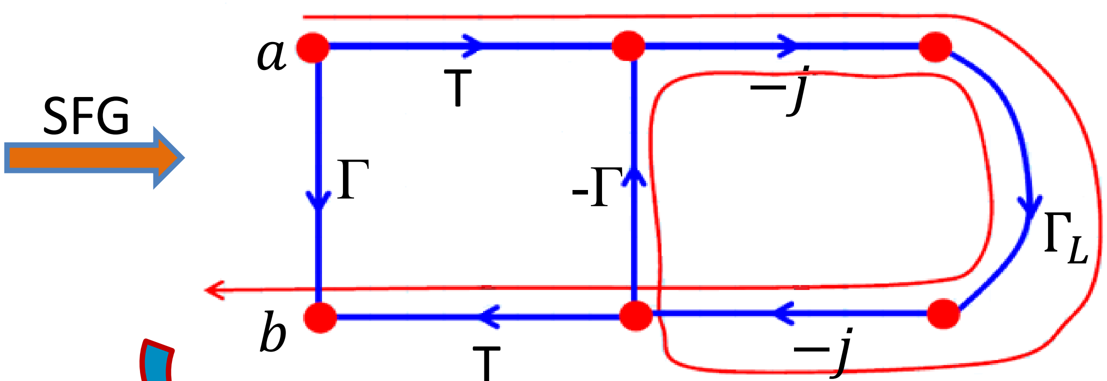
$$b = p_2 a$$



Multiple Reflection Viewpoint (contd.)

Path 3. However, a **portion** of this **second** wave is also **reflected** (Γ) back into the Z_1 transmission line at $z = -l$, where it again travels by $\beta l = 90^{\circ}$ to the load, is partially reflected (Γ_L), travels $\beta l = 90^{\circ}$ back to $z = -l$, and is partially transmitted into Z_0(T)—our **third** reflected wave!





Note that path 3 is **not** a direct path!

$$p_3 = Te^{-j90^\circ}\Gamma_L e^{-j90^\circ}(-\Gamma)e^{-j90^\circ}\Gamma_L e^{-j90^\circ}T = -T^2(\Gamma_L)^2\Gamma$$

Multiple Reflection Viewpoint (contd.)

Path n. We can see that this "bouncing" back and forth can go on **forever**, with each trip launching a **new** reflected wave into the Z_0 transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

Q: But, why then is $\Gamma = 0$?

A: Each reflected wave is a **coherent** wave. That is, they all oscillate at same frequency ω; the reflected waves differ only in terms of their **magnitude** and **phase**.

- Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—this summation results in our **propagation series**, a series that must converge for passive devices.

$$b = a\sum_{n=1}^{\infty} p_n$$

- It can be shown that the infinite propagation series for **this** quarter-wavelength structure **converges** to the closed-form expression:

$$\frac{b}{a} = \sum_{n=1}^{\infty} p_n = \frac{\Gamma - \Gamma^2\Gamma_L - T^2\Gamma_L}{1-\Gamma^2}$$

- Thus, the **input** reflection coefficient is:

$$\Gamma_{in} = \frac{b}{a} = \frac{\Gamma - \Gamma^2\Gamma_L - T^2\Gamma_L}{1-\Gamma^2}$$

Multiple Reflection Viewpoint (contd.)

- Using our definitions, it can be shown that the **numerator** of this expression is:

$$\Gamma - \Gamma^2\Gamma_L - T^2\Gamma_L = \frac{2\left(Z_1^2 - Z_0 R_L\right)}{\left(Z_1 + Z_0\right)\left(R_L + Z_1\right)}$$

- It is evident that the numerator (and therefore Γ_{in}) will be **zero if**:

$$Z_1^2 - Z_0 R_L = 0 \quad \Rightarrow \quad Z_1 = \sqrt{Z_0 R_L}$$

Just as we expected!

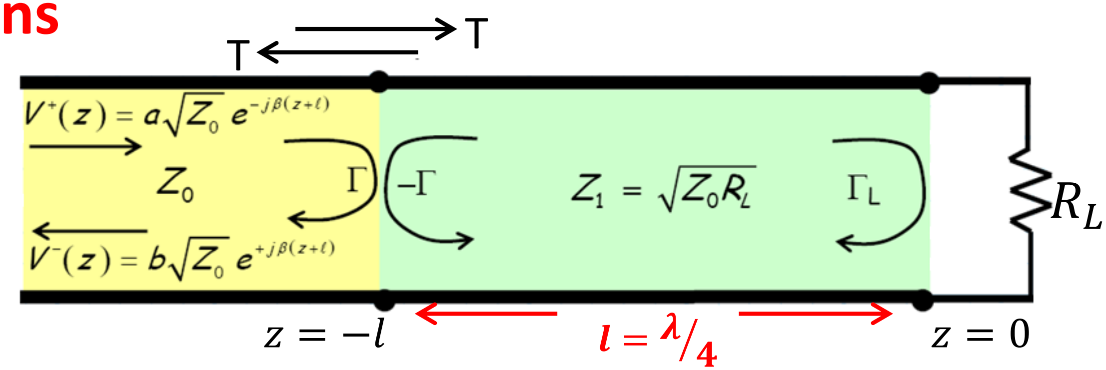
Physically, this result ensures that all the reflected waves add coherently together to produce a **zero value**!

Note **all** of our transmission line analysis has been **steady-state** analysis. We assume our signals are **sinusoidal**, of the form $\exp(j\omega t)$. This signal exists for **all time** t—the signal is assumed to have been "on" **forever**, and assumed to continue "on" forever.

In other words, in steady-state analysis, **all** the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is **zero**!

The Theory of Small Reflections

- Recall that we analysed a **quarter-wave** transformer using the multiple reflection view point.

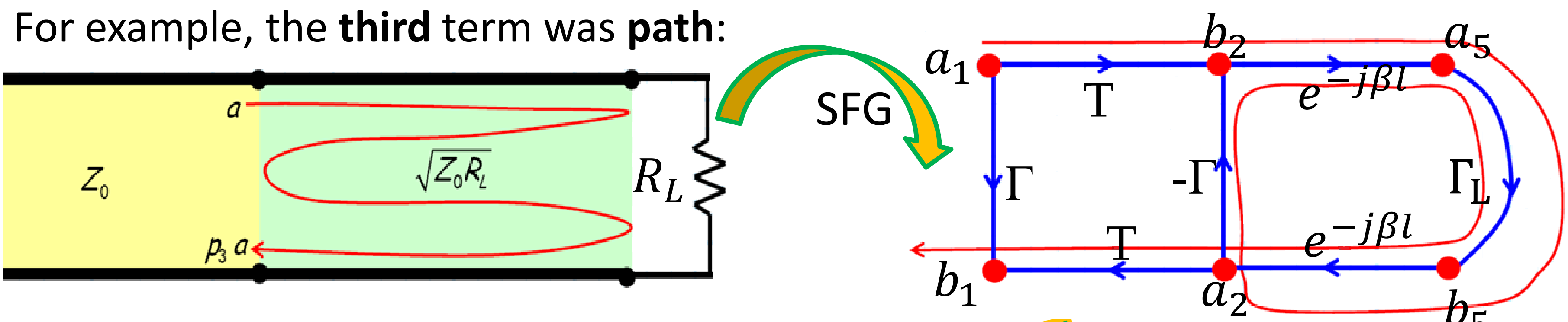


- We found that the solution could be written as an **infinite** summation of terms (the **propagation series**):

$$b = a\sum_{n=1}^{\infty} p_n$$

where each term had a specific **physical** interpretation, in terms of reflections, transmissions, and propagations.

- For example, the **third** term was **path**:



$$p_3 = T^2 (\Gamma_L)^2 \Gamma e^{-j2\beta l}$$

The Theory of Small Reflections (contd.)

- Now let's consider the **magnitude** of this path:

$$|p_3| = |T|^2 |\Gamma_L|^2 |\Gamma| \left|e^{-j2\beta l}\right| \quad \Rightarrow \quad |p_3| = |T|^2 |\Gamma_L|^2 |\Gamma|$$

- Recall that $\Gamma = \Gamma_L$ for a **properly designed** quarter-wave transformer:

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \quad \Rightarrow \quad |p_3| = |T|^2 |\Gamma_L|^3$$

- For the case where values R_L and Z_1 are numerically **"close",** $|\boldsymbol{R_L} - \boldsymbol{Z_1}| \ll |\boldsymbol{R_L} + \boldsymbol{Z_1}|$, the magnitude of the reflection coefficient will be **very** small:

$$|\Gamma_L| = \left|\frac{R_L - Z_1}{R_L + Z_1}\right| \ll 1.0$$

- As a result, the value $|\Gamma_L|^3$ will be **very**, **very**, **very** small.
- Moreover, we know (since the connector is **lossless**) that:

$$|\Gamma|^2 + |T|^2 = |\Gamma_L|^2 + |T|^2 = 1$$

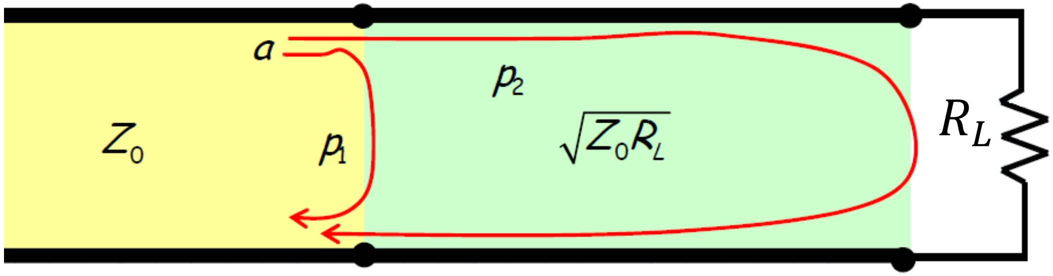
- We can thus conclude that the **magnitude** of path p_3 is likewise **very**, **very**, **very** small:

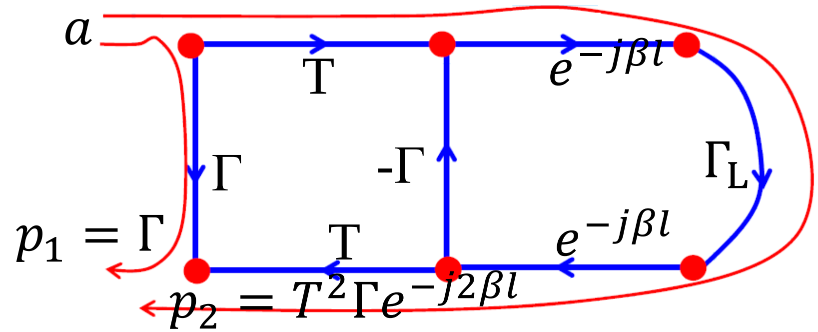
$$|p_3| = |T|^2 |\Gamma_L|^3 \approx |\Gamma_L|^3 \ll 1$$

This is a **classic case** where we can approximate the propagation series using only the **forward paths**!!

The Theory of Small Reflections (contd.)

- Recall there are **two** forward paths:





- Therefore **if** Z_0 and R_L are very **close** in value, the **approx** reflected wave using only the **direct paths** of the infinite series can be found from the SFG:

$$b \simeq (p_1 + p_2)a = \left(\Gamma + T^2\Gamma_L e^{j2\beta l}\right)a$$

- Now, if we likewise apply the **approximation** that $|T| \cong 1.0$, we conclude for this quarter wave transformer (at the design frequency):

$$b \simeq (p_1 + p_2)a = \left(\Gamma + \Gamma_L e^{j2\beta l}\right)a$$

This **approximation**, where we:

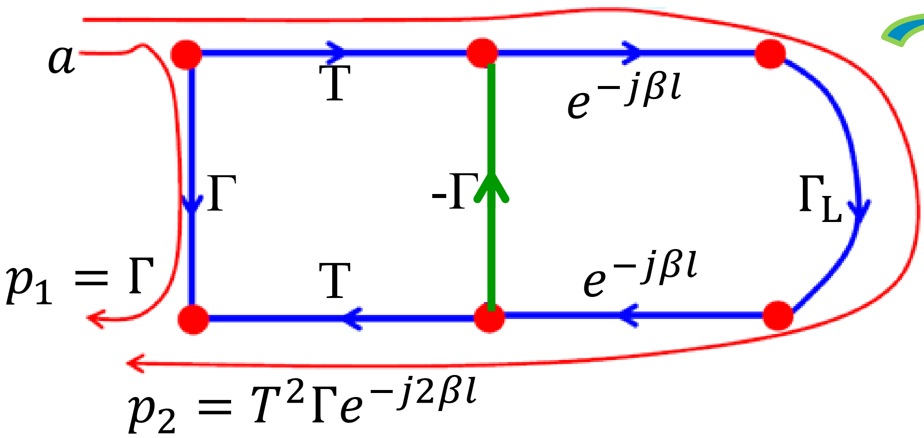
1. use only the **direct paths** to calculate the propagation series,

2. approximate the **transmission** coefficients as **one** (i.e., $|T| = 1.0$).

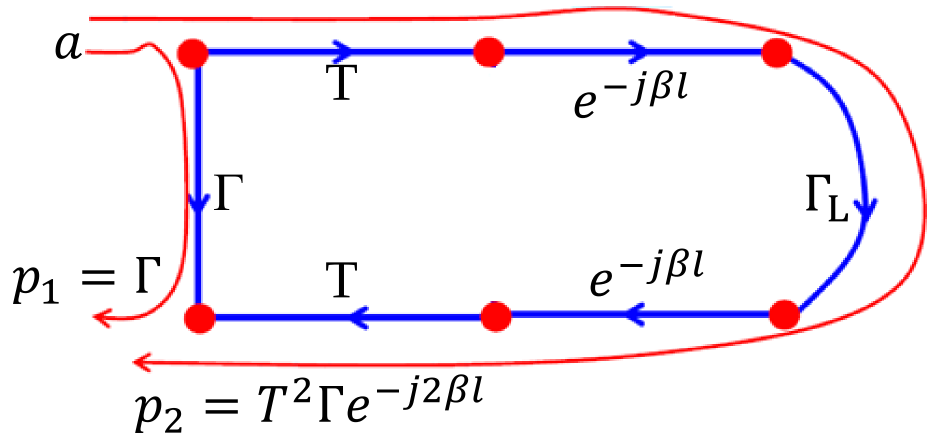
is known as the **Theory of Small Reflections**, and allows us to use the propagation series as an **analysis** tool (we don't have to consider an **infinite** number of terms!).

The Theory of Small Reflections (contd.)

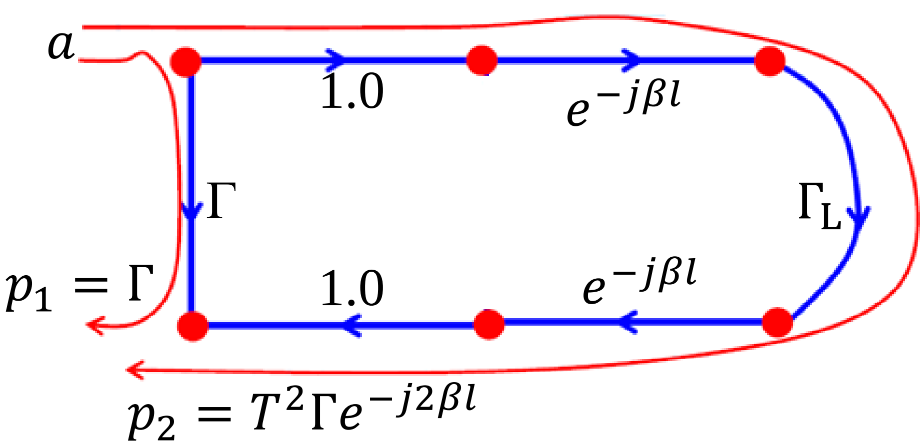
- Consider again the quarter-wave matching network SFG. Note there is **one branch** ($-\Gamma = S_{22}$ of the connector), that is **not included** in either **direct path**.



With respect to the theory of small reflections (where **only** direct paths are considered), this branch can be **removed** from the SFG **without affect**.



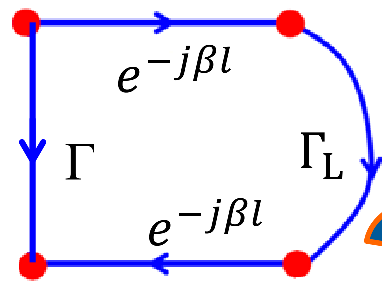
Moreover, the theory of small reflections implements the **approximation**, $|T| = 1.0$, so that the SFG becomes:



The Theory of Small Reflections (contd.)

- **Reducing** this SFG by combining the 1.0 branch and the $e^{-j\beta l}$ branch via the **series rule**, we get the following **approximate** SFG:

The approximate SFG when applying the theory of small reflections !



$$\Gamma_{in} = \frac{b}{a} \approx \Gamma + \Gamma_L e^{-j2\beta l}$$

Note this **approx** SFG provides **precisely** the results of the theory of small reflections!

Q: But wait! The quarter-wave transformer is a **matching** network, therefore $\Gamma_{in} = 0$. The **theory of small reflections**, however, provides the **approximate result**:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta l}$$

Is this **approximation** very **accurate**? How **close** is this **approximate** value to the correct answer of $\Gamma_{in} = 0$?

A: Let's find out!

- Recall that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, and so:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{j2\beta l} = \Gamma_L \left(1 + e^{-j2\beta l}\right)$$

The Theory of Small Reflections (contd.)

- Likewise, l = λ/4 (but **only** at the design frequency!) so that:

$$2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where **you** of course recall that $\beta = {}^{2\pi}/_{\lambda}$!

- Thus:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{j2\beta l} = \Gamma_L\left(1+e^{-j\pi}\right) = \Gamma_L(1-1) = 0$$

Q: Wow! The theory of small reflections appears to be a **perfect** approximation—**no error** at all!?!

A: Not so fast.

The **theory of small reflections** most definitely provides an **approximate** solution (e.g., it **ignores** most of the terms of the propagation series, and it **approximates** connector transmission as T = 1, when in fact T ≠ 1).

As a result, the solutions derived using the **theory of small reflections** will—generally speaking—exhibit **some** (hopefully small) **error**.

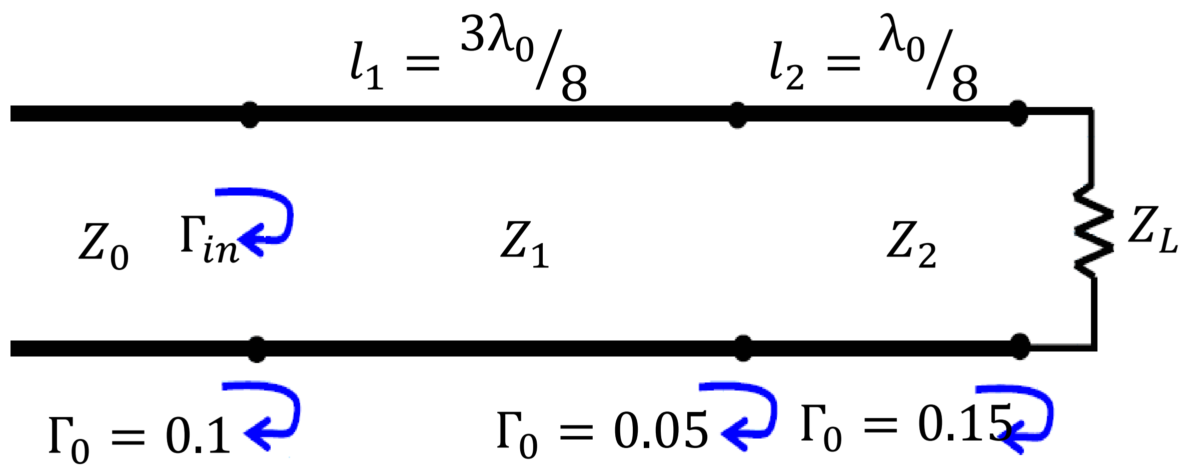
The Theory of Small Reflections (contd.)

We just got a bit **"lucky"** for the quarter-wave matching network; the "approximate" result $\Gamma_{in} = 0$ was exact for this one case!

The **theory of small reflections** is an **approximate** analysis tool!

Example – 1

- Use the **theory of small reflections** to determine a **numeric** value for the **input** reflection coefficient Γ_{in}, at the design frequency ω_0.



Note that the transmission line sections have **different lengths**!