Lecture – 9 Date: 01.02.2016

- Matched, Lossless, and Reciprocal 3-port Network
- Scattering Parameters and Circuit Symmetry
- Even-mode and Odd-mode Analysis
- Generalized S-Parameters
- Example
- T-Parameters

A Matched, Lossless, Reciprocal 3-Port Network

- Consider a 3-port device. Such a device would have a scattering matrix :

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

- Assuming the device is passive and made of simple (isotropic) materials, the device will be **reciprocal**, so that:

$$S_{21} = S_{12} \qquad S_{31} = S_{13} \qquad S_{23} = S_{32}$$

- Similarly, if it is **matched**, we know that: $S_{11} = S_{22} = S_{33} = 0$

- As a result, a **matched**, **reciprocal** device would have a scattering matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{21} & 0 & S_{32} \\ S_{31} & S_{32} & 0 \end{bmatrix}$$

- if we wish for this network to be **lossless**, the scattering matrix must be **unitary**, and therefore:

$$|S_{21}|^2 + |S_{31}|^2 = 1 \qquad S_{31}^* S_{32} = 0$$

$$|S_{12}|^2 + |S_{32}|^2 = 1 \qquad S_{21}^* S_{32} = 0$$

$$|S_{13}|^2 + |S_{23}|^2 = 1 \qquad S_{31}^* S_{31} = 0$$

A Matched, Lossless, Reciprocal 3-Port Network (contd.)

- Since each complex value S is represented by **two real numbers** (i.e., real and imaginary parts), the unitary equations result in **9** real equations. The problem is, the 3 complex values S_{21}, S_{31} and S_{32} are represented by only **6** real unknowns.

We have **over constrained** our problem ! There are **no unique solutions** to these equations !

As unlikely as it might seem, this means that a matched, lossless, reciprocal **3-port** device of **any** kind is a **physical impossibility**!

You **can** make a lossless reciprocal 3-port device, **or** a matched reciprocal 3-port device, **or even** a matched, lossless (but non-reciprocal) 3-port network.

But try as you might, you **cannot** make a lossless, matched, **and** reciprocal three port component!

Matched, Lossless, Reciprocal 4-Port Network

Guess what! I have determined that—unlike a **3-port** device—a matched, lossless, reciprocal **4-port** device **is** physically possible! In fact, I've found **two** general solutions!

- The first solution is referred to as the **symmetric** solution:

$$\mathbf{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

- Note for the symmetric solution, every row and every column of the scattering matrix has the **same** four values (i.e., α, jβ, and two zeros)!
- The second solution is referred to as the **anti-symmetric** solution.

$$\mathbf{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Note that for anti-symmetric solution, **two** rows and **two** columns have the same four values (i.e., α, β, and two zeros), while the **other** two row and columns have (slightly) **different** values (α, -β, and two zeros)

Matched, Lossless, Reciprocal 4-Port Network (contd.)

- It is **quite** evident that each of these solutions are **matched** and **reciprocal**. However, to ensure that the solutions are indeed **lossless**, we must place an **additional** constraint on the values of α, β. Recall that a **necessary** condition for a lossless device is:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \qquad \textbf{For all } \mathbf{n}$$

- For the **symmetric** case, we find: $|\alpha|^2 + |\beta|^2 = 1$
- Similarly, for the **anti-symmetric** case, we find: $|\alpha|^2 + |\beta|^2 = 1$
- It is evident that if the scattering matrix is **unitary** (i.e., lossless), the values α and β **cannot** be independent, but must be **related** as:

$$|\alpha|^2 + |\beta|^2 = 1$$

- **Generally** speaking, we can find that α ≥ β. Given the constraint on these two values, we can thus conclude that:

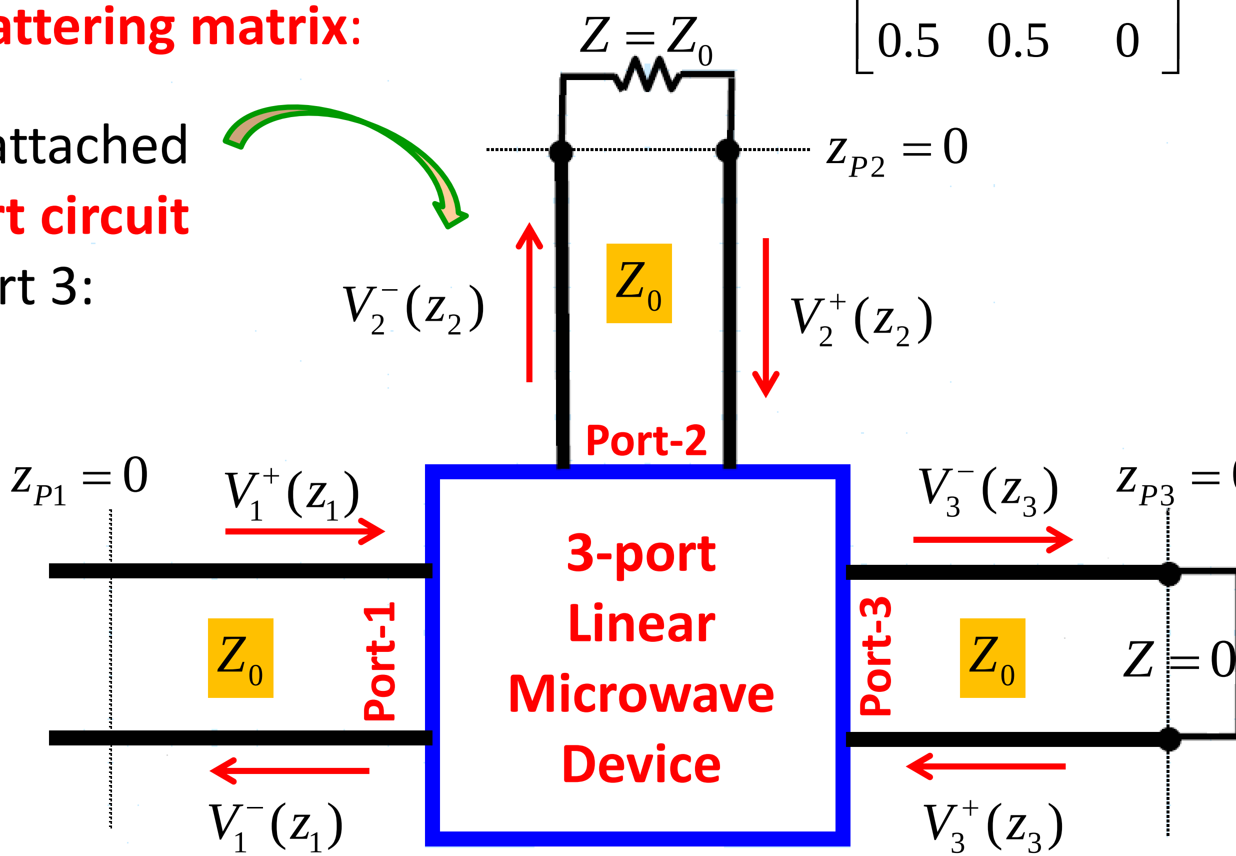
$$0 \le |\beta| \le \frac{1}{\sqrt{2}} \qquad \frac{1}{\sqrt{2}} \le |\beta| \le 1$$

Example – 1

- Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\boldsymbol{S} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0 & 0.2 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

- A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 = \frac{V_1^-(z_{P1})}{V_1^+(z_{P1})}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} = \frac{V_2^-(z_{P2})}{V_1^+(z_{P1})}$$

Example – 1 (contd.)

Solution:

I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:

$$\Gamma_1 = \frac{V_1^-}{V_1^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_2^-}{V_1^+} = S_{21} = 0.5$$

NO!!! The above solution is **not correct**!

Remember, $V_1^-/V_1^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit**.

Therefore: $\Gamma_1 = \frac{V_1^-}{V_1^+} \neq S_{11}$ **and similarly:** $T_{21} = \frac{V_2^-}{V_1^+} \neq S_{21}$

Example – 1 (contd.)

- To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_1^- = 0.2V_2^+ + 0.5V_3^+$$

$$V_2^- = 0.5V_1^+ + 0.2V_3^+$$

$$V_3^- = 0.5V_1^+ + 0.5V_2^+$$

- **and** the **two** equations provided by the **attached loads**:

$$V_2^+ = 0$$

$$V_3^+ = -V_3^-$$

Solve those five expressions to find:

$$\Gamma_1 = \frac{V_1^-}{V_1^+} = -0.25$$

$$T_{21} = \frac{V_2^-}{V_1^+} = 0.4$$

Example – 2

- Consider a **two-port device** with Z_0 =50Ω and scattering matrix (at some specific frequency ω_0):

$$S(\omega=\omega_0)=\begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

- Say that the transmission line connected to **port 2** of this device is terminated in a **matched** load, and that the wave **incident** on **port 1** is:

$$V_1^+(z_1) = -j2e^{-j\beta z_1}$$ **where** $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$

2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$

3. the net power flowing into port 1

Solution: **1.** Given the **incident** wave on port 1 is: $V_1^+(z_1) = -j2e^{-j\beta z_1}$

- we can conclude (since z_{1P} = 0):

$$V_1^+(z_1 = z_{1P}) = -j2e^{-j\beta z_{1P}} = -j2e^{-j\beta(0)} = -j2$$

Example – 2 (contd.)

- since port 2 is **matched** (and **only** because its matched!), we find:

$$V_1^-(z_1 = z_{1P}) = S_{11}V_1^+(z_1 = z_{1P}) = 0.1(-j2) = -j0.2$$

- The voltage at port 1 is thus:

$$V_1(z_1 = z_{1P}) = V_1^+(z_1 = z_{1P}) + V_1^-(z_1 = z_{1P}) = -j2 + (-j0.2) = -j2.2 = 2.2e^{j(-\pi/2)}$$

- Similarly, since port 2 is **matched**: $V_2^+(z_2 = z_{2P}) = 0$

- Therefore: $V_2^-(z_2 = z_{2P}) = S_{21}V_1^+(z_1 = z_{1P}) = j0.7(-j2) = 1.4$

- The voltage at port 2 is thus:

$$V_2(z_2 = z_{2P}) = V_2^+(z_2 = z_{2P}) + V_2^-(z_2 = z_{2P}) = 0 + 1.4 = 1.4 = 1.4e^{-j0}$$

Example – 2 (contd.)

2. **The port currents** can be easily determined from the results of the previous section

$$I_1(z_1=z_{1P}) = I_1^+(z_1=z_{1P}) - I_1^-(z_1=z_{1P}) = \frac{V_1^+(z_1=z_{1P})}{Z_0} - \frac{V_1^-(z_1=z_{1P})}{Z_0}$$

$$\Rightarrow I_1(z_1=z_{1P}) = -j\frac{2.0}{50} + j\frac{0.2}{50} = -j\frac{1.8}{50} = -j0.036 = 0.036e^{-j\pi/2}$$

$$I_2(z_2=z_{2P}) = I_2^+(z_2=z_{2P}) - I_2^-(z_2=z_{2P}) = \frac{V_2^+(z_2=z_{2P})}{Z_0} - \frac{V_2^-(z_2=z_{2P})}{Z_0}$$

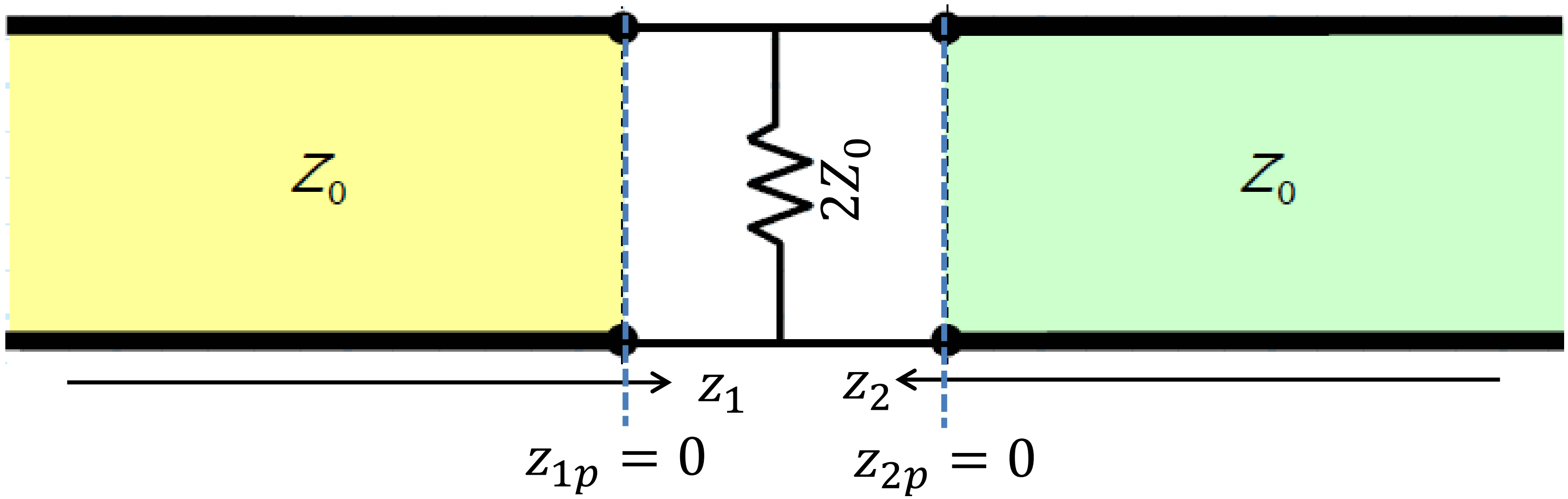
$$\Rightarrow I_2(z_2=z_{2P}) = \frac{0}{50} - \frac{1.4}{50} = -\frac{1.4}{50} = -0.028 = 0.028e^{+j\pi}$$

3. The **net power** flowing into port 1 is: $\Delta P_1 = P_1^+ - P_1^-$

$$\Rightarrow \Delta P_1 = \frac{\left|V_1^+\right|^2}{2Z_0} - \frac{\left|V_1^-\right|^2}{2Z_0} \qquad \Rightarrow \Delta P_1 = \frac{(2)^2 - (0.2)^2}{2(50)} = 0.0396 \text{ Watts}$$

Example – 3

- determine the **scattering matrix** of this two-port device:



Q: OK, but how can we **determine** the scattering matrix of a device?
A: We must carefully apply our **transmission line theory**!

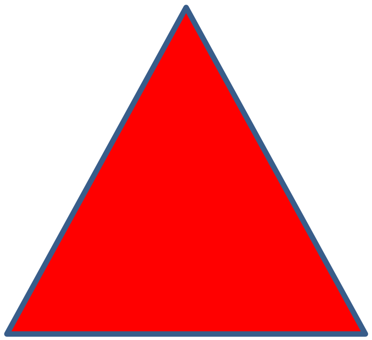
Q: Determination of the Scattering Matrix of a multi-port device would seem to be particularly laborious. Is there any way to simplify the process?
A: Many (if not most) of the useful devices made by us humans exhibit a high degree of **symmetry**. This can greatly **simplify** circuit analysis—if we **know how** to exploit it!

Q: Is there any **other** way to use circuit symmetry to our advantage?
A: Absolutely! One of the most **powerful** tools in circuit analysis is **Odd-Even Mode** analysis.

Circuit Symmetry

- One of the most powerful concepts in for evaluating circuits is that of symmetry. **Normal** humans have a **conceptual** understanding of symmetry, based on an **aesthetic** perception of structures and figures.
- On the other hand, **mathematicians** (as they are wont to do) have defined symmetry in a very precise and unambiguous way. Using a branch of mathematics called **Group Theory**, first developed by the young genius **Évariste Galois** (1811-1832), **symmetry** is defined by a set of operations (a group) that leaves an object **unchanged**.
- Initially, the symmetric "objects" under consideration by Galois were **polynomial functions**, but group theory can likewise be applied to evaluate the symmetry of **structures**.

- For example, consider an ordinary **equilateral triangle**; we find that it is a highly **symmetric** object!

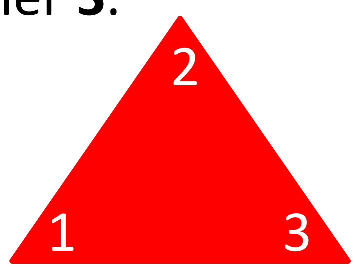


Circuit Symmetry (contd.)

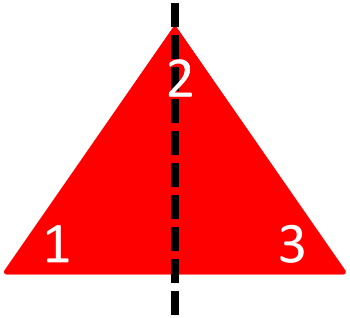
Q: Obviously this is true. We don't need a mathematician to tell us that!

A: Yes, but **how** symmetric is it? How does the symmetry of an equilateral triangle **compare** to that of an isosceles triangle, a rectangle, or a square?

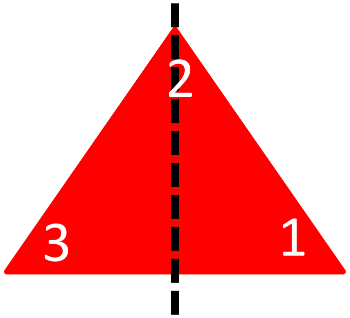
- To determine its level of symmetry, let's first label each corner as corner **1**, corner **2**, and corner **3**.



- First, we note that the triangle exhibits a plane of **reflection symmetry**:



- Thus, if we "**reflect**" the triangle across this plane we get:



Circuit Symmetry (contd.)

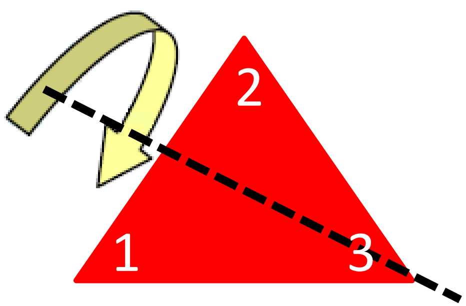
Note that although corners 1 and 3 have changed places, the triangle itself remains **unchanged**—that is, it has the same **shape**, same **size**, and same **orientation** after reflecting across the symmetric plane!

- Mathematicians say that these two triangles are **congruent.**
- Note that we can write this reflection operation as a **permutation** (an exchange of position) of the corners, defined as:

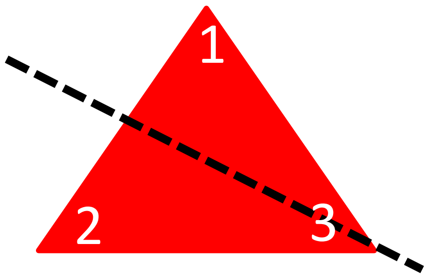
$$1 \rightarrow 3$$
$$2 \rightarrow 2$$
$$3 \rightarrow 1$$

Q: But wait! Isn't there **more** than just **one** plane of reflection symmetry?

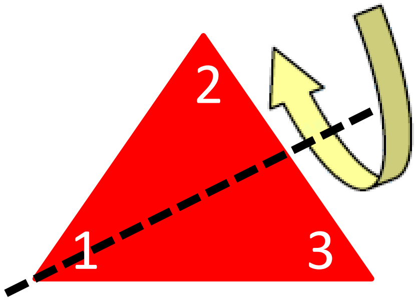
A: Definitely! There are **two more**:



$$1 \rightarrow 2$$
$$2 \rightarrow 1$$
$$3 \rightarrow 3$$



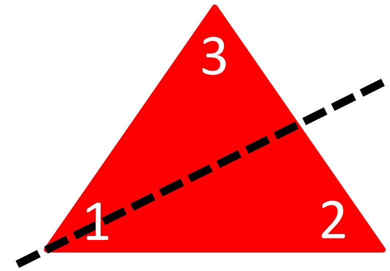
Circuit Symmetry (contd.)



$1 \rightarrow 1$

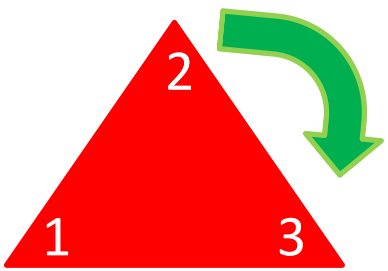
$2 \rightarrow 3$

$3 \rightarrow 2$



In addition, an equilateral triangle exhibits **rotation symmetry!**

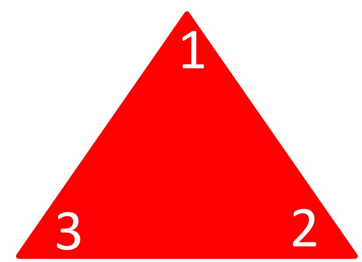
- **Rotating** the triangle 120° clockwise also results in a **congruent** triangle:



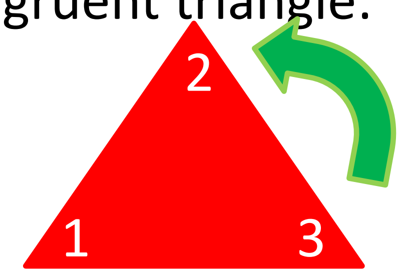
$1 \rightarrow 2$

$2 \rightarrow 3$

$3 \rightarrow 1$



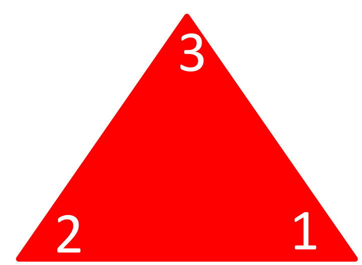
- Likewise, rotating the triangle 120° **counter-clockwise** results in a congruent triangle:



$1 \rightarrow 3$

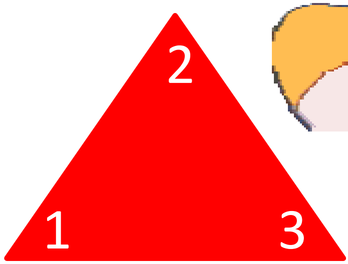
$2 \rightarrow 1$

$3 \rightarrow 2$



Circuit Symmetry (contd.)

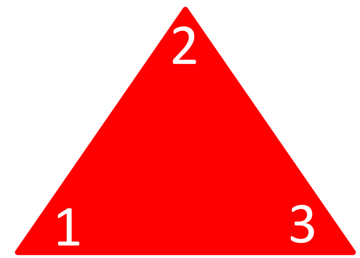
- Additionally, there is **one more** operation that will result in a congruent triangle—do **nothing**!



$$1 \rightarrow 1$$

$$2 \rightarrow 2$$

$$3 \rightarrow 3$$



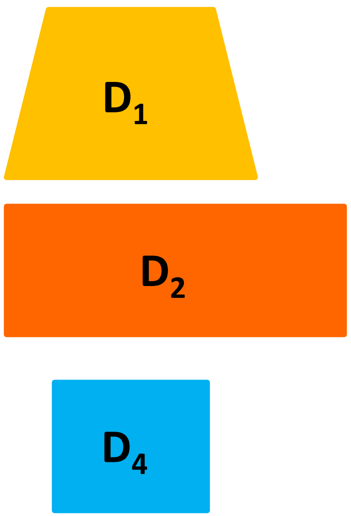
This seemingly **trivial** operation is known as the **identity operation**, and is an element of **every** symmetry group.

These 6 operations form the **dihedral symmetry group D_3** which has **order six** (i.e., it consists of six operations). An object that remains **congruent** when operated on by any and all of these six operations is said to have **D_3** symmetry.

An equilateral triangle has **D_3** symmetry!

Circuit Symmetry (contd.)

- By applying a similar analysis to an isosceles trapezoid, rectangle, and square, we find that:



An isosceles trapezoid has $\mathbf{D_1}$ symmetry, a dihedral group of **order 2**.

A rectangle has $\mathbf{D_2}$ symmetry, a dihedral group of **order 4**.

A square has $\mathbf{D_4}$ symmetry, a dihedral group of **order 8**.

Thus, a square is the **most** symmetric object of the four we have discussed; the isosceles trapezoid is the **least**.

Q: Well that's all just fascinating—but just what the heck does this have to do with **RF circuits!?!**

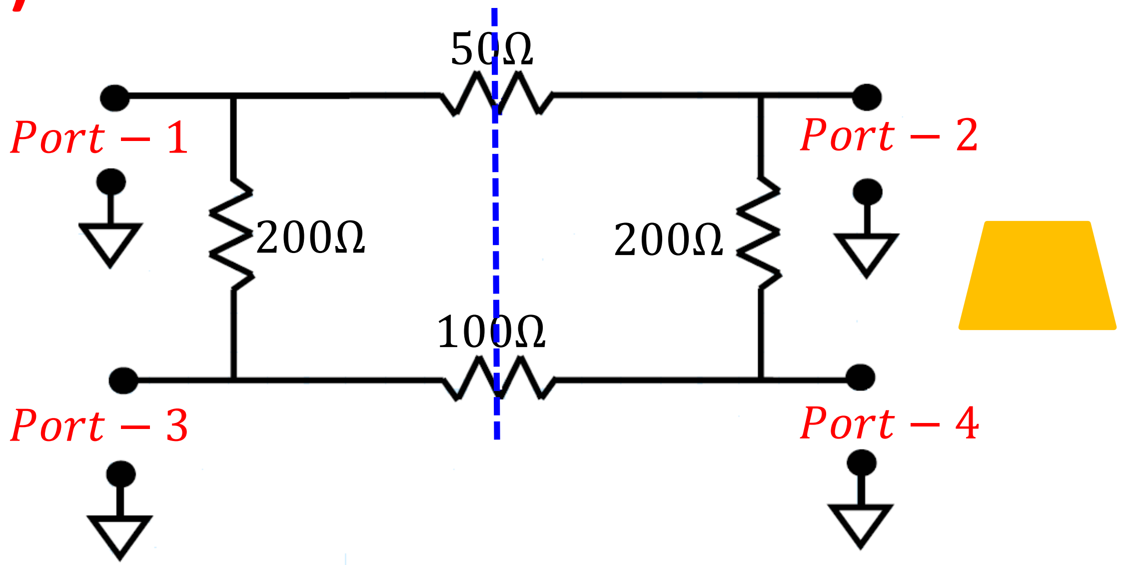
A: Plenty! **Useful circuits** often display high levels of symmetry.

Circuit Symmetry (contd.)

- For example consider these $\mathbf{D_1}$ symmetric multi-port circuits:

  $1 \rightarrow 2$ $3 \rightarrow 4$

  $2 \rightarrow 1$ $4 \rightarrow 3$

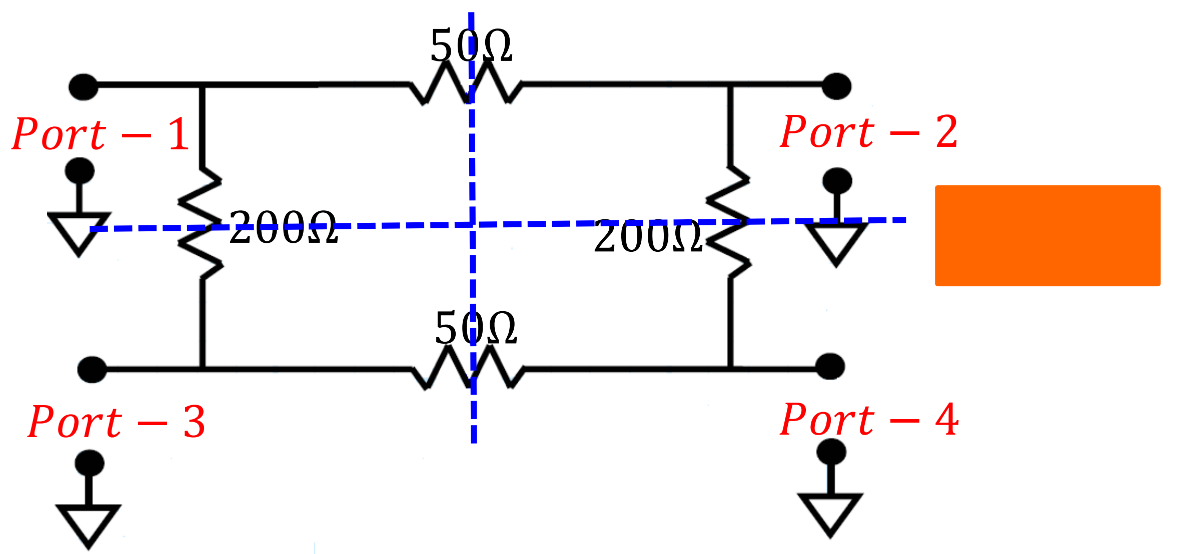


- Or this circuit with $\mathbf{D_2}$ symmetry: which is **congruent** under these permutations:

$1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 2$

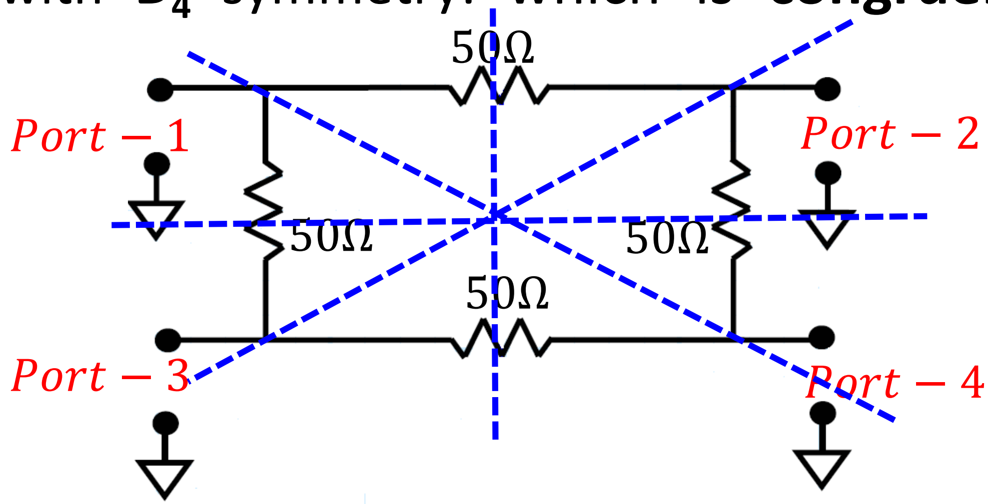
$1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 3$

$1 \rightarrow 4, 2 \rightarrow 3, 3 \rightarrow 2, 4 \rightarrow 1$



Circuit Symmetry (contd.)

- Or this circuit with **D$_4$** symmetry: which is **congruent** under these permutations:



$$1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 2$$

$$1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 3$$

$$1 \rightarrow 4, 2 \rightarrow 3, 3 \rightarrow 2, 4 \rightarrow 1$$

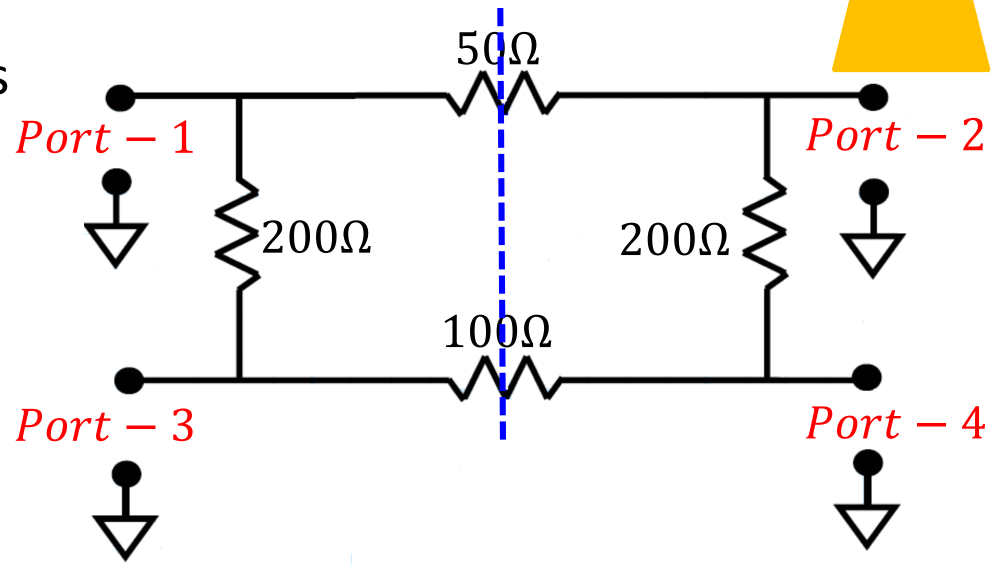
$$1 \rightarrow 4, 2 \rightarrow 2, 3 \rightarrow 3, 4 \rightarrow 1$$

$$1 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 2, 4 \rightarrow 4$$

The **importance** of this can be seen when considering the scattering matrix, impedance matrix, or admittance matrix of these networks.

Circuit Symmetry (contd.)

- For **example**, consider again this **symmetric circuit**:
- This four-port network has a single plane of **reflection symmetry** (i.e., $\mathbf{D_1}$ symmetry), and thus is congruent under the permutation:



$$1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 3$$

- So, since (for example) 1→2, we find that for this circuit:

$S_{11} = S_{22}$ $\quad Z_{11} = Z_{22}$ $\quad Y_{11} = Y_{22}$ **must** be true!

- Or, since 1→2 and 3→4 we find:

$$S_{13} = S_{24} \quad Z_{13} = Z_{24} \quad Y_{13} = Y_{24}$$

$$S_{31} = S_{42} \quad Z_{31} = Z_{42} \quad Y_{31} = Y_{42}$$

Circuit Symmetry (contd.)

- Continuing for **all** elements of the permutation, we find that for this symmetric circuit, the scattering matrix **must** have **this** form:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{13} & S_{14} \\ S_{21} & S_{11} & S_{14} & S_{13} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

and the **impedance** and **admittance** matrices would likewise have this same form.

- Note there are just **8** independent elements in this matrix. If we also consider **reciprocity** (a constraint independent of symmetry) we find that $S_{31} = S_{13}$ and $S_{41} = S_{14}$, and the matrix reduces further to one with just **6** independent elements:

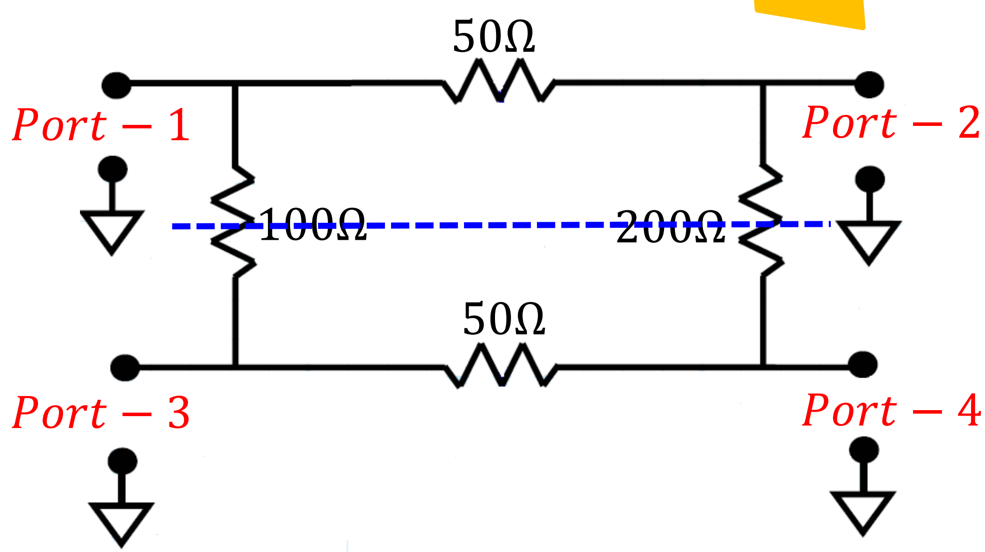
$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

Circuit Symmetry (contd.)

- Or, for circuits with **this D_1** symmetry:

$$1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 2$$

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{22} \end{bmatrix}$$

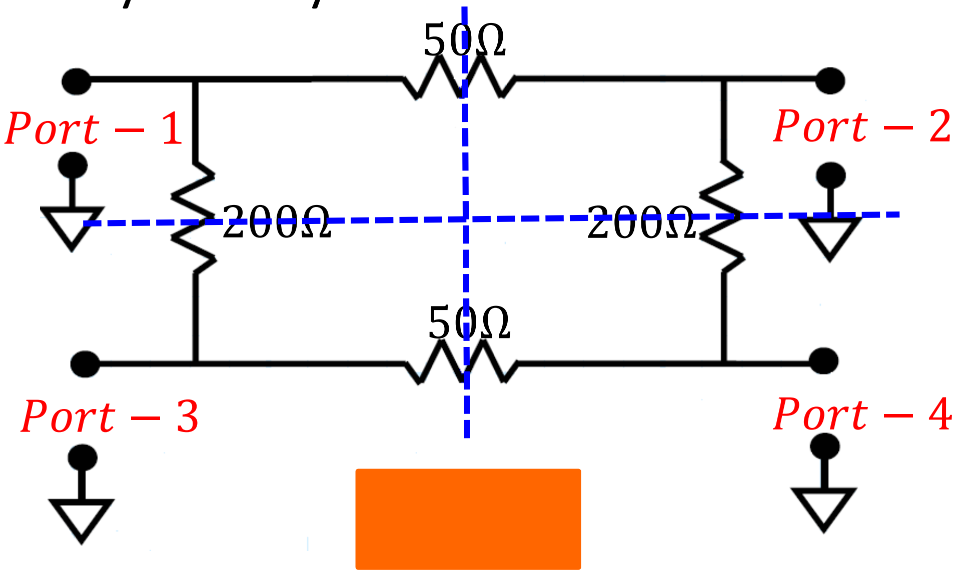


Q: Interesting. But **why do we care**?

A: This will greatly **simplify** the analysis of this symmetric circuit, as we need to determine **only** six matrix elements!

Circuit Symmetry (contd.)

- For a circuit with $\mathbf{D_2}$ symmetry:



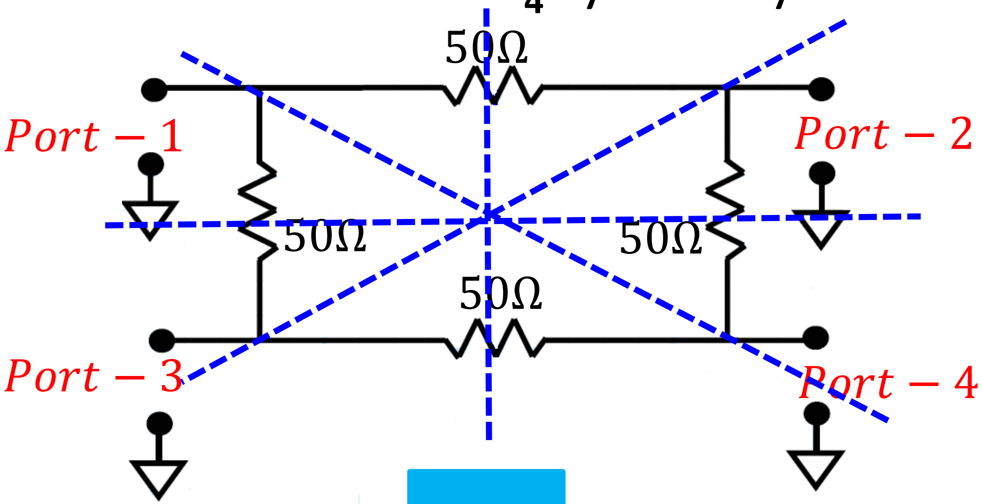
- we find that the impedance (or scattering, or admittance) matrix has the form:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{11} \end{bmatrix}$$

Note: there are just **four** independent values!

Circuit Symmetry (contd.)

- For a circuit with $\mathbf{D_4}$ symmetry:



- we find that the admittance (or scattering, or impedance) matrix has the form:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{21} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{21} \\ S_{21} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{21} & S_{21} & S_{11} \end{bmatrix}$$

Note: there are just **three** independent values!

Circuit Symmetry (contd.)

- One more interesting thing (yet **another** one!); recall that we earlier found that a matched, lossless, reciprocal **4-port** device must have a scattering matrix with one of **two forms**:

$$S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Symmetric

$$S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Anti-symmetric

- The "symmetric solution" has the **same form** as the scattering matrix of a circuit with **D2** symmetry!

$$S = \begin{bmatrix} 0 & S_{21} & S_{31} & 0 \\ S_{21} & 0 & 0 & S_{31} \\ S_{31} & 0 & 0 & S_{21} \\ 0 & S_{31} & S_{21} & 0 \end{bmatrix}$$

Q: Does this mean that a matched, lossless, reciprocal four-port device with the "symmetric" scattering matrix **must** exhibit **D_2** symmetry?

A: That's **exactly** what it means!

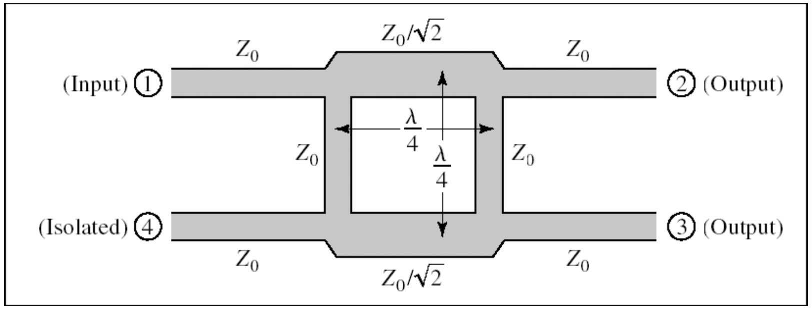
Circuit Symmetry (contd.)

- Not only can we determine from the **form** of the scattering matrix **whether** a particular design is possible (e.g., a matched, lossless, reciprocal 3-port device is impossible), we can also determine the **general structure** of a possible solutions (e.g. the circuit must have $\mathbf{D_2}$ symmetry).

- Likewise, the "anti-symmetric" matched, lossless, reciprocal four-port network **must** exhibit $\mathbf{D_1}$ symmetry!

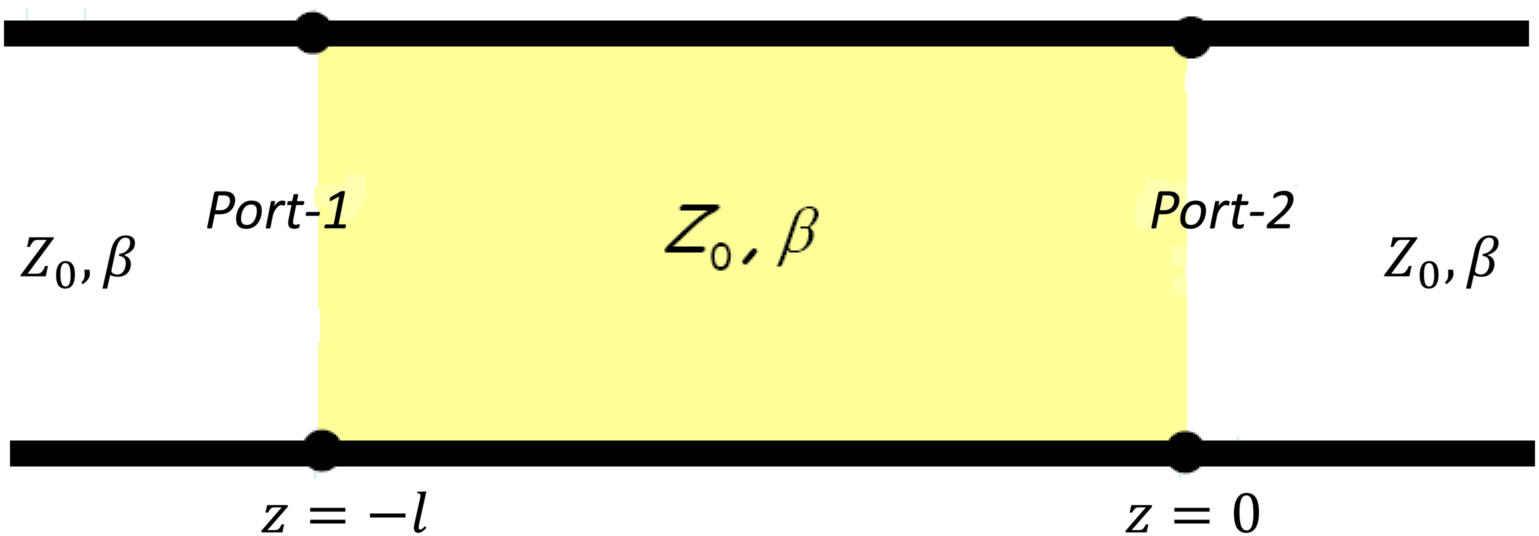
$$S = \begin{bmatrix} 0 & S_{21} & S_{31} & 0 \\ S_{21} & 0 & 0 & -S_{31} \\ S_{31} & 0 & 0 & S_{21} \\ 0 & -S_{31} & S_{21} & 0 \end{bmatrix}$$

We'll see just what these symmetric, matched, lossless, reciprocal four-port circuits actually are later in the course!



Example – 4

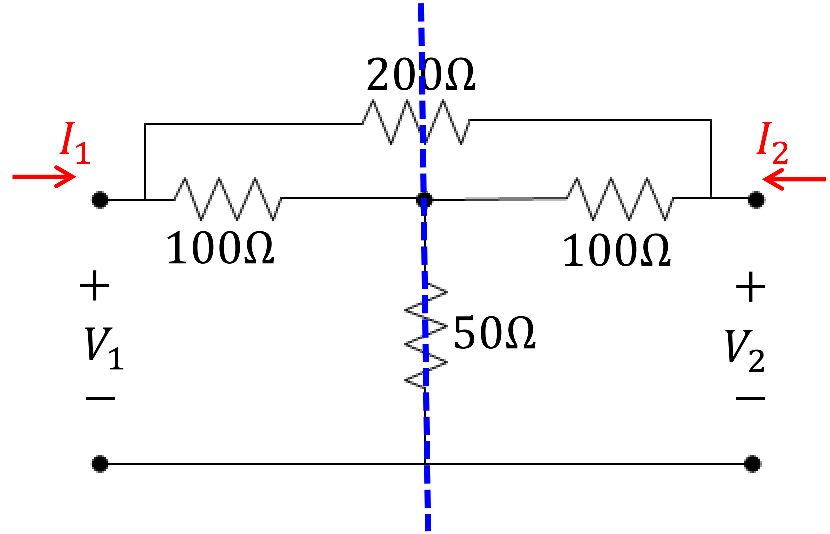
- determine the scattering matrix of the simple two-port device shown below:



$$S = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix}$$

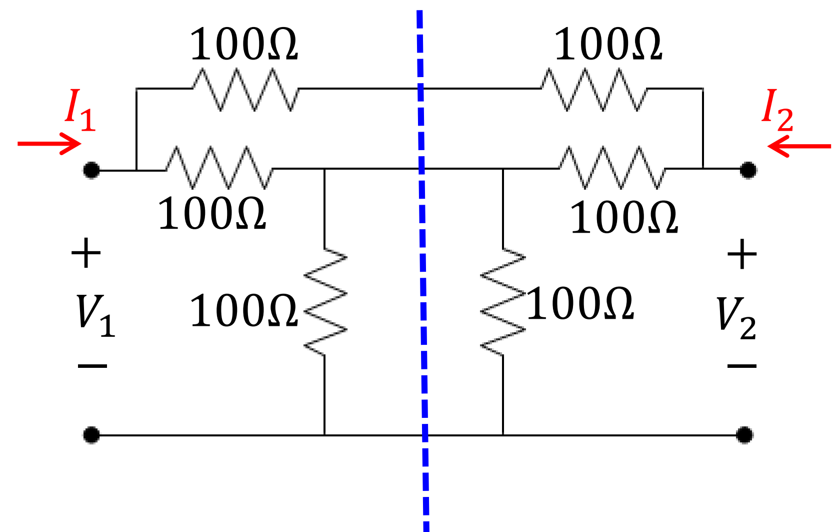
Symmetric Circuit Analysis

- Consider this **D$_1$** symmetric **two-port** device:



Q: Yikes! The plane of reflection symmetry slices through two resistors. What can we do about that?

A: Resistors are easily split into two equal pieces: the 200Ω resistor into two 100Ω resistors in **series**, and the 50Ω resistor as two 100 Ω resistors in **parallel**.



- Recall that the **symmetry** of this 2-port device leads to **simplified** network matrices:

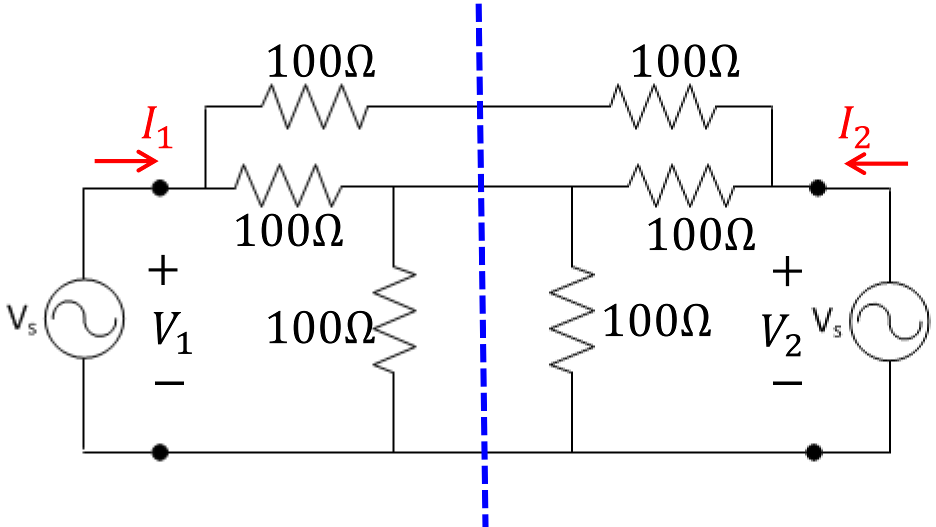
$$S = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{11} \end{bmatrix}$$

Symmetric Circuit Analysis (contd.)

Q: can circuit symmetry likewise simplify the procedure of **determining** these elements? In other words, can symmetry be used to **simplify circuit analysis**?

A: You bet!

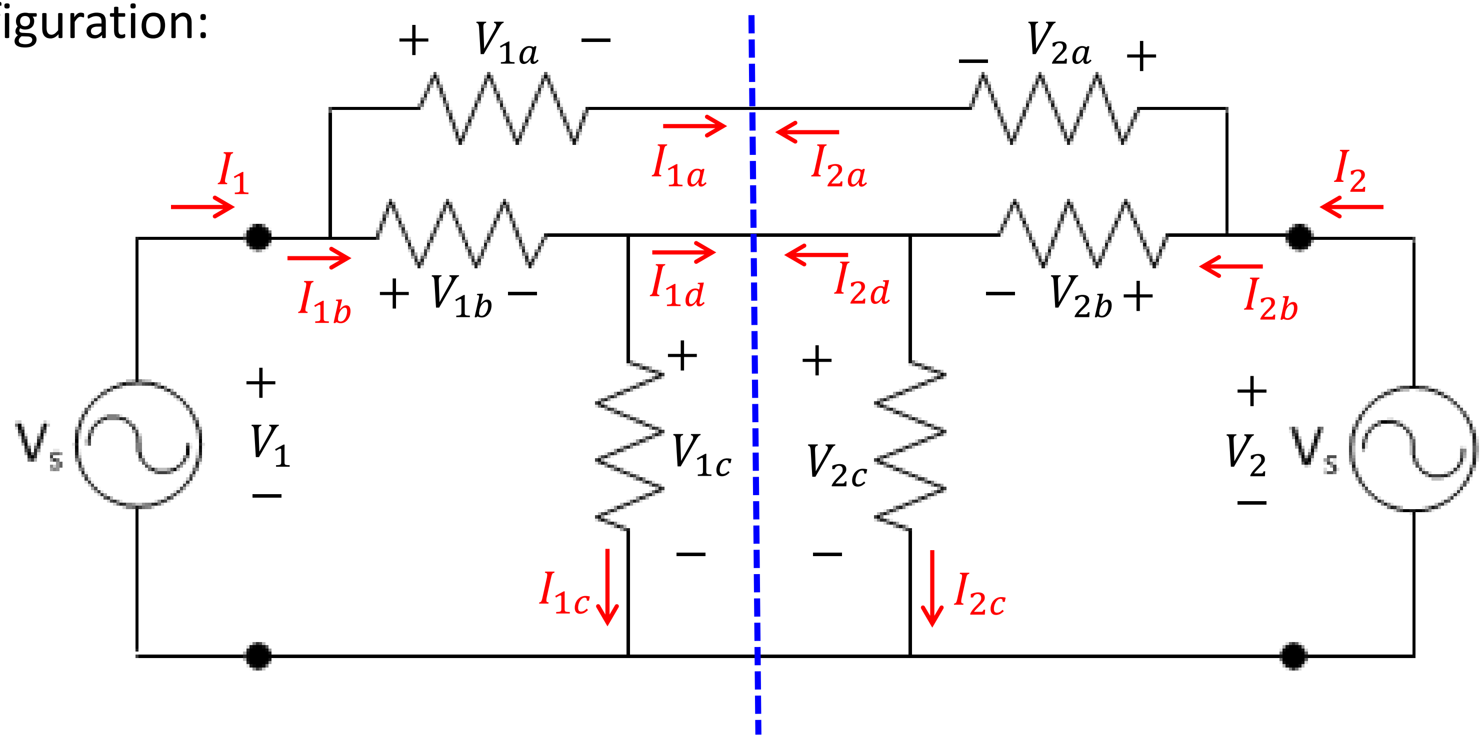
- First, consider the case where we **attach sources** to circuit in a way that **preserves** the circuit **symmetry**:



But remember! In order for **symmetry to be preserved**, the source values on both sides (i.e, V_s) must be **identical**!

Symmetric Circuit Analysis (contd.)

- Now, consider the **voltages** and **currents** within this circuit under this symmetric configuration:



- Since this circuit possesses **bilateral** (reflection) symmetry (1→2, 2→1), symmetric currents and voltages must be equal:

$$V_1 = V_2 \qquad V_{1a} = V_{2a} \qquad V_{1b} = V_{2b} \qquad V_{1c} = V_{2c}$$

$$I_1 = I_2 \qquad I_{1a} = I_{2a} \qquad I_{1b} = I_{2b} \qquad I_{1c} = I_{2c} \qquad I_{1d} = I_{2d}$$

Symmetric Circuit Analysis (contd.)

Q: Wait! This **can't** possibly be correct! Look at currents I_{1a} and I_{2a}, as well as currents I_{1d} and I_{2d}. From KCL, **this** must be true:

$$I_{1a} = -I_{2a}$$

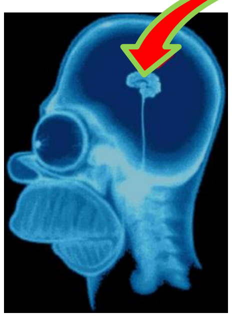
$$I_{1d} = -I_{2d}$$

- Yet **you** say that **this** must be true: $I_{1a} = I_{2a}$ $\quad I_{1d} = I_{2d}$

There is an obvious contradiction here! There is no way that both sets of equations can simultaneously be correct, is there?

A: Actually there **is**! There is **one** solution that will satisfy **both** sets of equations: $I_{1a} = I_{2a} = 0$ $\quad I_{1d} = I_{2d} = 0$

The currents are **zero**!



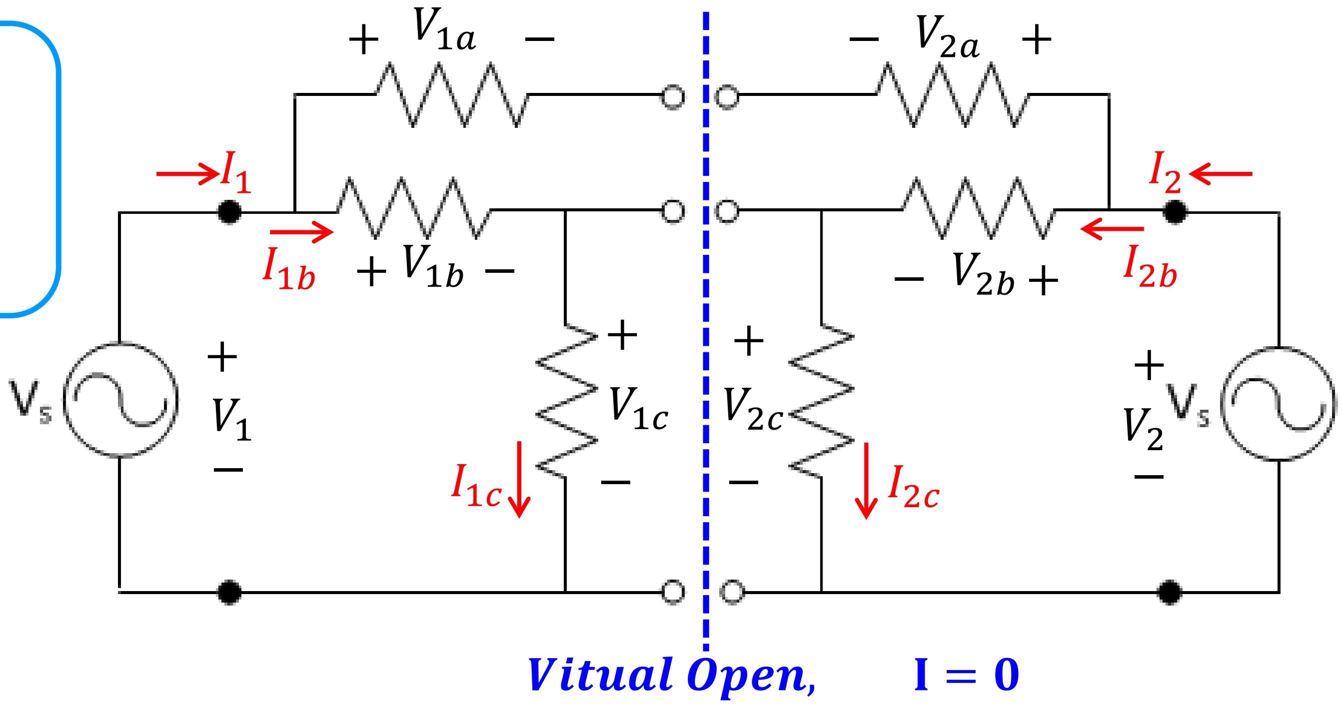
If you **think** about it, this makes **perfect sense**! The result says that **no current** will flow from one side of the symmetric circuit into the other.

- If current **did** flow across the symmetry plane, then the circuit symmetry would be **destroyed**—one side would effectively become the "**source side**", and the other the "**load side**" (i.e., the source side delivers current to the load side).

Symmetric Circuit Analysis (contd.)

- Thus, **no current** will flow **across** the reflection symmetry plane of a **symmetric circuit**—the symmetry plane thus acts as a **open circuit**!

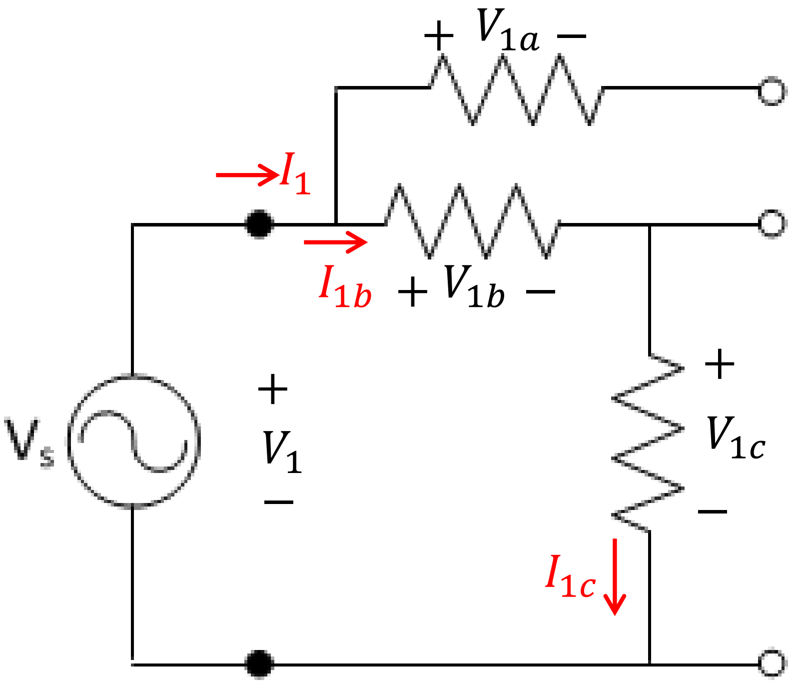
The plane of symmetry thus becomes a **virtual open**!



Symmetric Circuit Analysis (contd.)

Q: So what?

A: So what! This means that our circuit can be **split apart** into **two separate** but **identical** circuits. Solve **one** half-circuit, and you have **solved** the other!



$V_1 = V_2 = V_s$ $V_{1a} = V_{2a} = 0$

$V_{1b} = V_{2b} = V_s/2$ $V_{1c} = V_{2c} = V_s/2$

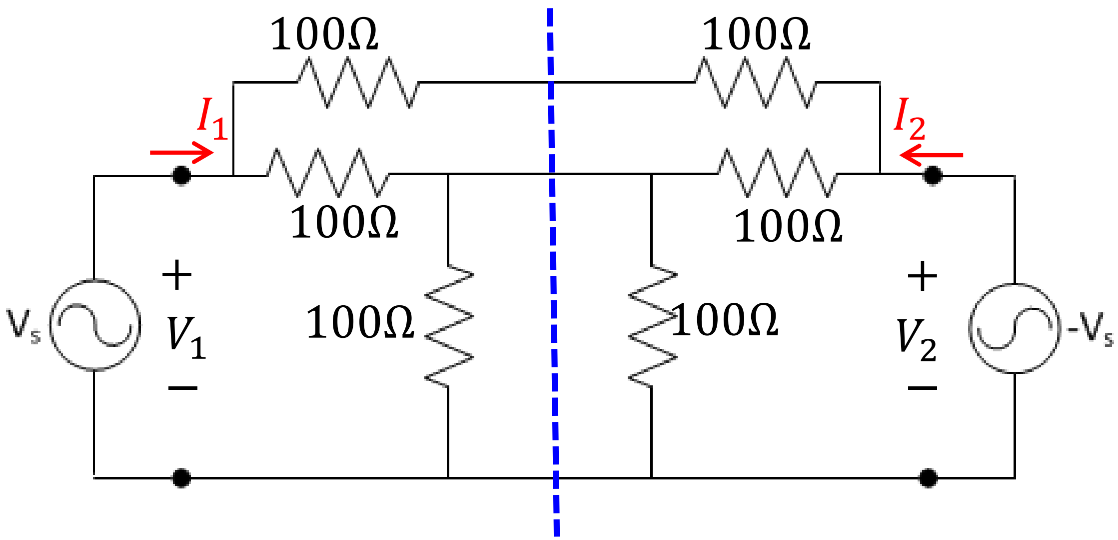
$I_1 = I_2 = V_s/200$ $I_{1a} = I_{2a} = 0$

$I_{1b} = I_{2b} = V_s/200$ $I_{1c} = I_{2c} = V_s/200$

$I_{1d} = I_{2d} = 0$

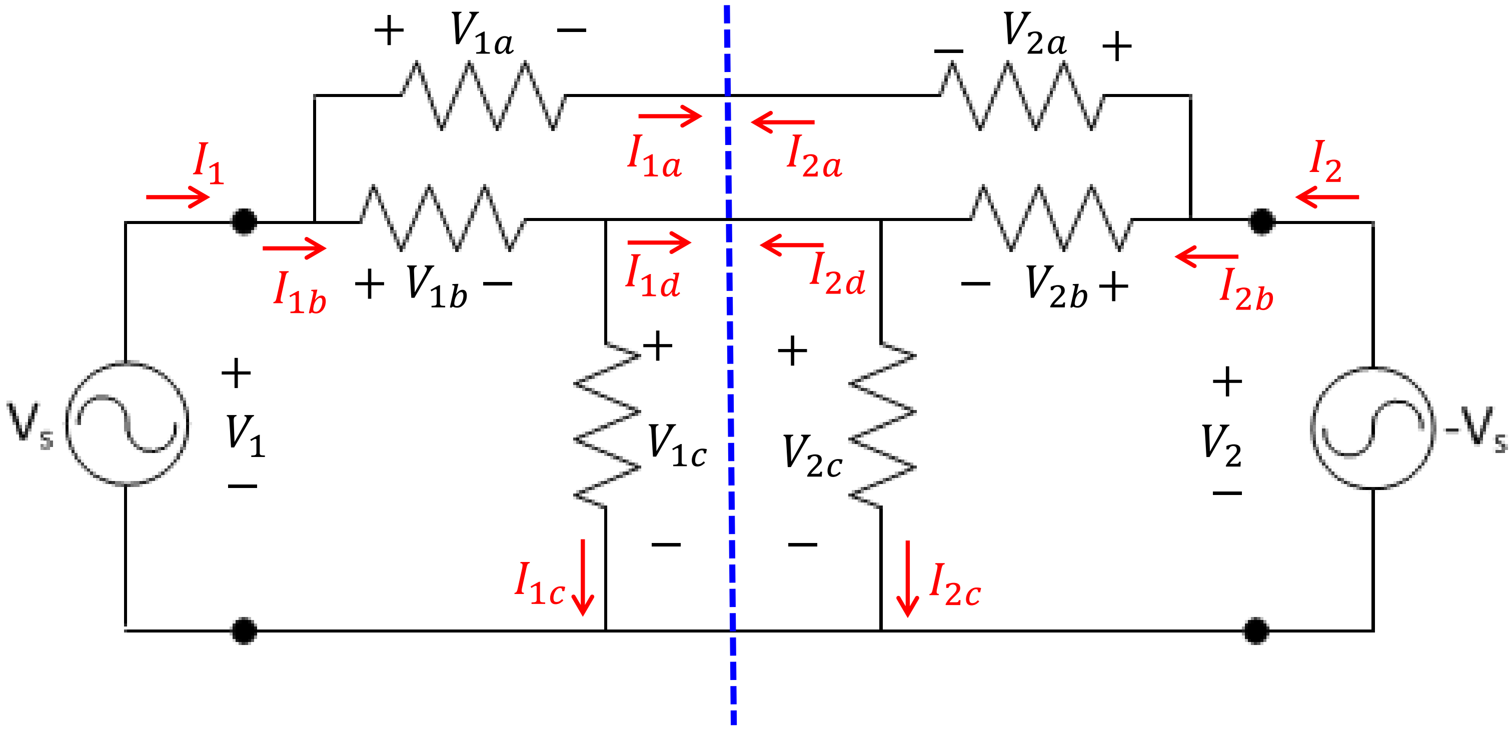
Asymmetric Circuit Analysis

- Now, consider **another** type of symmetry, where the sources are **equal** but **opposite** (i.e., **180 degrees** out of phase).



This situation still preserves the **symmetry** of the circuit— **somewhat.** The **voltages** and **currents** in the circuit will now posses **odd symmetry**—they will be **equal but opposite** (180 degrees out of phase) at symmetric points across the symmetry plane.

Asymmetric Circuit Analysis (contd.)

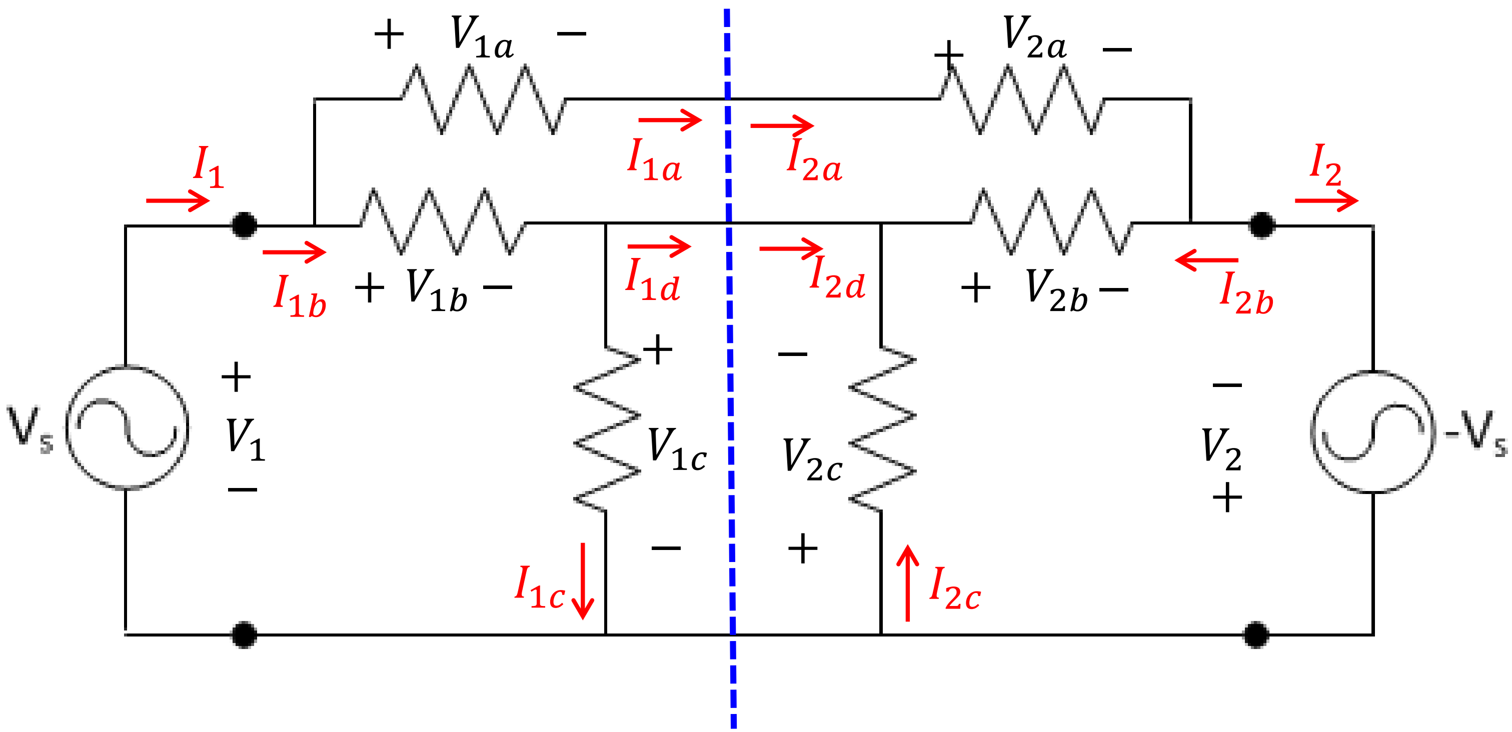


$$V_1 = -V_2 \qquad V_{1a} = -V_{2a} \qquad V_{1b} = -V_{2b} \qquad V_{1c} = -V_{2c}$$

$$I_1 = -I_2 \qquad I_{1a} = -I_{2a} \qquad I_{1b} = -I_{2b} \qquad I_{1c} = -I_{2c} \qquad I_{1d} = -I_{2d}$$

Asymmetric Circuit Analysis (contd.)

- Perhaps it would be easier to **redefine** the circuit variables as:



$V_1 = V_2 \qquad V_{1a} = V_{2a} \qquad V_{1b} = V_{2b} \qquad V_{1c} = V_{2c}$

$I_1 = I_2 \qquad I_{1a} = I_{2a} \qquad I_{1b} = I_{2b} \qquad I_{1c} = I_{2c} \qquad I_{1d} = I_{2d}$

Q: But wait! **Again** I see a problem. By **KVL** it is evident that: $V_{1c} = -V_{2c}$

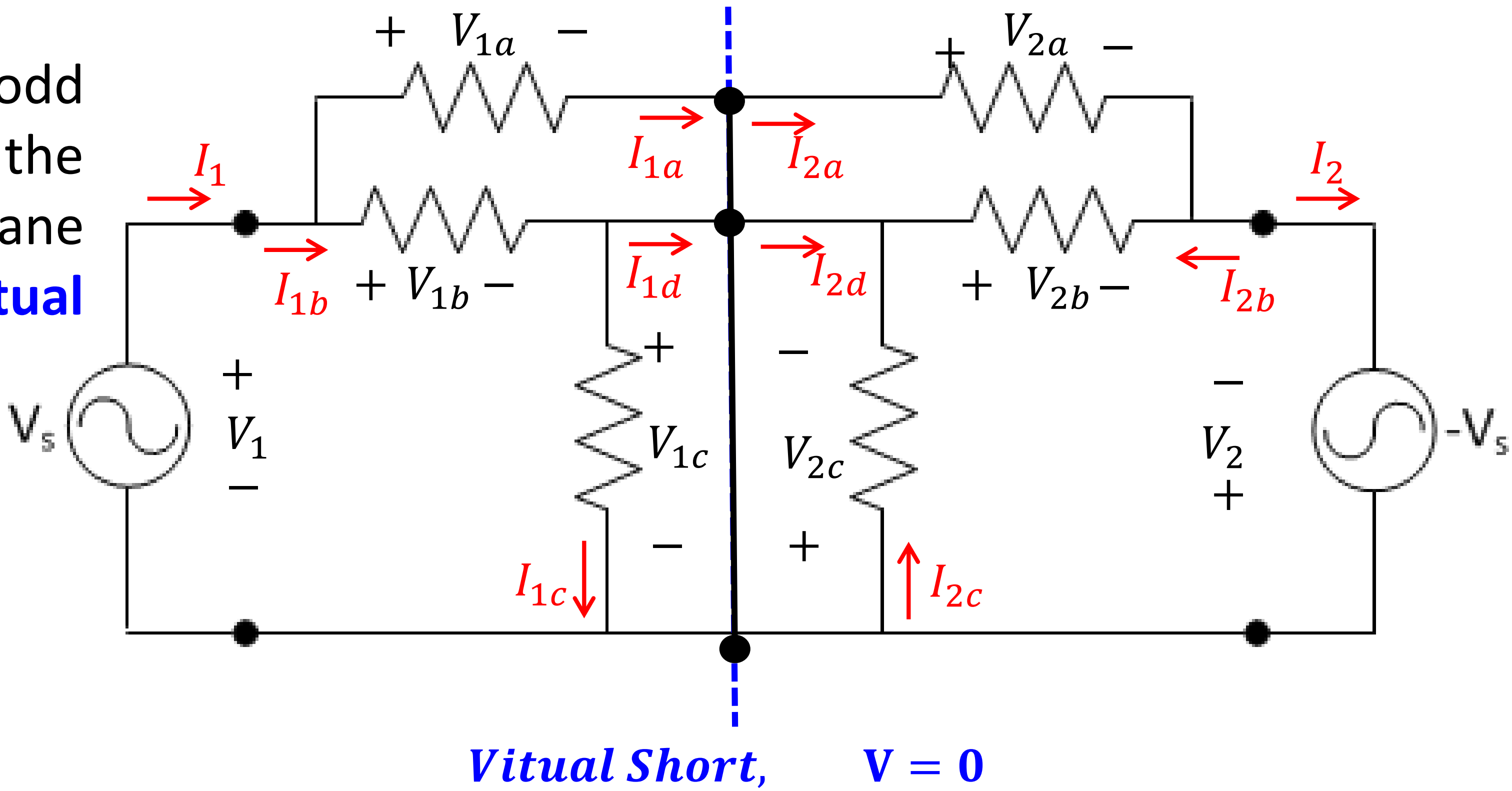
Yet **you** say that $V_{1c} = V_{2c}$ must be true!

Asymmetric Circuit Analysis (contd.)

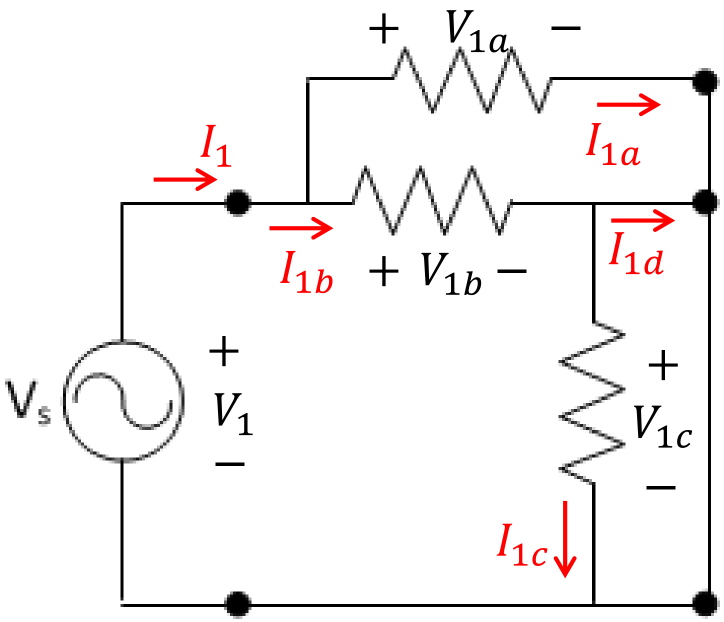
A: Again, the solution to **both** equations is **zero**! $V_{1c} = V_{2c} = 0$

For the case of **odd symmetry**, the symmetric plane must be a plane of **constant potential** (i.e., constant voltage)—just like a **short circuit**!

- Thus, for odd symmetry, the symmetric plane forms a **virtual short**.



Asymmetric Circuit Analysis (contd.)



$$V_1 = V_s$$

$$V_{1b} = V_s$$

$$V_{1a} = V_s$$

$$V_{1c} = 0$$

$$I_1 = {}^{V_s}/_{50}$$

$$I_{1a} = {}^{V_s}/_{100}$$

$$I_{1b} = {}^{V_s}/_{100}$$

$$I_{1c} = 0$$

$$I_{1d} = {}^{V_s}/_{100}$$

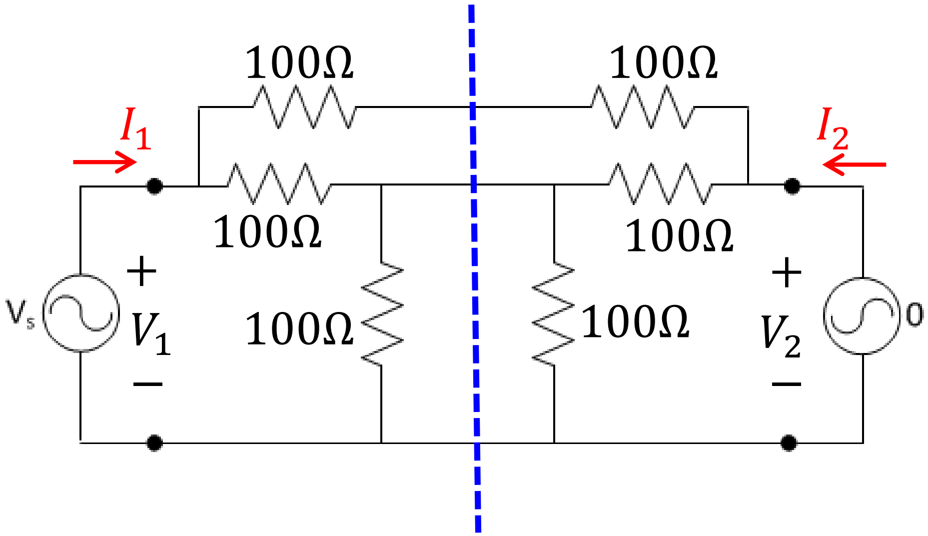
Odd/Even Mode Analysis

Q: Although symmetric **circuits** appear to be plentiful in microwave engineering, it seems **unlikely** that we would often encounter symmetric **sources** . Do virtual shorts and opens typically ever occur?

A: One word—**superposition**!

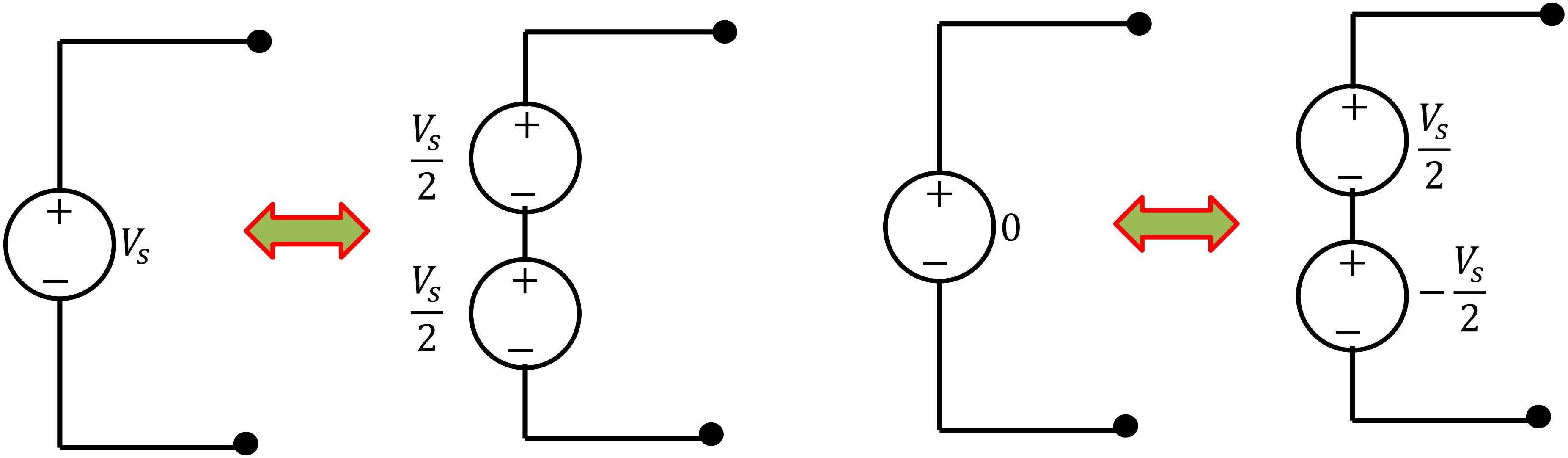
If the elements of our circuit are **independent** and **linear**, we can apply superposition to analyze **symmetric circuits** when **non-symmetric** sources are attached.

- For example, say we wish to determine the **admittance matrix** of this circuit. We would place a **voltage source at port 1**, and a **short circuit at port 2**—a set of **asymmetric** sources if there ever was one!



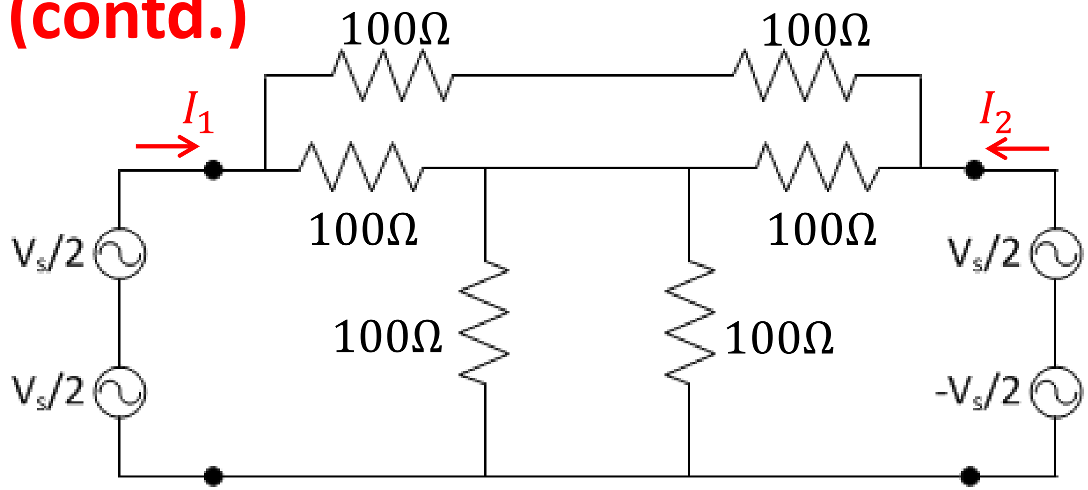
Odd/Even Mode Analysis (contd.)

- Here's the really **neat** part. We find that the source on port 1 can be modelled as **two equal** voltage sources in series, whereas the source at port 2 can be modelled as **two equal but opposite** sources in series.

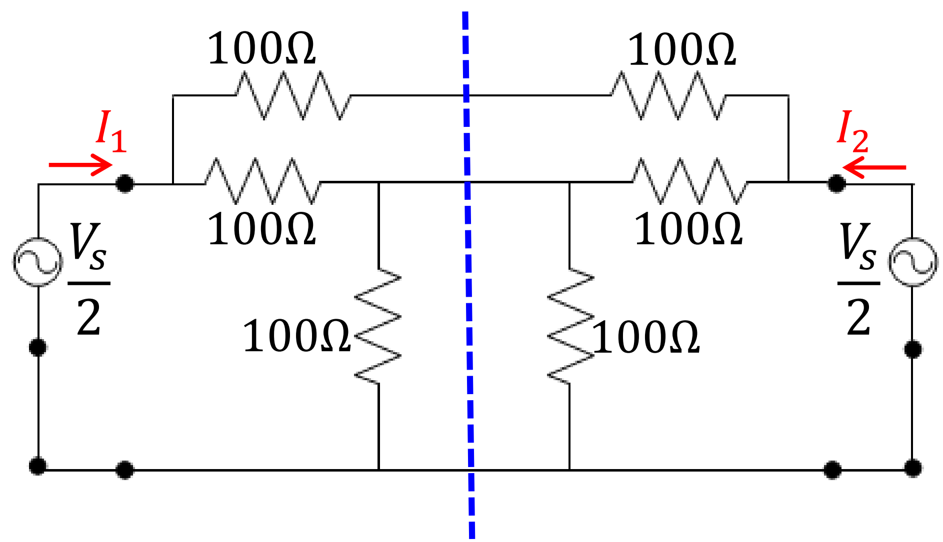


Odd/Even Mode Analysis (contd.)

- Therefore an **equivalent** circuit is:



- Now, the **above** circuit (due to the sources) is obviously **asymmetric—no** virtual ground, **nor** virtual short is present. But, let's say we **turn off** (i.e., set to V =0) the **bottom** source on **each side** of the circuit:



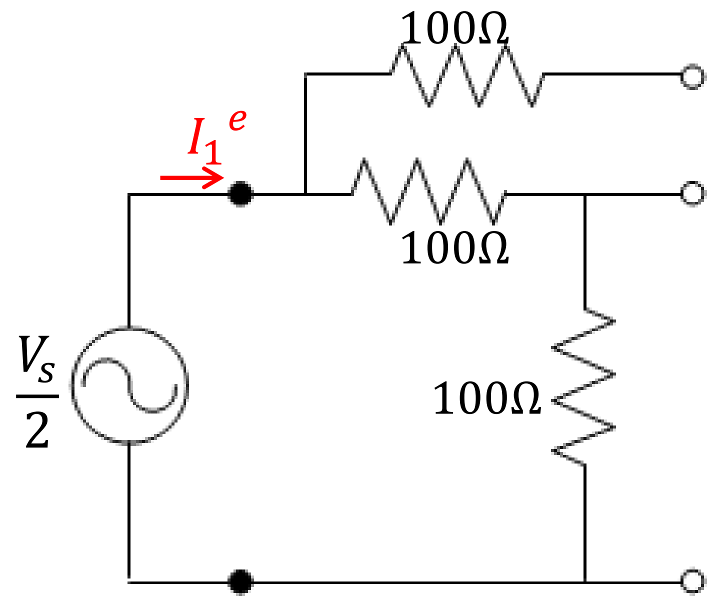
Our **symmetry** has been **restored**! The symmetry plane is a **virtual open**.

Odd/Even Mode Analysis (contd.)

- The circuit is referred to as its **even mode**, and analysis of it is known as the **even mode analysis**. The solutions are known as the even mode **currents** and **voltages**!

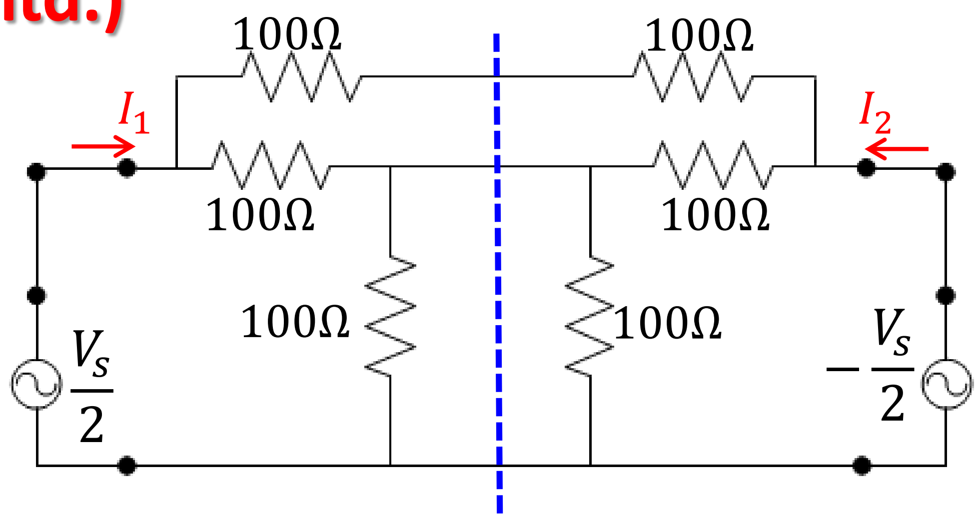
- Evaluating the resulting **even mode** half circuit we find:

$$I_1^e = \frac{V_s}{2}\frac{1}{200} = \frac{V_s}{400} = I_2^e$$



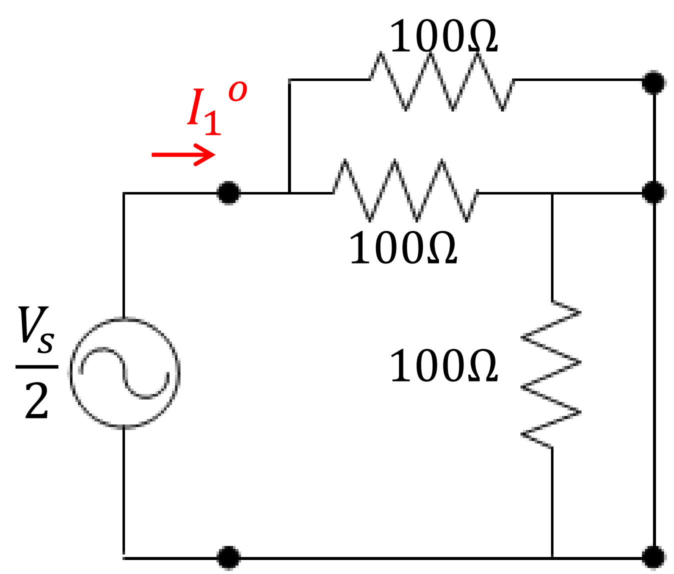
Odd/Even Mode Analysis (contd.)

- Now, let's turn the bottom sources **back on**—but turn **off** the **top two**!
- We now have a circuit with **odd symmetry**—the symmetry plane is a **virtual short**!



- This circuit is referred to as its **odd mode**, and analysis of it is known as the **odd mode analysis**. The solutions are known as the odd mode **currents** and **voltages**!
- Evaluating the resulting **odd mode** half circuit we find:

$$I_1^o = \frac{V_s}{2}\frac{1}{50} = \frac{V_s}{100} = -I_2^o$$



Odd/Even Mode Analysis (contd.)

Q: But what good is this "even mode" and "odd mode" analysis? After all, the source on port 1 is $V_{s1} = V_s$, and the source on port 2 is $V_{s2} = 0$. What are the currents $I_1 = I_2$ for **these** sources?

A: Recall that these sources are the **sum** of the even and odd mode sources:

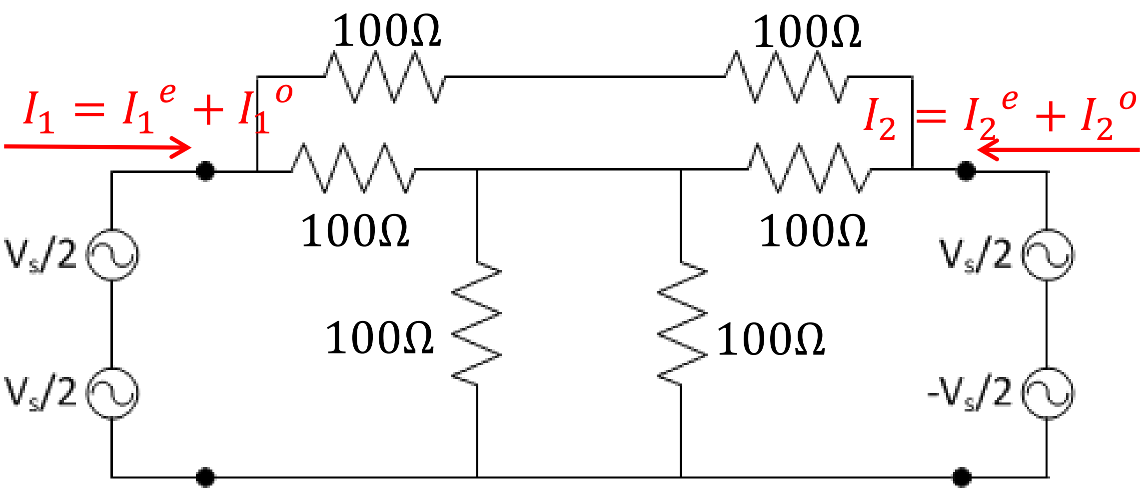
First Source: $V_s = \frac{V_s}{2} + \frac{V_s}{2}$ Second Source: $V_s = \frac{V_s}{2} - \frac{V_s}{2}$

- and thus—since all the devices in the circuit are **linear**—we know from superposition that the currents I_1 and I_2 are simply the **sum** of the **odd** and **even** mode currents !

$$I_1 = I_1^e + I_1^o$$

$$I_2 = I_2^e + I_2^o$$

Odd/Even Mode Analysis (contd.)



- Thus, **adding** the odd and even mode analysis results together:

$$I_1 = I_1^e + I_1^o = \frac{V_s}{400} + \frac{V_s}{100} = \frac{V_s}{80}$$

$$I_2 = I_2^e + I_2^o = \frac{V_s}{400} - \frac{V_s}{100} = -\frac{3V_s}{400}$$

- And then the **admittance parameters** for this two port network is:

$$y_{11} = \frac{I_1}{V_{s1}}\Big|_{V_{s2}=0} = \frac{V_s}{80}\frac{1}{V_s} = \frac{1}{80}$$

$$y_{21} = \frac{I_2}{V_{s1}}\Big|_{V_{s2}=0} = -\frac{3V_s}{400}\frac{1}{V_s} = \frac{-3}{400}$$

- And from the **symmetry** of the device we know:

$$y_{22} = y_{11} = \frac{1}{80}$$

$$y_{12} = y_{21} = \frac{-3}{400}$$

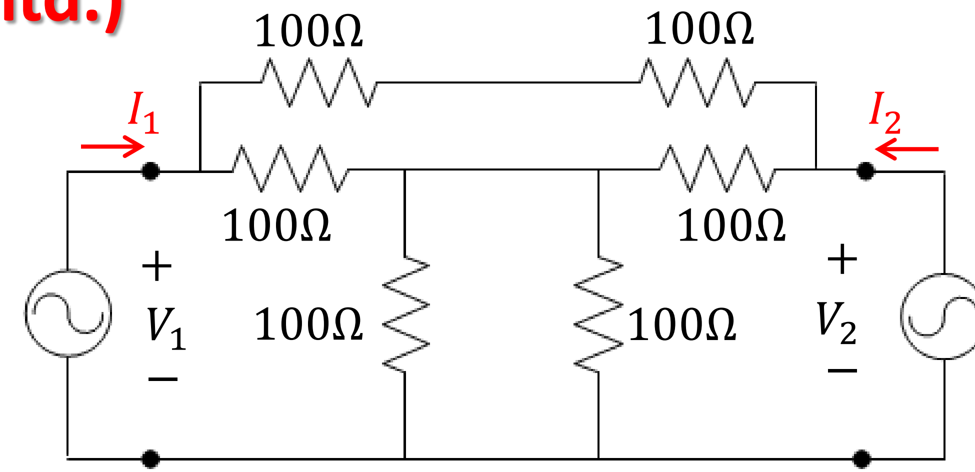
- Thus, the full **admittance matrix** is:

$$Y = \begin{bmatrix} 1/80 & -3/400 \\ -3/400 & 1/80 \end{bmatrix}$$

Odd/Even Mode Analysis (contd.)

Q: What happens if **both** sources are **non-zero**? Can we use symmetry then?

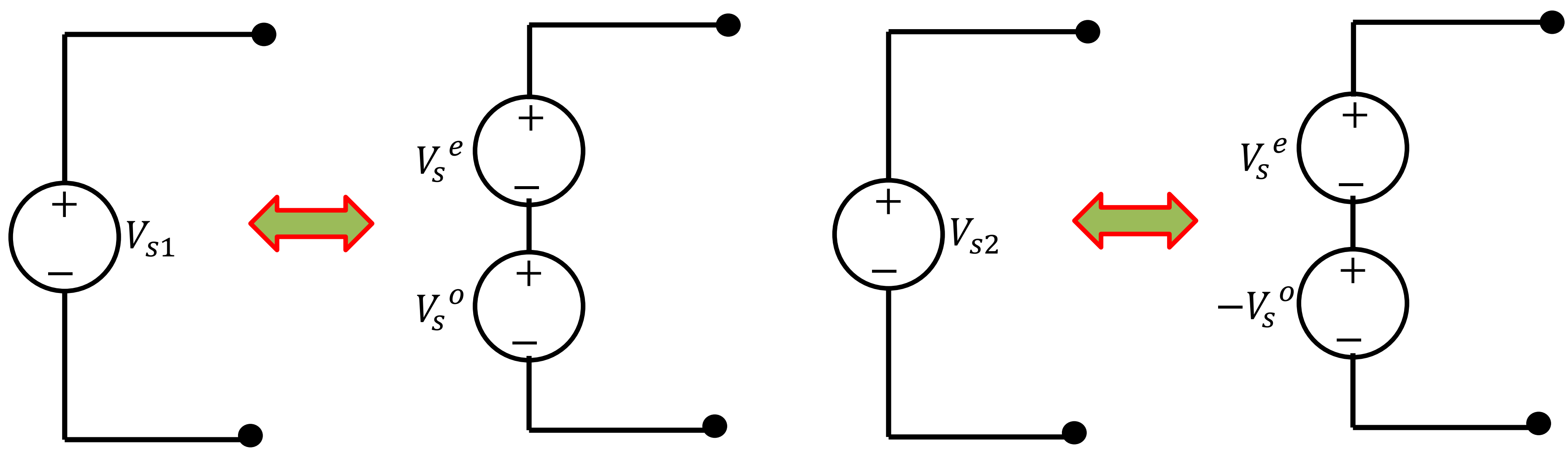
A: Absolutely! Consider this problem, where **neither** source is equal to zero:



- In this case we can define an even mode and an odd mode source as:

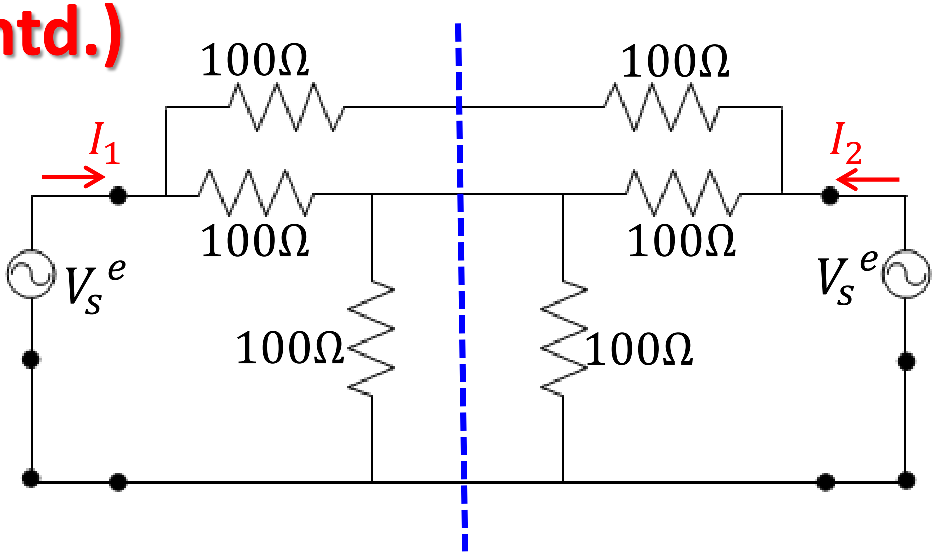
$$V_s^e = \frac{V_{s1} + V_{s2}}{2}$$

$$V_s^o = \frac{V_{s1} - V_{s2}}{2}$$

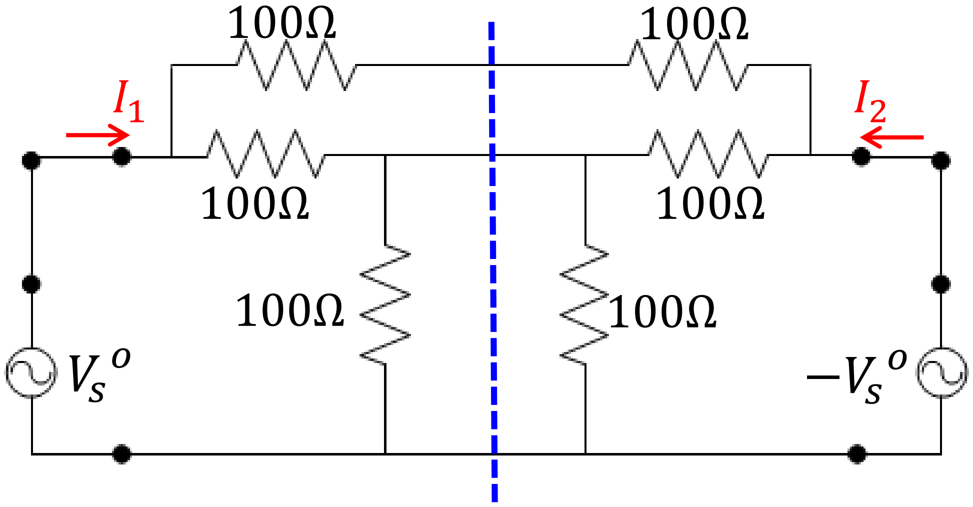


Odd/Even Mode Analysis (contd.)

- We can then analyze the **even mode** circuit:



- And then the **odd mode** circuit:



And then combine these results in a **linear superposition**!

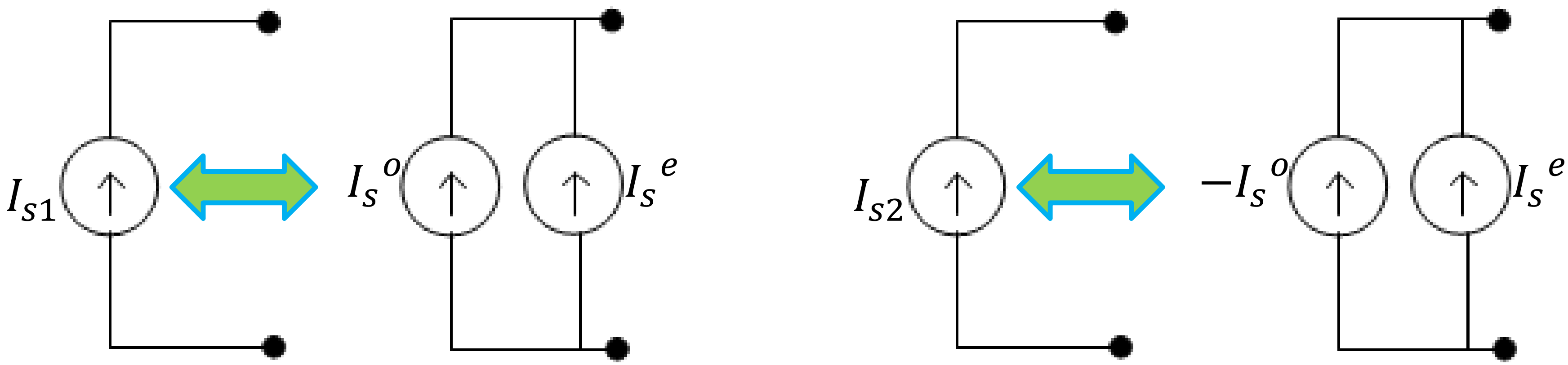
Odd/Even Mode Analysis (contd.)

Q: What about **current sources**? Can I likewise consider them to be a **sum** of an odd mode source and an even mode source?

A: Yes, but be **very** careful! The current of two source will add if they are placed in **parallel**—**not** in series! Therefore:

$$I_s^e = \frac{I_{s1} + I_{s2}}{2}$$

$$I_s^o = \frac{I_{s1} - I_{s2}}{2}$$

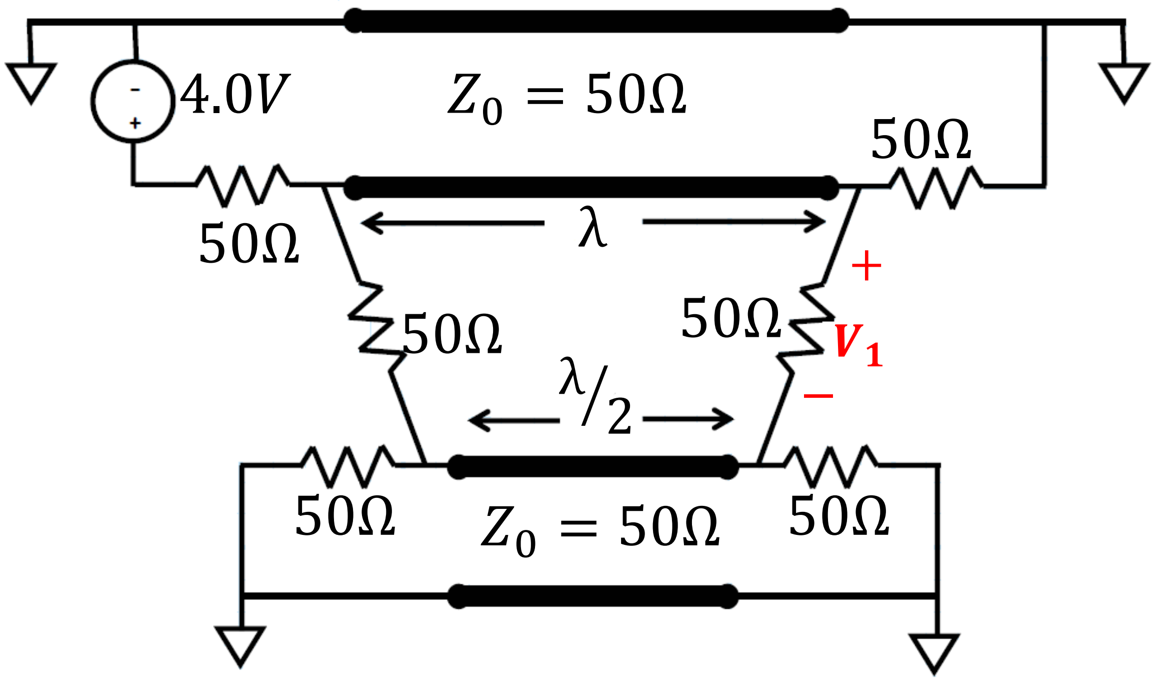


- One **final** word (I promise!) about circuit symmetry and even/odd mode analysis: **precisely** the **same** concept exits in **electronic circuit** design!

Specifically, the **differential** (odd) and **common** (even) **mode** analysis of bilaterally symmetric electronic circuits, such as **differential amplifiers**!

Example – 2

- Carefully (**very** carefully) consider the **symmetric** circuit below:

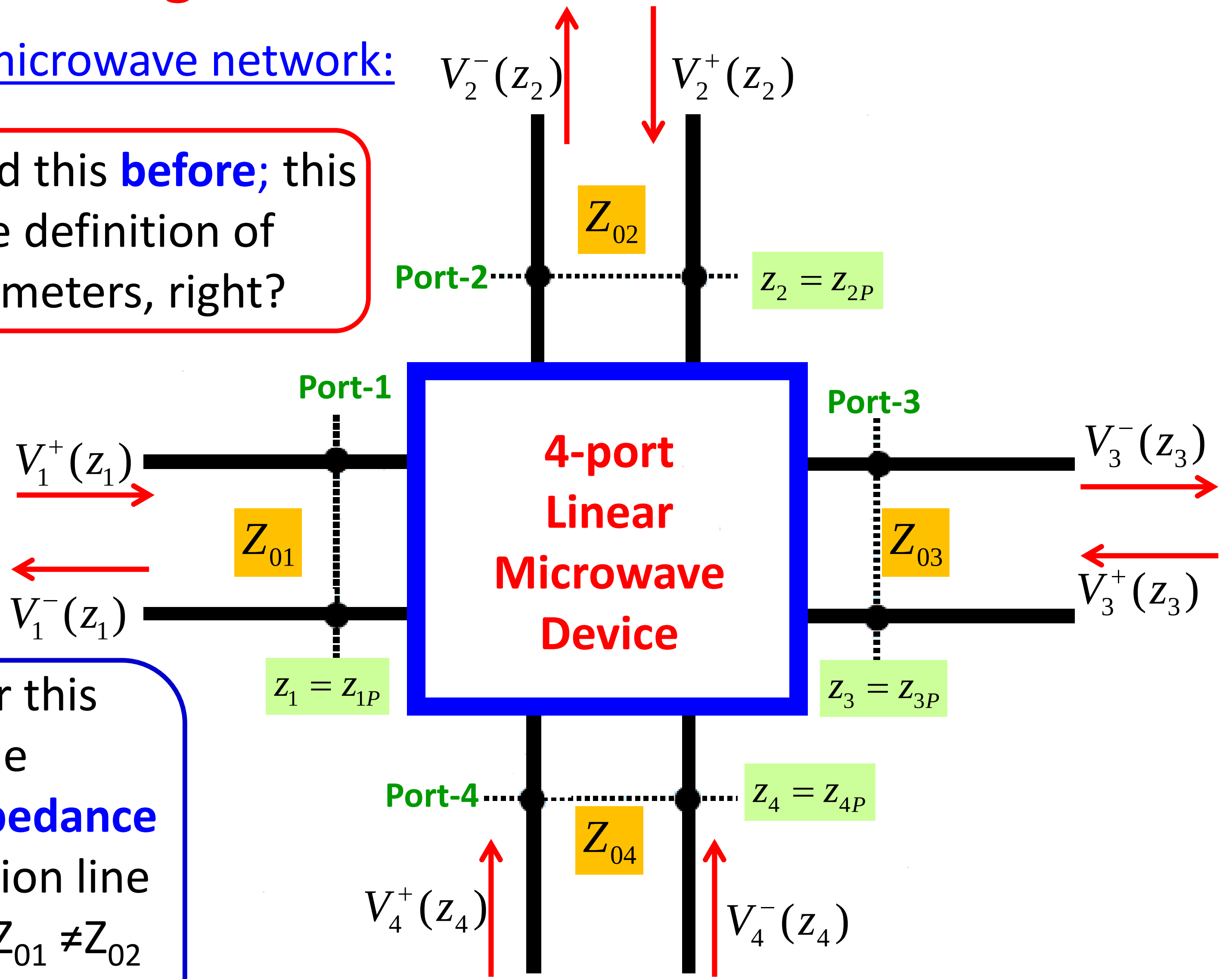


Use **odd-even mode analysis** to determine the value of voltage V_1.

Generalized Scattering Parameters

Consider now **this** microwave network:

Boring! We studied this **before**; this will lead to the definition of scattering parameters, right?



Not exactly. For this network, the **characteristic impedance** of each transmission line is **different** (i.e., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$)!

Generalized Scattering Parameters (contd.)

Yikes! You said scattering parameters are **dependent** on transmission line characteristic impedance Z_0. If these values are **different** for each port, **which** Z_0 do we use?

For this **general** case, we must use **generalized scattering parameters**! First, we define a slightly new form of complex wave amplitudes

$$a_n = \frac{V_n^+}{\sqrt{Z_{0n}}} \qquad b_n = \frac{V_n^-}{\sqrt{Z_{0n}}}$$

- **The key things to note are:**

a → **variable a** (e.g., a_1, a_2, …) denotes the complex amplitude of an **incident (i.e., plus)** wave.

b ← **variable b** (e.g., a_1, a_2, …) denotes the complex amplitude of an **exiting (i.e., minus)** wave.

Generalized Scattering Parameters (contd.)

We now get to **rewrite** all our transmission line knowledge in terms of these generalized complex amplitudes!

- First, our two propagating wave amplitudes (i.e., plus and minus) are **compactly** written as:

$$V_n^+ = a_n \sqrt{Z_{0n}} \qquad V_n^- = b_n \sqrt{Z_{0n}}$$

- Therefore:

$$V_n^+(z_n) = a_n \sqrt{Z_{0n}}.e^{-j\beta z_n} \qquad V_n^-(z_n) = b_n \sqrt{Z_{0n}}.e^{+j\beta z_n}$$

$$\Gamma(z_n) = \frac{b_n}{a_n} e^{+j2\beta z_n}$$

Generalized Scattering Parameters (contd.)

- Similarly, the total voltage, current, and impedance at the **n^th^** port are:

$$V_n(z_n) = \sqrt{Z_{0n}}\left(a_n.e^{-j\beta z_n} + b_n e^{+j\beta z_n}\right)$$

$$I_n(z_n) = \frac{\left(a_n.e^{-j\beta z_n} - b_n e^{+j\beta z_n}\right)}{\sqrt{Z_{0n}}}$$

$$Z(z_n) = \frac{a_n.e^{-j\beta z_n} + b_n e^{+j\beta z_n}}{a_n.e^{-j\beta z_n} - b_n e^{+j\beta z_n}}$$

- Assuming that our port planes are defined with $z_{nP} = 0$, we can determine the total voltage, current, and impedance **at port n** as:

$$V_n \doteq V_n(z_n = 0) = \sqrt{Z_{0n}}\left(a_n + b_n\right)$$

$$I_n \doteq I_n(z_n = 0) = \frac{\left(a_n - b_n\right)}{\sqrt{Z_{0n}}}$$

$$Z_n \doteq Z(z_n = 0) = \frac{a_n + b_n}{a_n - b_n}$$

Generalized Scattering Parameters (contd.)

- Similarly, the **power** associated with each wave is:

$$P_n^+ = \frac{|V_n^+|^2}{2Z_{0n}} = \frac{|a_n|^2}{2}$$

$$P_n^- = \frac{|V_n^-|^2}{2Z_{0n}} = \frac{|b_n|^2}{2}$$

- As such, the power **delivered to** port n (i.e., the power **absorbed by** port n) is:

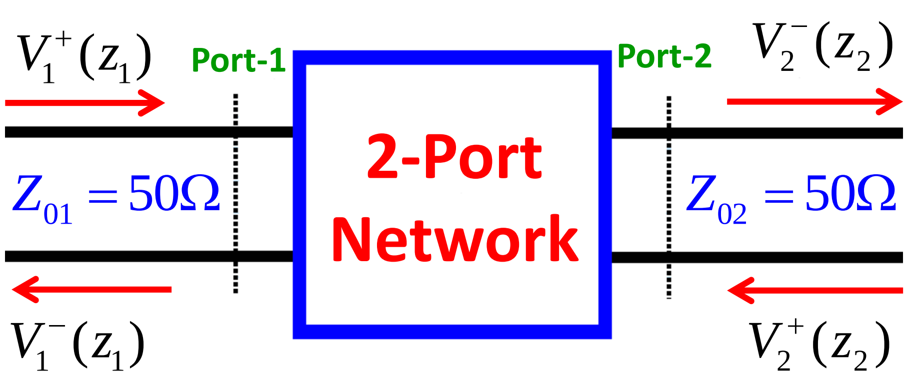
$$P_n = P_n^+ - P_n^- = \frac{|a_n|^2 - |b_n|^2}{2}$$

So what's the **big deal**? This is yet **another way** to express transmission line activity.
Do we **really** need to know this, or is this simply a strategy for making the next quiz **even harder**?

Generalized Scattering Parameters (contd.)

- You may have noticed that this notation (a_n, b_n) provides descriptions that are a bit **"cleaner"** and more symmetric between current and voltage.
- However, the **main reason** for this notation is for evaluating the **scattering parameters** of a device with **dissimilar** transmission line impedance (e.g., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$).
- For these cases we must use **generalized scattering parameters**:

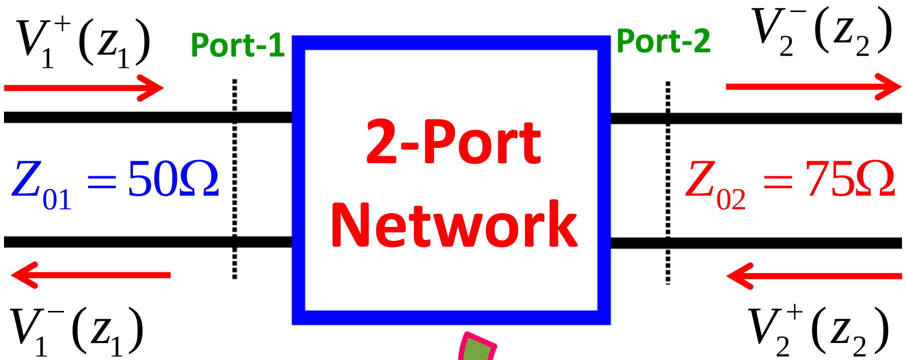
$$S_{mn} = \frac{V_m^-}{V_n^+}\frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} \quad \text{when } V_k^+(z_k) = 0 \text{ for all } k \neq n)$$



Here, $Z_{01} = Z_{02}$

$$S_{21} = \frac{V_2^-}{V_1^+} \quad \text{when, } V_2^+ = 0$$

Generalized Scattering Parameters (contd.)



Here, $Z_{01} \neq Z_{02}$

$$S_{21} \neq \frac{V_2^-}{V_1^+} \quad \text{when, } V_2^+ = 0$$

Instead

$$S_{21} = \frac{V_2^-}{V_1^+}\frac{\sqrt{50}}{\sqrt{75}} \quad \text{when, } V_2^+ = 0$$

- Note that the generalized scattering parameters can be more **compactly** written in terms of our **new** wave amplitude notation:

$$S_{mn} = \frac{V_m^-}{V_n^+}\frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} = \frac{b_m}{a_n} \quad \text{when } a_k = 0 \text{ for all } k \neq n)$$

Remember, this is the **generalized** form of scattering parameter—it **always** provides the correct answer, **regardless** of the values of Z_{0m} or Z_{0n}!

Generalized Scattering Parameters (contd.)

- But **why can't** we define the scattering parameter as $S_{mn} = V_m^-/V_n^+$, **regardless** of Z_{0m} or Z_{0n}?? **Who says** we must define it with those **awful** Z_{0n} values in there?

Recall that a lossless device will **always** have a **unitary** scattering matrix. As a result, the scattering parameters of a lossless device will **always** satisfy, for example:

$$\sum_{m=1}^{N} \left| S_{mn} \right|^2 = 1$$

This is true only if generalized scattering parameters are used

The scattering parameters of a lossless device will form a unitary matrix **only** if defined as $S_{mn} = b_m/a_n$. If we use $S_{mn} = V_m^-/V_n^+$, the matrix will be unitary **only** if the connecting transmission lines have the **same** characteristic impedance.

Generalized Scattering Parameters (contd.)

- Do we really **care** if the matrix of a lossless device is unitary or not?

Absolutely! we do!

lossless device ⇔ unitary scattering matrix

This relationship is a very powerful one. It allows us to **identify** lossless devices, and it allows us to determine **if** specific lossless devices are **even possible**!

Example – 3

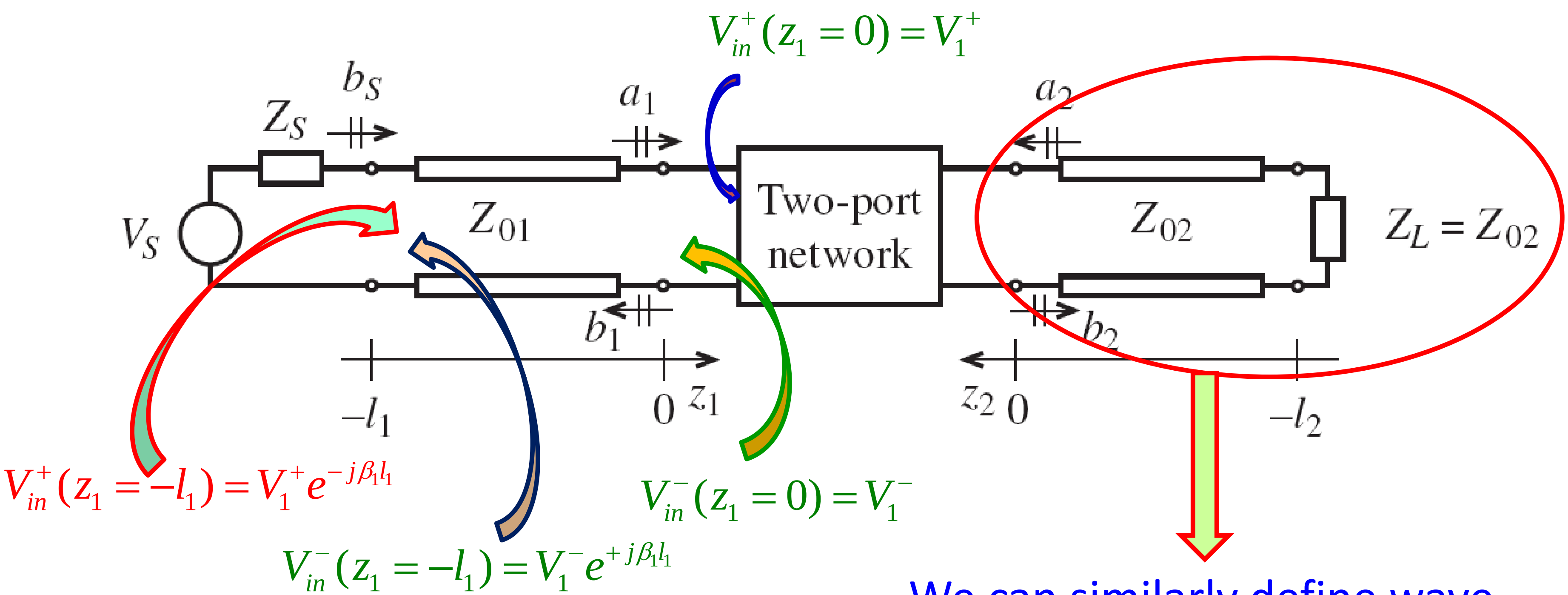
- let's consider a **perfect connector**—an electrically **very small** two-port device that allows us to connect the ends of different transmission lines together.

Determine the S-matrix of this ideal connector:

1. First case: it connects two transmission lines with same characteristic impedance of Z_0.
2. Second case: it connects two transmission lines with characteristic impedances of Z_{01} and Z_{02} respectively.

Shifting of Planes

- It is not often easy or feasible to match network ports for determination of S-parameters→ in such a situation S-parameters are determined through transmission lines of finite length.
- Let us consider a 2-port network to understand these situations.



We can similarly define wave functions on the output

Shifting of Planes (contd.)

- The equations can be combined to form following matrix

$$\begin{Bmatrix} V_{in}^{+}(-l_1) \\ V_{out}^{+}(-l_2) \end{Bmatrix} = \begin{bmatrix} e^{-j\beta_1 l_1} & 0 \\ 0 & e^{-j\beta_2 l_2} \end{bmatrix} \begin{Bmatrix} V_1^{+} \\ V_2^{+} \end{Bmatrix}$$

Links the incident waves at the network ports shifted by TL segments

$$\begin{Bmatrix} V_{in}^{-}(-l_1) \\ V_{out}^{-}(-l_2) \end{Bmatrix} = \begin{bmatrix} e^{+j\beta_1 l_1} & 0 \\ 0 & e^{+j\beta_2 l_2} \end{bmatrix} \begin{Bmatrix} V_1^{-} \\ V_2^{-} \end{Bmatrix}$$

Links the incident waves at the network ports shifted by TL segments

- We can also deduce that S-parameters are linked to the generalized coefficients a_n and b_n (which in turn can be expressed through voltages) through following expressions (if we assume $Z_{01} = Z_{02}$)

$$\begin{Bmatrix} V_1^{-} \\ V_2^{-} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} V_1^{+} \\ V_2^{+} \end{Bmatrix}$$

Shifting of Planes (contd.)

- Simplification of these three matrix expression results in:

$$\begin{Bmatrix} V_{in}^{-}(-l_1) \\ V_{out}^{-}(-l_2) \end{Bmatrix} = \begin{bmatrix} e^{+j\beta_1 l_1} & 0 \\ 0 & e^{+j\beta_2 l_2} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e^{+j\beta_1 l_1} & 0 \\ 0 & e^{+j\beta_2 l_2} \end{bmatrix} \begin{Bmatrix} V_{in}^{+}(-l_1) \\ V_{out}^{+}(-l_2) \end{Bmatrix}$$

S-parameters of the shifted network **$[S]^{SHIFT}$**

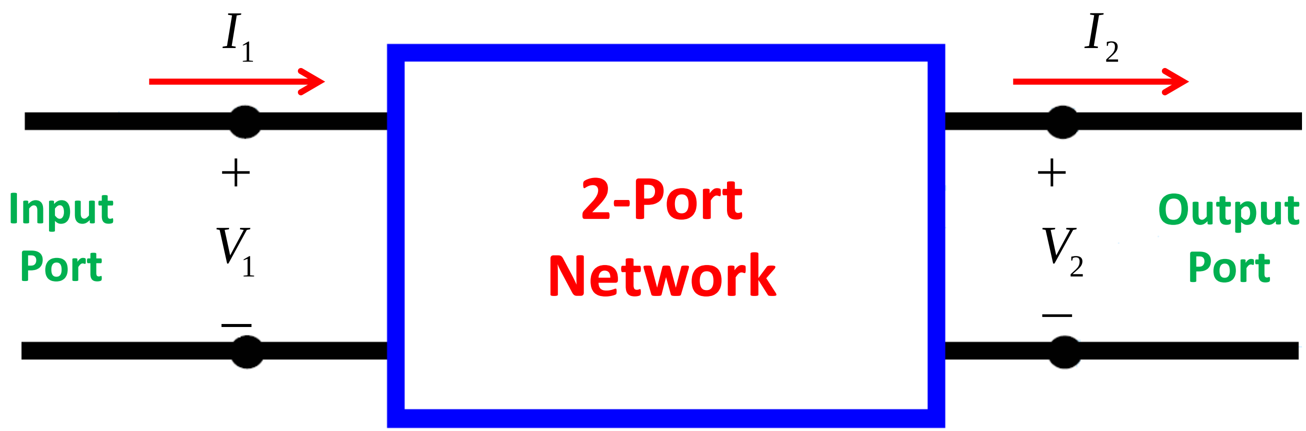
Physical Meaning

$$[S]^{SHIFT} = \begin{bmatrix} S_{11} e^{+j2\beta_1 l_1} & S_{12} e^{+j(\beta_1 l_1 + \beta_2 l_2)} \\ S_{21} e^{+j(\beta_1 l_1 + \beta_2 l_2)} & S_{22} e^{+j2\beta_2 l_2} \end{bmatrix}$$

The first term **(S_{11})** reveals that we have to take into account twice the travel time for the incident voltage to reach port-1 and, upon reflection, return to the end of the TL segment. Similarly for **S_{22}** at port-2. The cross terms **(S_{12} and S_{21})** require additive phase shifts associated with TL segments at port-1 and port-2

The Transmission Matrix

- If a network has **two** ports, then we can **alternatively** define the voltages and currents at each port as:



For such a network, we can relate the input port parameters (I_1 and V_1) and output port parameters (I_2 and V_2) using transmission parameters also known as ABCD parameters

$$V_1 = AV_2 + BI_2$$
$$I_1 = CV_2 + DI_2$$

$$\begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix}$$

Transmission Matrix 'T'

The Transmission Matrix (contd.)

- Similar to the impedance and admittance matrices, we determine the elements of the transmission matrix using **shorts** and **opens**.

- Note when $I_2 = 0$ then: $V_1 = AV_2$ ⟹ $A = \frac{V_1}{V_2}$ **A is unitless (i.e., it is a coefficient)**

- Note when $V_2 = 0$ then: $V_1 = BI_2$ ⟹ $B = \frac{V_1}{I_2}$ **B has unit of impedance (i.e., Ohms)**

- Note when $I_2 = 0$ then: $I_1 = CV_2$ ⟹ $C = \frac{I_1}{V_2}$ **C has unit of admittance (i.e., mhos)**

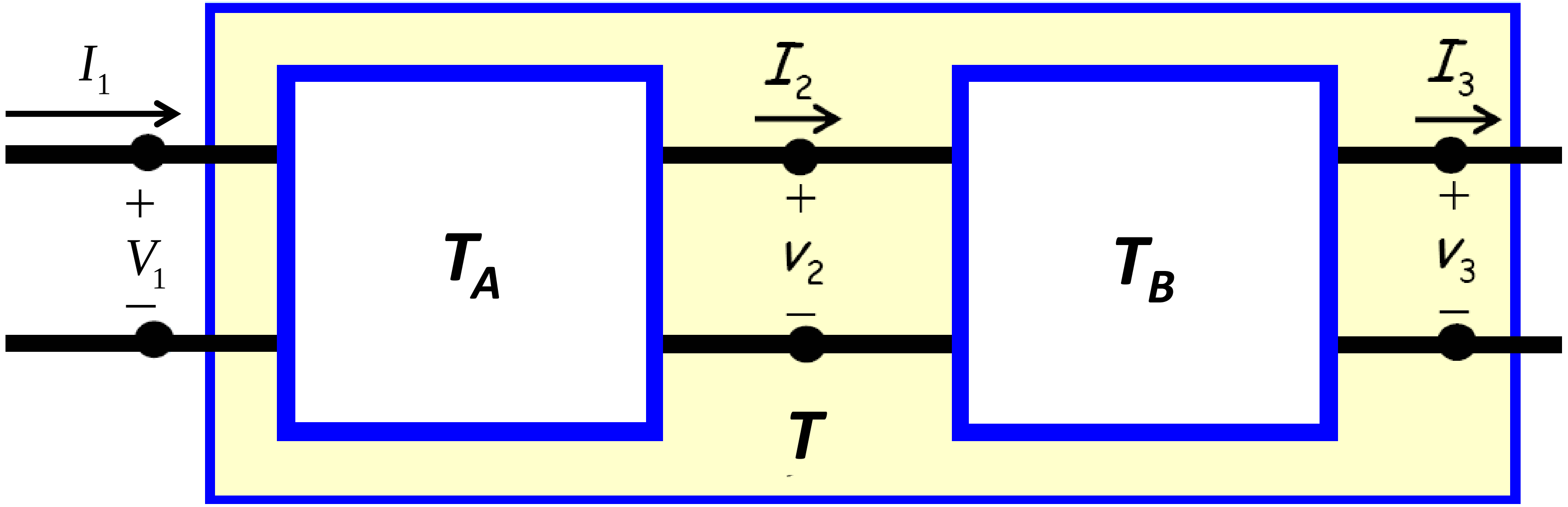
- Note when $V_2 = 0$ then: $I_1 = DI_2$ ⟹ $D = \frac{I_1}{I_2}$ **D is unitless (i.e., it is a coefficient)**

The Transmission Matrix (contd.)

Crying out loud! We already have the impedance matrix, the scattering matrix, **and** the admittance matrix. **Why** do we need the transmission matrix? Is it somehow **uniquely** useful?

The Transmission Matrix (contd.)

- Let us consider the case where a 2-port network is created by connecting (i.e., cascading) **two** networks:



$$\begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \boldsymbol{T}_A \begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix}$$

$$\begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix} = \boldsymbol{T}_B \begin{Bmatrix} V_3 \\ I_3 \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \boldsymbol{T} \begin{Bmatrix} V_3 \\ I_3 \end{Bmatrix}$$

The Transmission Matrix (contd.)

- Combining the first two equations we get: $\begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \boldsymbol{T}_A \begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix} = \boldsymbol{T}_A \boldsymbol{T}_B \begin{Bmatrix} V_3 \\ I_3 \end{Bmatrix}$

- Combining this combined relationship to the third we get:

$$\begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \boldsymbol{T}_A \begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix} = \boldsymbol{T}_A \boldsymbol{T}_B \begin{Bmatrix} V_3 \\ I_3 \end{Bmatrix} = \boldsymbol{T} \begin{Bmatrix} V_3 \\ I_3 \end{Bmatrix} \quad \Rightarrow \quad \boldsymbol{T} = \boldsymbol{T}_A \boldsymbol{T}_B$$

- Similarly, for N cascaded networks, the **total** transmission matrix **T** can be determined as the product of all N networks!

$$T = T_1 T_2 T_3 \ldots\ldots\ldots\ldots T_N = \prod_{n=1}^{N} T_n$$

- Note this result is **only** true for the **transmission** matrix **T**. **No** equivalent result exists for **S** ,**Z** ,**Y** !
- Thus, the transmission matrix can greatly simplify the analysis of complex networks constructed from two-port devices. We find that the T matrix is particularly useful when creating design software for CAD applications.