Lecture – 6

Date: 21.01.2016

- Smith Chart – Construction
- Smith Chart – Geography
- Smith Chart – Outer Scales

The Smith Chart (contd.)

- Let us revisit the generalized reflection coefficient formulation:

$$\Gamma(z) = |\Gamma_0| e^{j\theta_0} e^{j2\beta z} = \Gamma_r + j\Gamma_i$$

- Therefore, the normalized impedance can be formulated as:

$$z'(z) = r + jx = \frac{Z(z)}{Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)} = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

$$\Rightarrow \left((1-\Gamma_r) - j\Gamma_i\right)(r + jx) = (1+\Gamma_r) + j\Gamma_i$$

- The separation of real and imaginary part results in:

$$r(1-\Gamma_r) + x\Gamma_i = (1+\Gamma_r) \quad \text{Real}$$

$$x(1-\Gamma_r) - r\Gamma_i = \Gamma_i \quad \text{Imaginary}$$

- Simplification and then elimination of reactance (x) from these two give:

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}$$

Similar equation to circle of radius l, centered at (p, q):

$$(\Gamma_r - p)^2 + (\Gamma_i - q)^2 = l^2$$

The Smith Chart (contd.)

center: $\left(p,q\right)=\left(\frac{r}{1+r},0\right)$ **and** **radius:** $l=\frac{1}{1+r}$

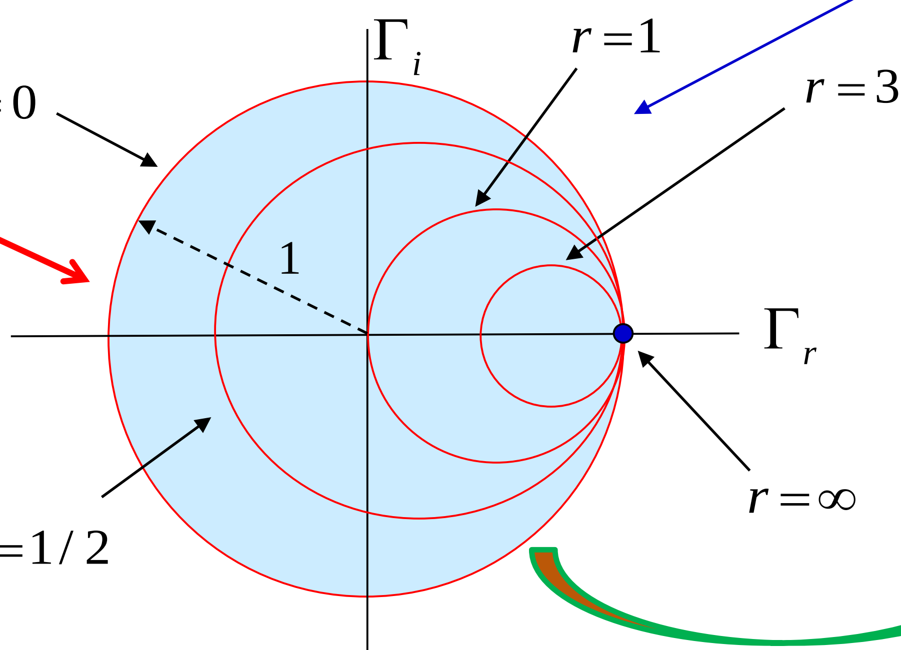
Observations:

- **For r =0:** $p^2 + q^2 = 1$; (p, q) = (0, 0) and l = 1
- **For r =1/2:** $(p - 1/3)^2 + q^2 = (2/3)^2$; (p, q) = (1/3, 0) and l = 2/3
- **For r =1:** $(p - 1/2)^2 + q^2 = (1/2)^2$; (p, q) = (1/2, 0) and l = 1/2
- **For r =3:** $(p – 3/4)^2 + q^2 = (1/4)^2$; (p, q) = (3/4, 0) and l = 1/4

Circles of distinct centre and radii

Note: $p+l=1$

Because of $(q – 0)^2$ term, all the constant resistance (r) circles have centers on this line



This approach enables mapping of any realizable vertical line (representing **r**) in the complex Γ-plane

The Smith Chart (contd.)

- For the mapping of horizontal lines of the normalized impedance plane to Γ-plane, let us simplify and eliminate resistance (*r*) from these:

$$r(1-\Gamma_r)+x\Gamma_i=(1+\Gamma_r) \quad \text{Real}$$

$$x(1-\Gamma_r)-r\Gamma_i=\Gamma_i \quad \text{Imaginary}$$

$$(\Gamma_r-1)^2+\left(\Gamma_i-\frac{1}{x}\right)^2=\left(\frac{1}{x}\right)^2$$

center: $(p,q)=(1,1/x)$

radius: $l=\frac{1}{|x|}$

Note:

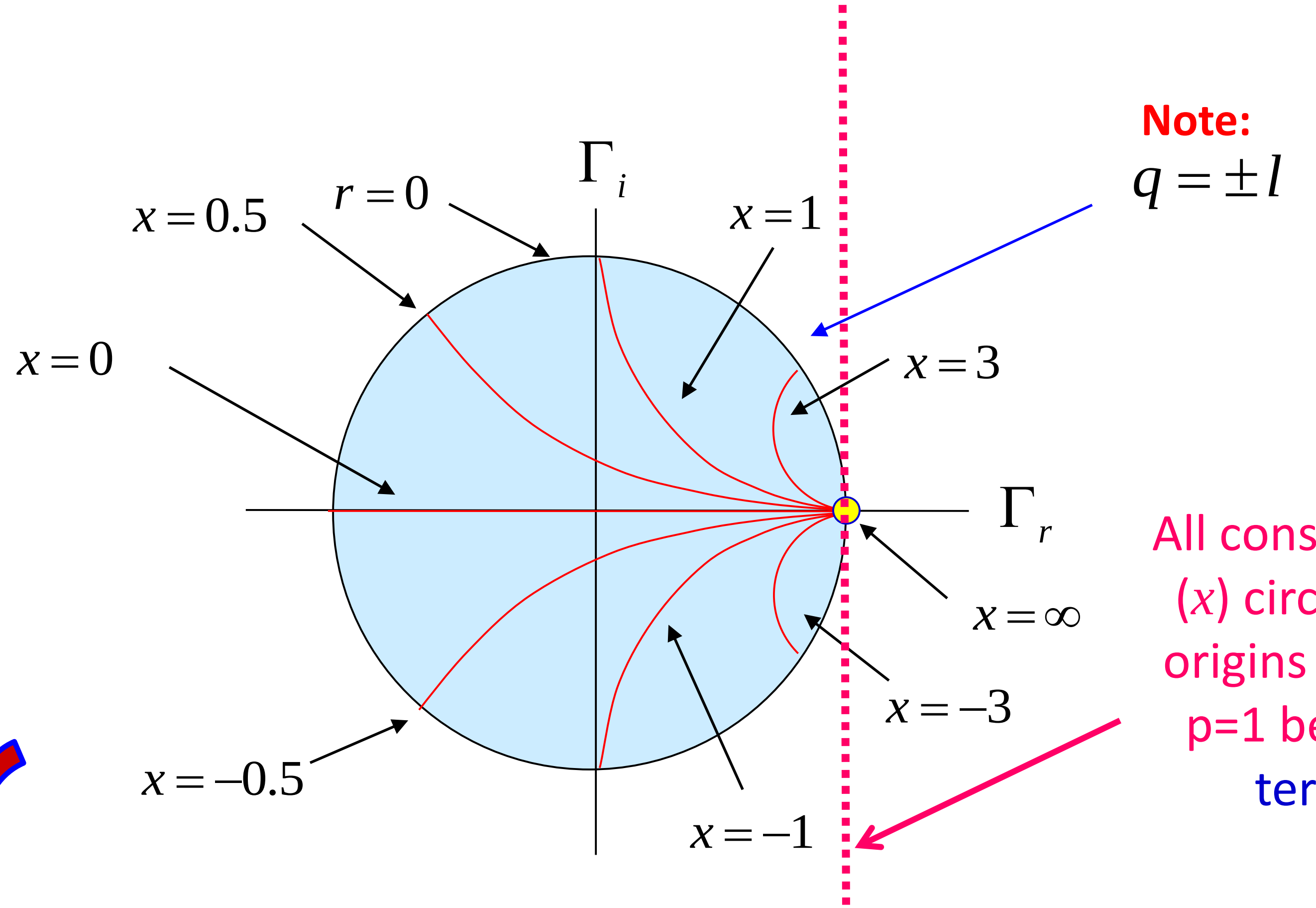
$q=\pm l$

Observations:

- **For x =1:** (p – 1)^2 + (q – 1)^2 = (1)^2; (p, q) = (1, 1) and l = 1
- **For x =-1:** (p – 1)^2 + (q + 1)^2 = (1)^2; (p, q) = (1, -1) and l = 1
- **For x =1/2:** (p – 1)^2 + (q – 2)^2 = (2)^2; (p, q) = (1, 2) and l = 2
- **For x =-1/2:** (p – 1)^2 + (q + 2)^2 = (2)^2; (p, q) = (1, -2) and l = 2

Circles of distinct centre and radii

The Smith Chart (contd.)



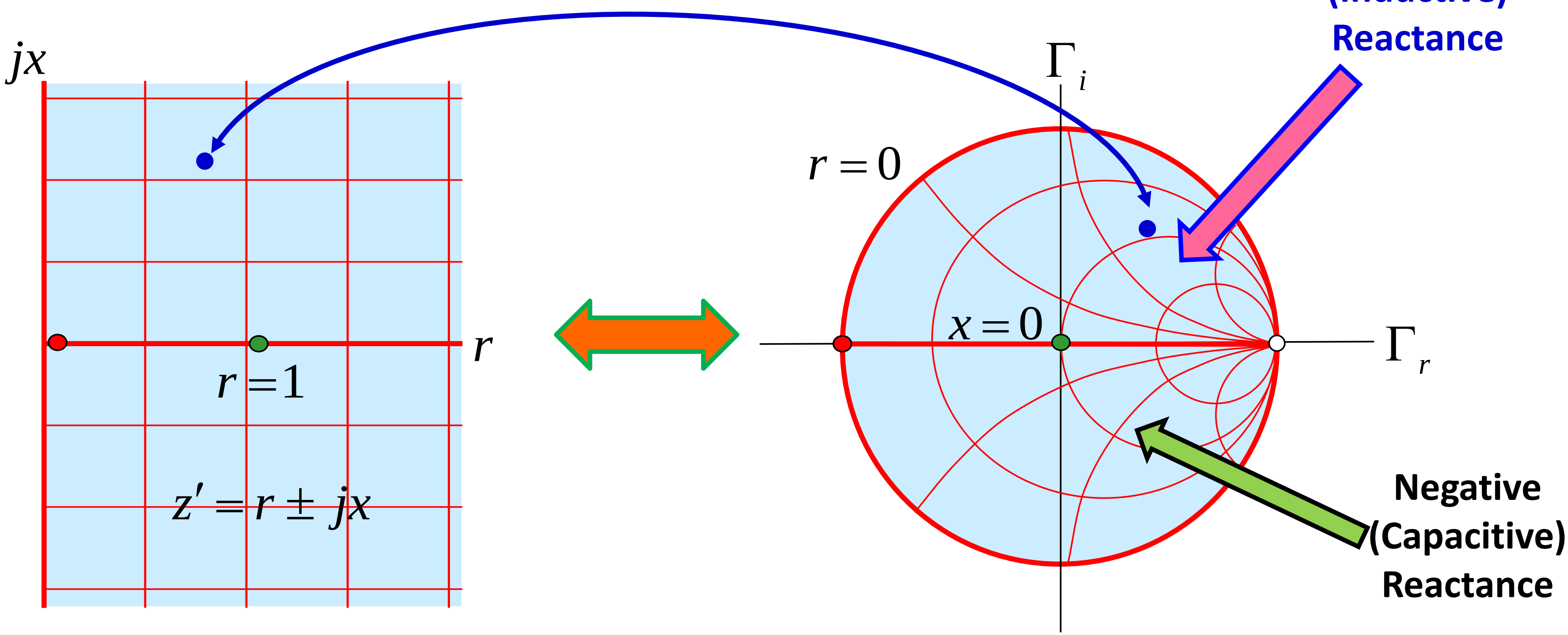
All constant reactance (x) circles have their origins along this line p=1 because of the term $(p - 1)^2$

This approach enables mapping of any realizable horizontal line (representing x) in the complex Γ-plane

The Smith Chart (contd.)

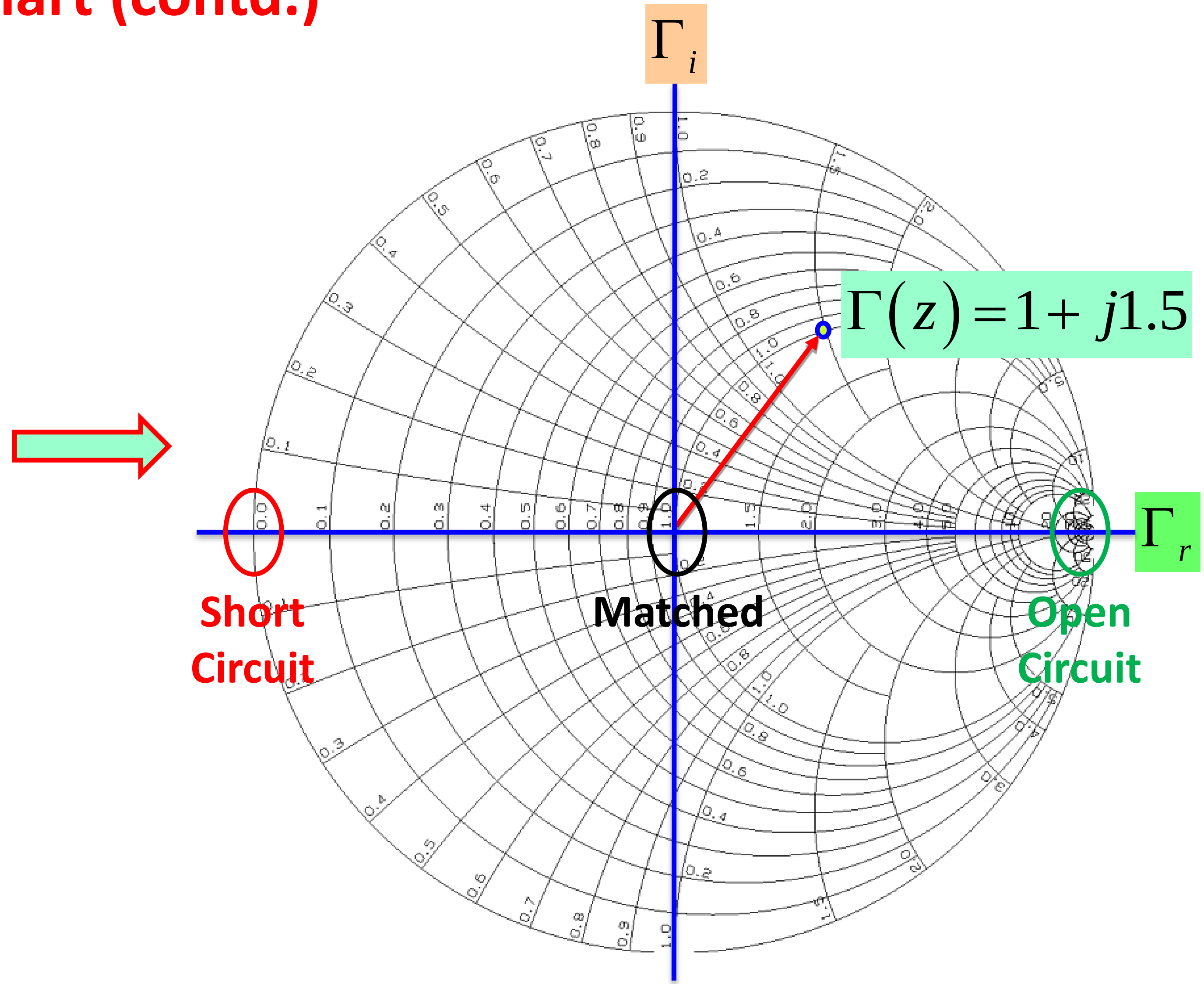
- Combination of these constant resistance and reactance circles define the mappings from normalized impedance (z') plane to Γ-plane and is called as Smith chart.

$$z'(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} \quad \Longleftrightarrow \quad \Gamma(z) = \frac{z'(z)-1}{z'(z)+1}$$



The Smith Chart (contd.)

Actual Smith chart



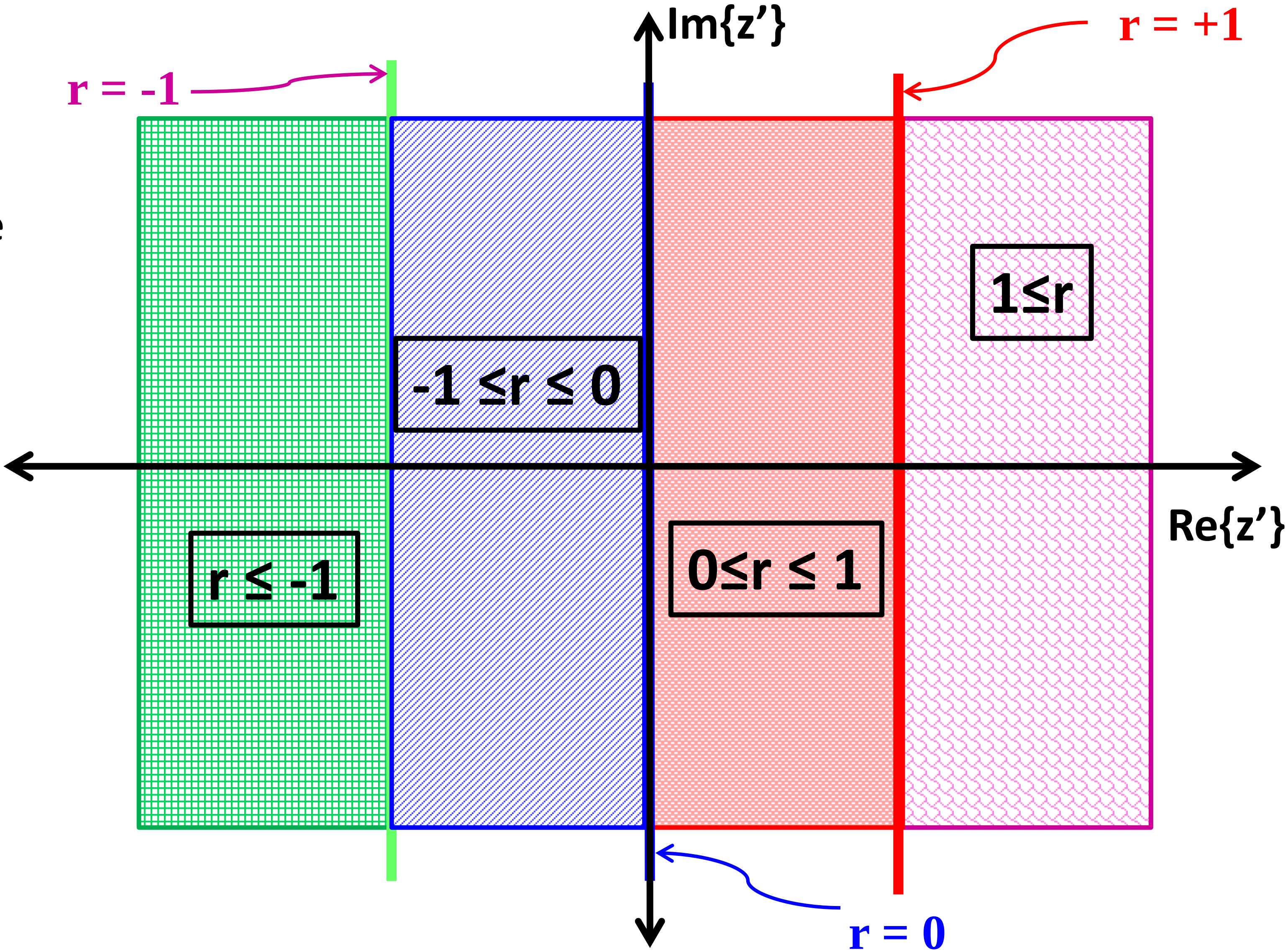
The Smith Chart (contd.) – Geography

- We have located specific **points** on the complex impedance plane, such as a **short circuit** or a **matched load**
- We've also identified **contours**, such as r =1 or x =1.5

We can likewise identify **whole regions (!)** of the complex **impedance** plane, providing a bit of a **geography lesson** of the complex impedance plane

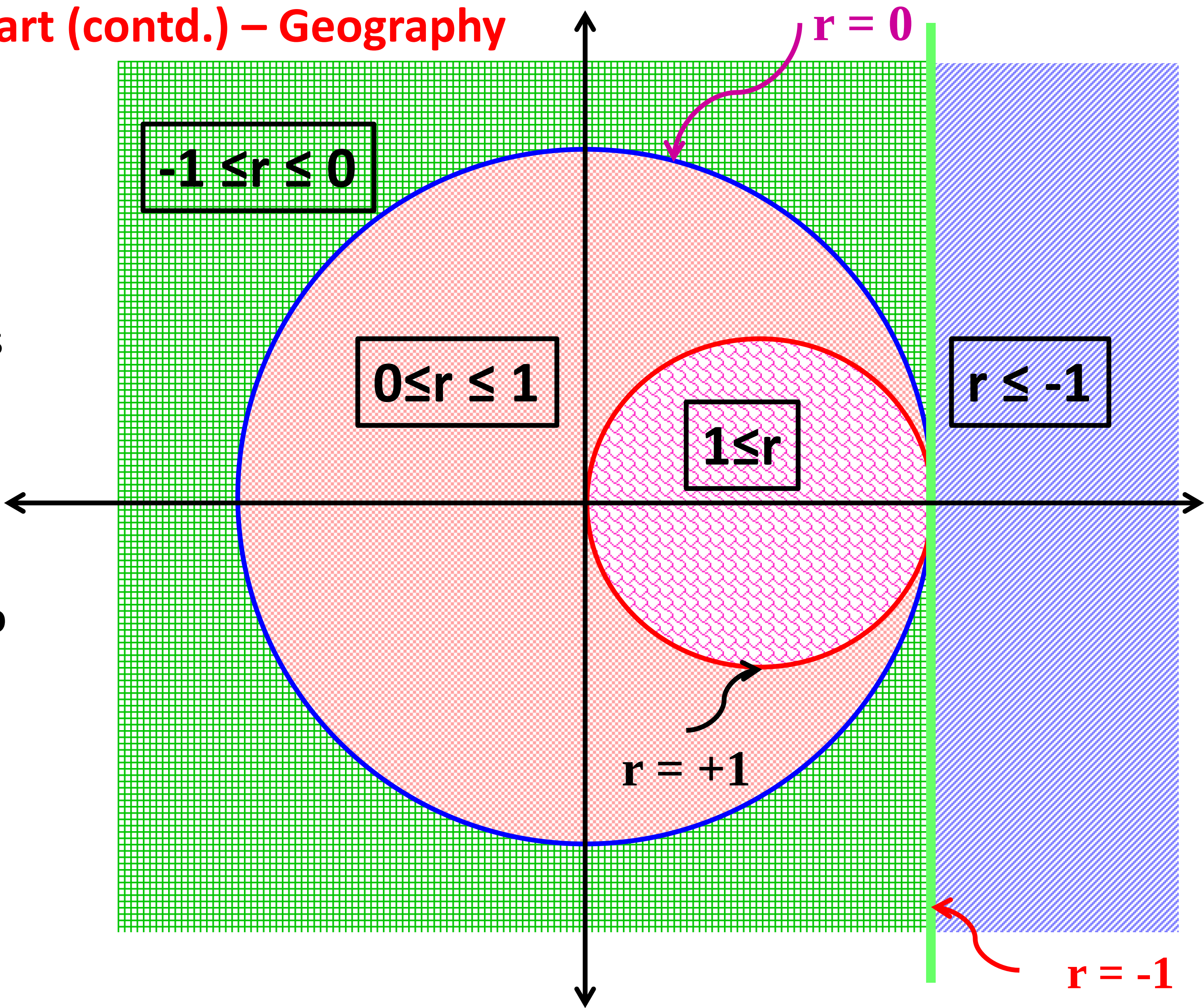
The Smith Chart (contd.) – Geography

For example, we can divide the complex impedance plane into **four** regions based on normalized **resistance** value **r**:



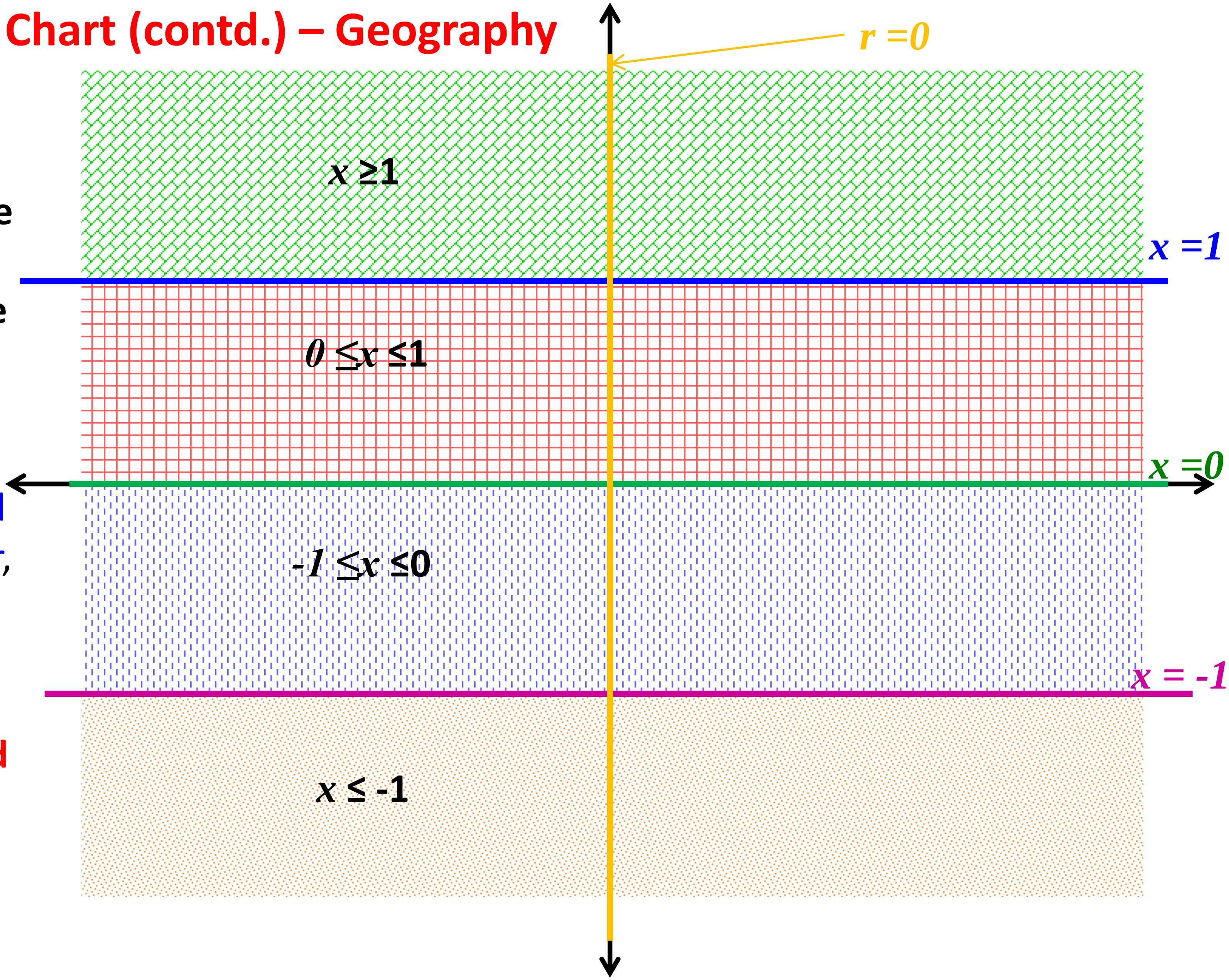
The Smith Chart (contd.) – Geography

Just like points and contours, these regions of the complex impedance plane can be **mapped onto the complex gamma plane!**



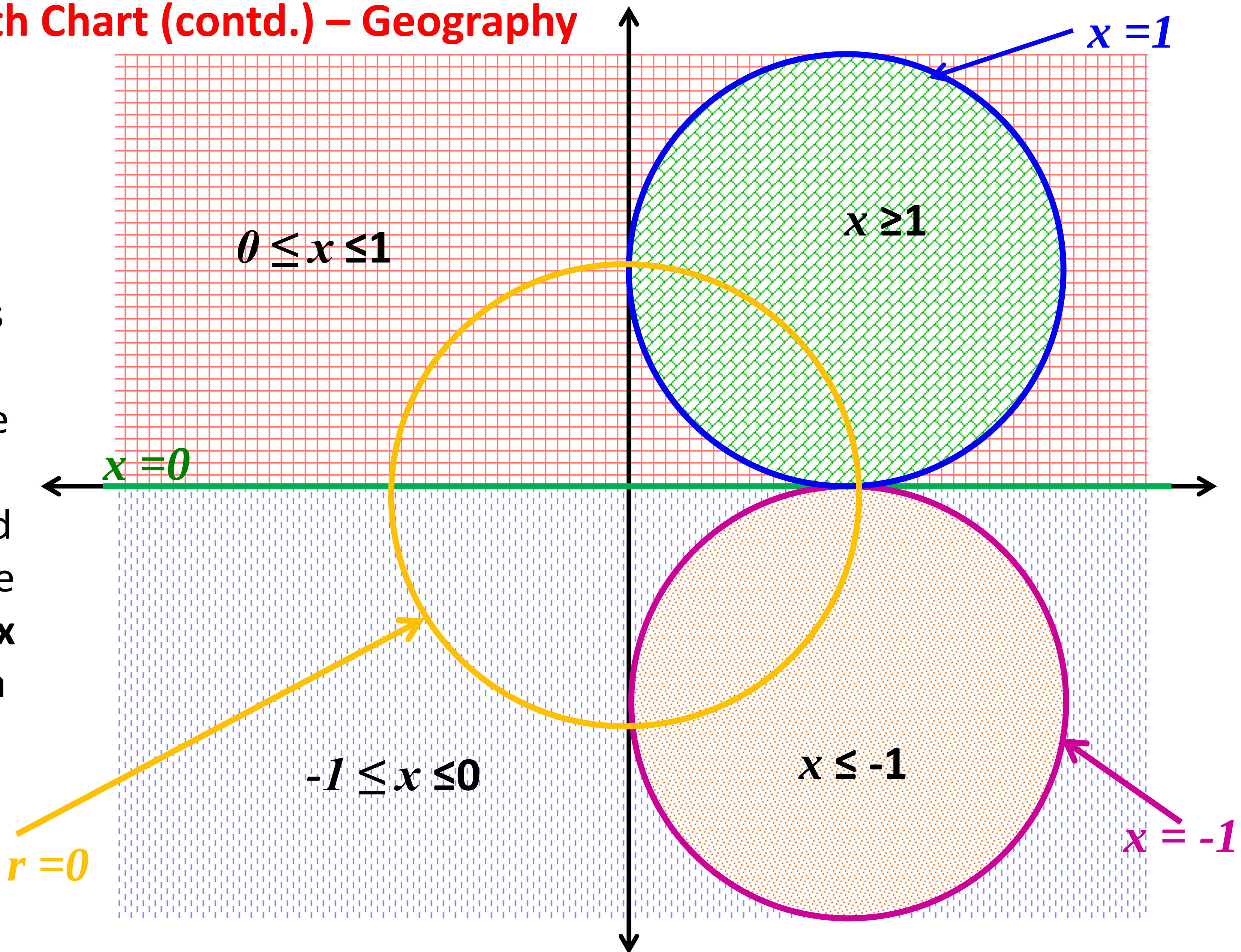
The Smith Chart (contd.) – Geography

Instead of dividing the complex impedance plane into regions based on normalized resistance r, we could divide it based on **normalized reactance x**:



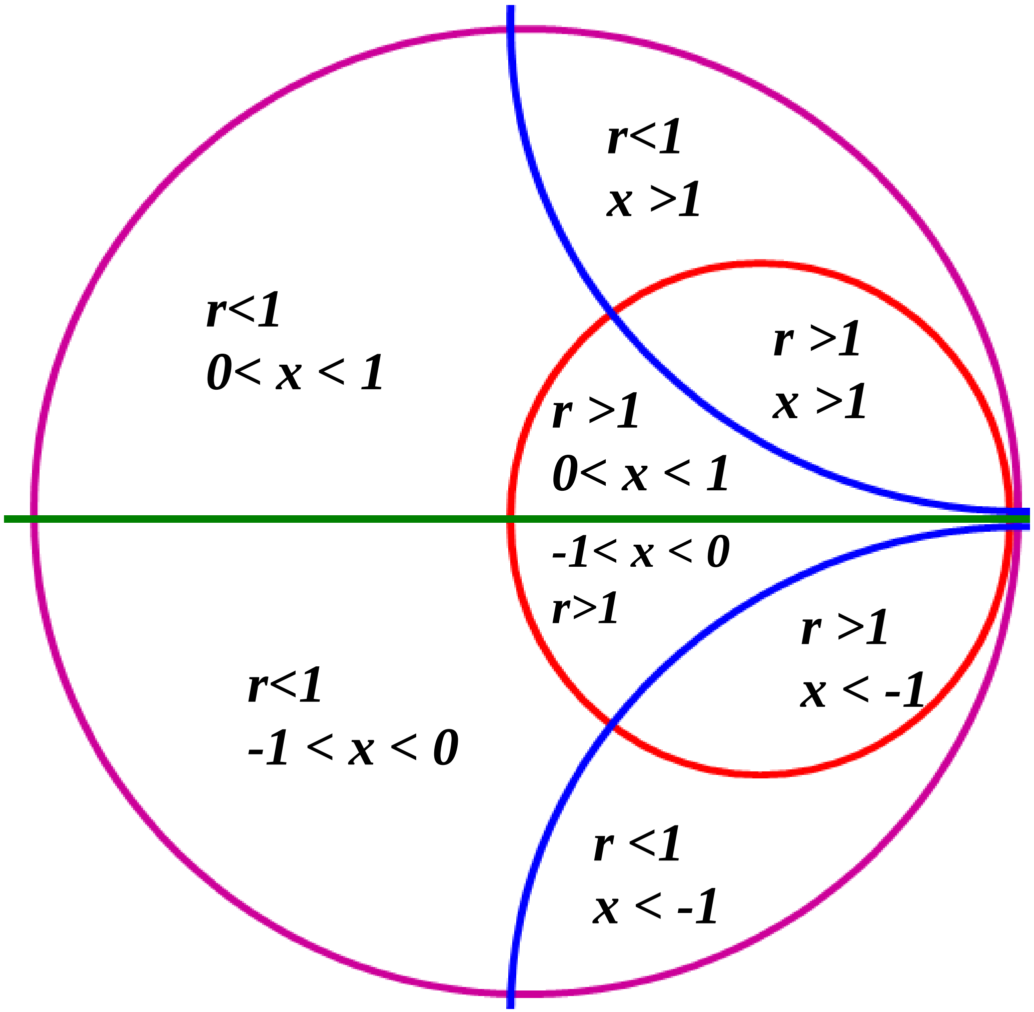
The Smith Chart (contd.) – Geography

These four regions can likewise be mapped onto the **complex gamma plane:**



The Smith Chart (contd.) – Geography

- the four resistance regions and the four reactance regions combine to from **16 separate regions on the complex impedance** and complex gamma planes!
- **Eight of these sixteen regions lie in the valid region (i.e.,** r > 0)
- Make sure **you can locate the eight impedance regions on a Smith Chart—this understanding of Smith Chart geography** will help you understand your design and analysis results!

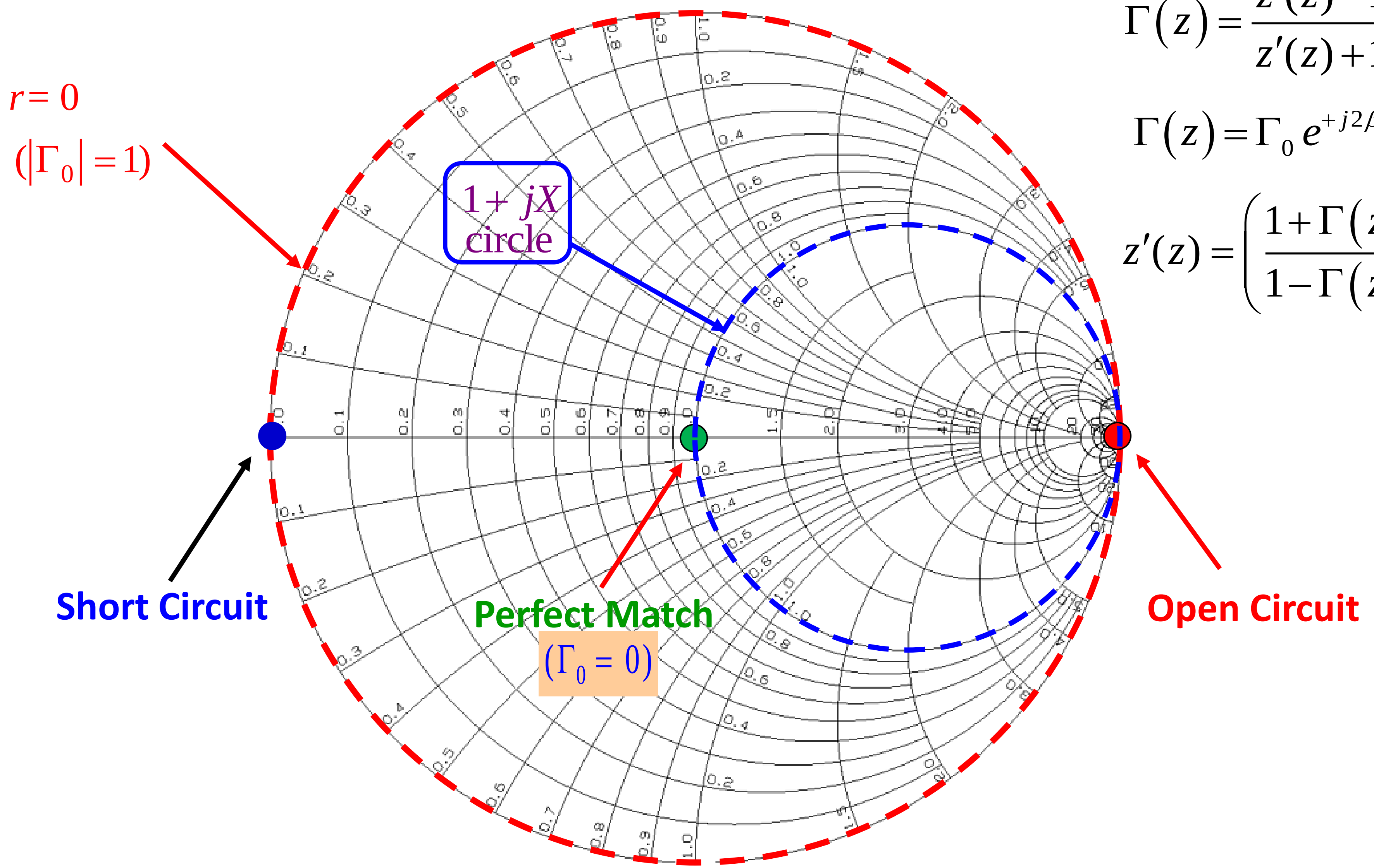


The Smith Chart – Important Points

$$\Gamma(z) = \frac{z'(z)-1}{z'(z)+1}$$

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

$$z'(z) = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$



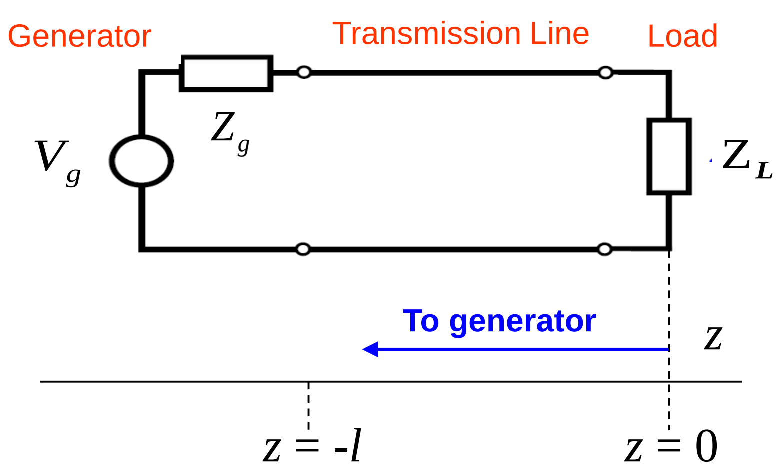
The Smith Chart (contd.)

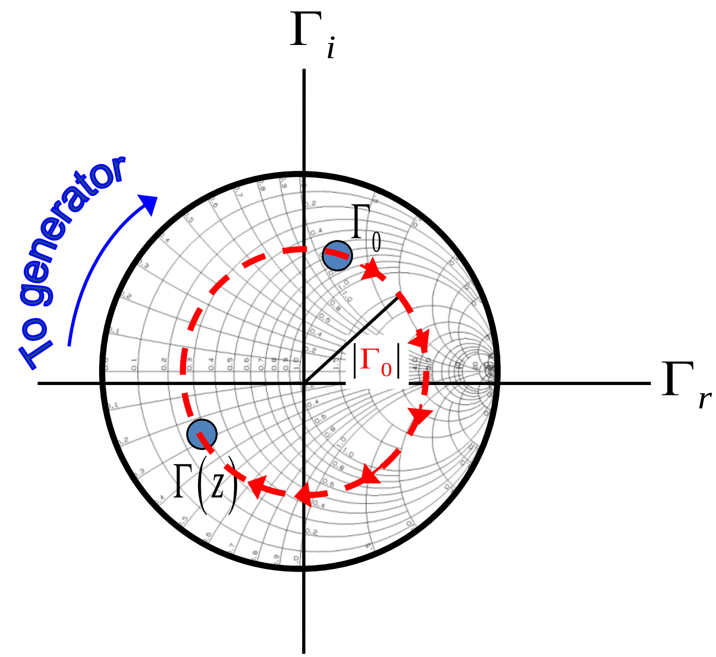
$$\boxed{z'(z) = \frac{1+\Gamma_0\, e^{+2j\beta z}}{1-\Gamma_0\, e^{+2j\beta z}}}$$

$$\Gamma(z) = \Gamma_0\, e^{+j2\beta z}$$

movement in negative z direction $\Leftrightarrow$ clockwise motion on circle of constant $|\Gamma_0|$

(toward generator)





angle change = $2\beta z$

The Smith Chart (contd.)

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

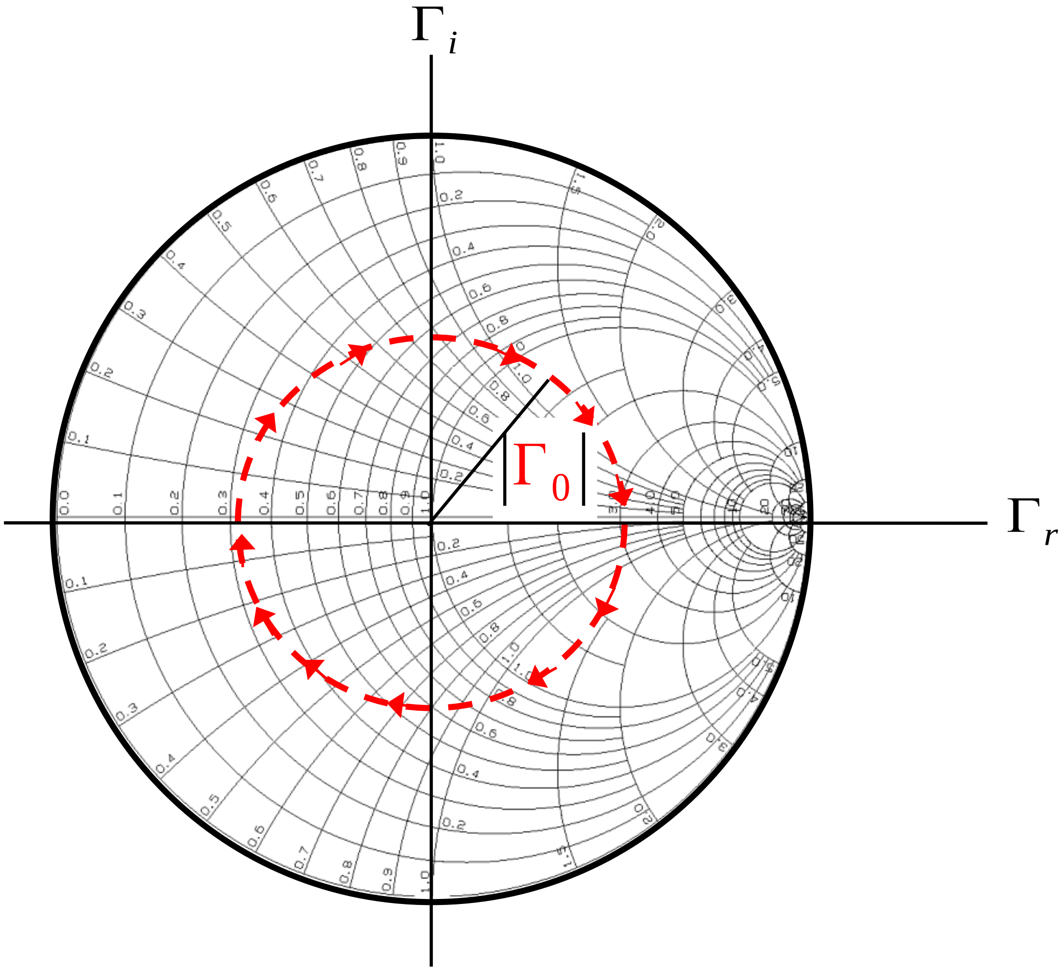
We go completely around the Smith chart when

$$z = \lambda / 2$$

$$2\beta z = 2\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{2}\right) = 2\pi$$

In general:

$$2\beta z = 2\left(\frac{2\pi}{\lambda}\right)(z) = 4\pi\left(\frac{z}{\lambda}\right)$$



Note: the Smith chart already has wavelength scales on the periphery for your convenience.

The Smith Chart (contd.)

Reciprocal Property

$$z'(z) = \left(\frac{1 + \Gamma_0 e^{+j2\beta z}}{1 - \Gamma_0 e^{+j2\beta z}} \right)$$

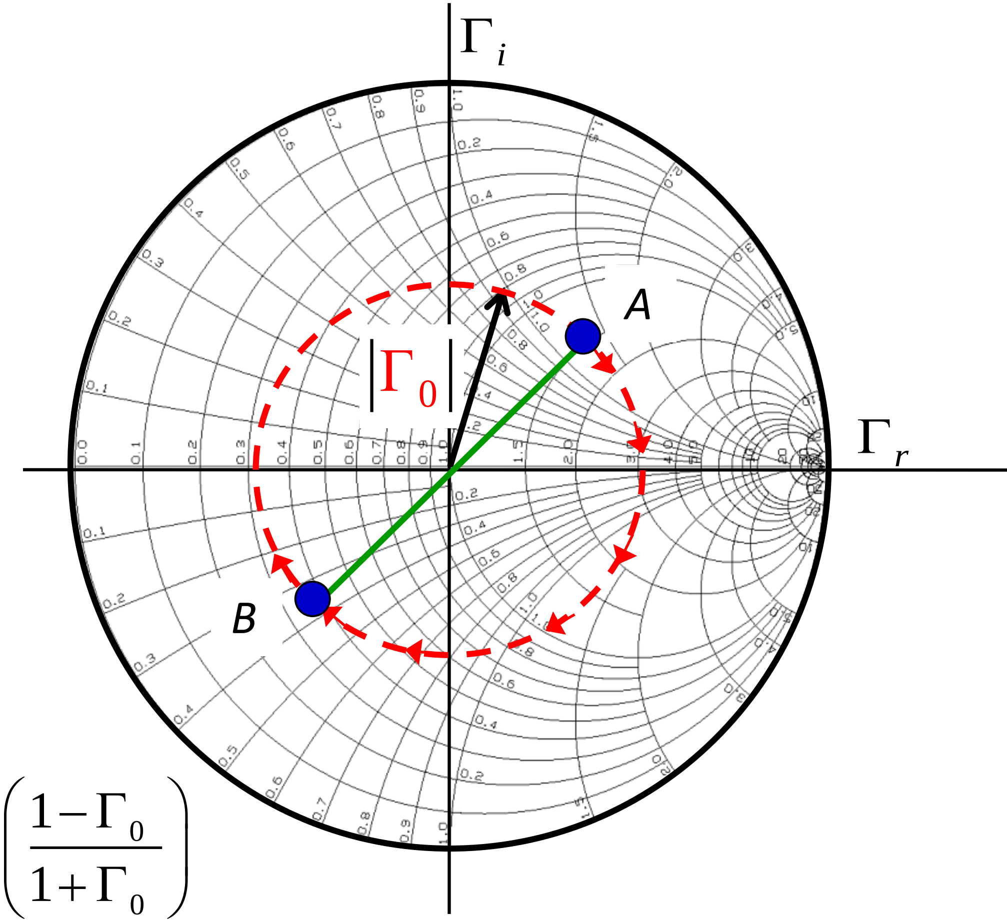
- Go half-way around the Smith chart:

$-l = \lambda / 4$

$$2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\left(-\frac{\lambda}{4}\right) = -\pi$$

$$z'(z=0) = \left(\frac{1+\Gamma_0}{1-\Gamma_0}\right) \qquad z'(z=-l) = \left(\frac{1-\Gamma_0}{1+\Gamma_0}\right)$$

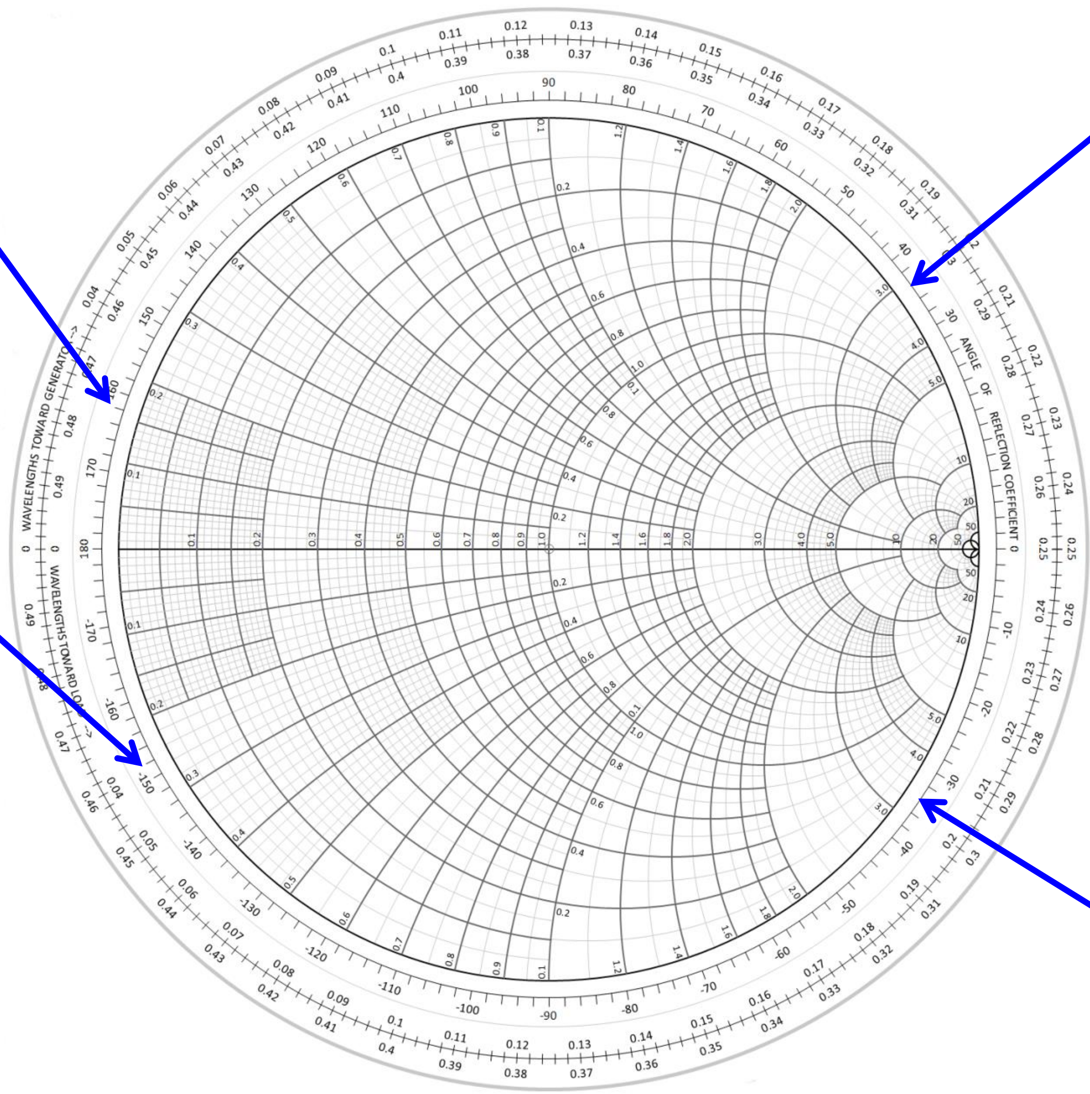
$$z'(z=0) = \frac{1}{z'(z=-l)}$$

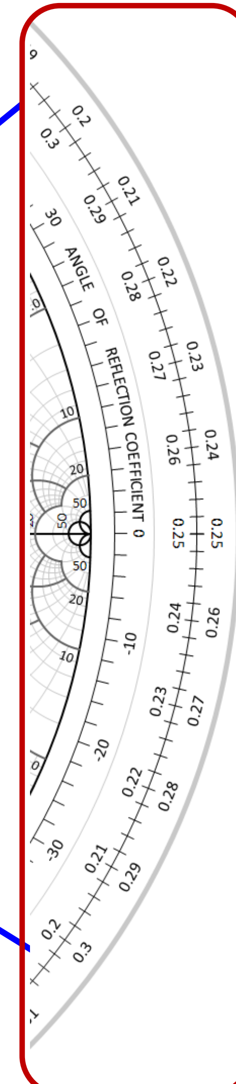


$$\boxed{z'(A) = \frac{1}{z'(B)}} \Rightarrow \boxed{z'(A) = y'(B)}$$

The Smith Chart – Outer Scale

Note that around the **outside** of the Smith Chart there is a scale indicating the **phase angle**, from 180^o to -180^o.





The Smith Chart – Outer Scale (contd.)

- Recall however, for a **terminated** transmission line, the reflection coefficient function is:

$$\Gamma(z) = \Gamma_0 \, e^{+j2\beta z} = |\Gamma_0| e^{+j(2\beta z + \theta_0)}$$

- Thus, the **phase** of the reflection coefficient function depends on transmission line **position z** as:

$$\theta_\Gamma(z) = 2\beta z + \theta_0 = 2\left(\frac{2\pi}{\lambda}\right) z + \theta_0 = 4\pi\left(\frac{z}{\lambda}\right) + \theta_0$$

- As a result, a **change** in line position z (i.e., Δz) results in a **change** in reflection coefficient phase θ_Γ (i.e., $\Delta\theta_\Gamma$):

$$\Delta\theta_\Gamma = 4\pi\left(\frac{\Delta z}{\lambda}\right)$$

- E.g., a change of position equal to one-quarter wavelength **Δz =λ/4** results in a phase change of π radians—we rotate **half-way** around the complex Γ-plane (otherwise known as the Smith Chart).

The Smith Chart – Outer Scale (contd.)

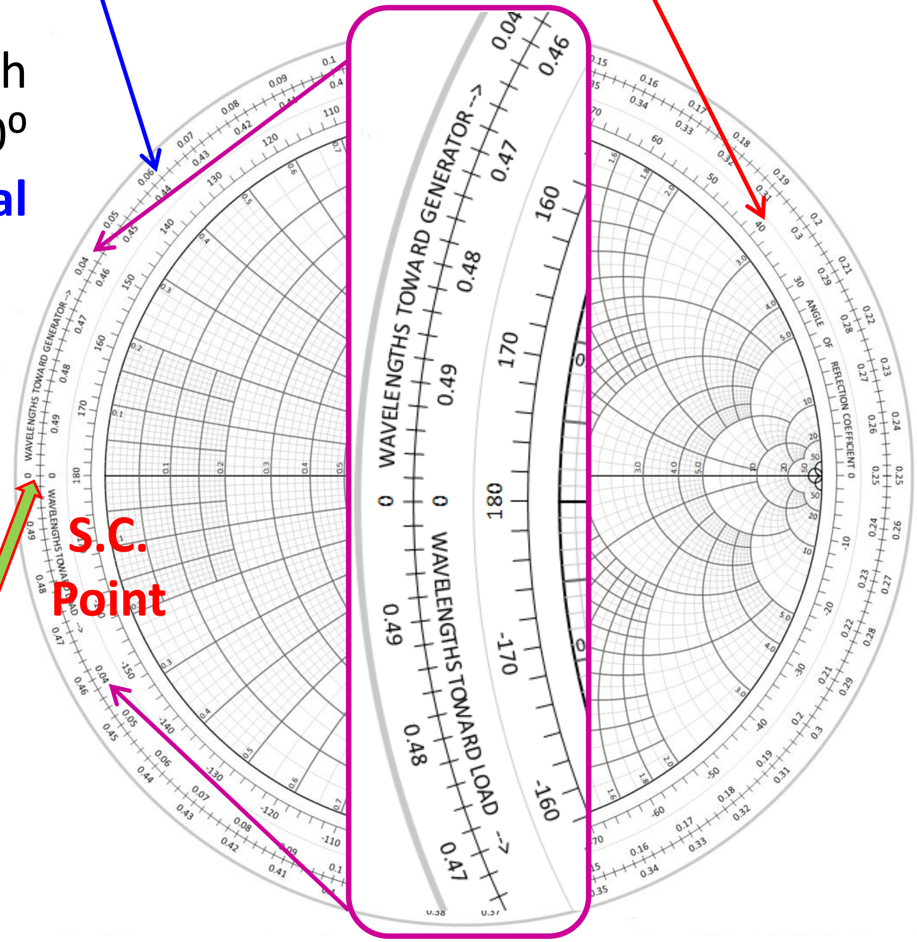
- The Smith Chart then has a **second scale** (besides θ_Γ) that surrounds it—one that relates TL position in **wavelengths** ($\Delta z/\lambda$) to the θ_Γ:

- Since the phase scale on the Smith Chart extends from $-180^o < \theta_\Gamma < 180^o$ (i.e., $-\pi < \theta_\Gamma < \pi$), this **electrical length scale** extends from:

$$\boxed{0 < z/\lambda < 0.5}$$

- Note, for this mapping the reflection coefficient phase at location z = 0 is $\theta_\Gamma = -\pi$. Therefore, $\theta_0 = -\pi$, and we find that:

$$\boxed{\Gamma_0 = |\Gamma_0| e^{+j\theta_0} = |\Gamma_0| e^{-j\pi} = -|\Gamma_0|}$$



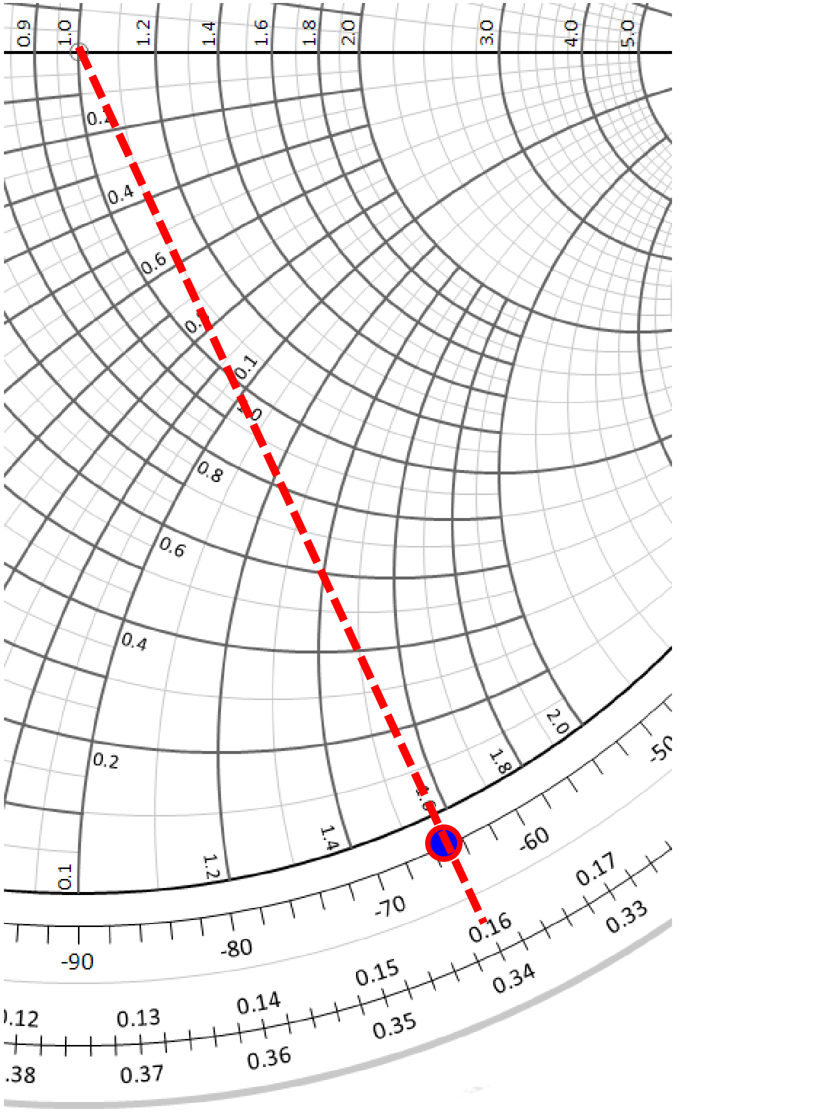
The Smith Chart – Outer Scale (contd.)

- **Example:** say you're at some location z = z_1 along a TL. The value of the **reflection coefficient** at that point happens to be:

$$\Gamma(z=z_1)=0.685e^{-j65^\circ}$$

- Finding the **phase angle** of θ_Γ = -65° on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

$$0.160\lambda$$



Note: this tells us **nothing** about the location z = z_1. This does **not** mean that z_1 =0.160λ , for example!

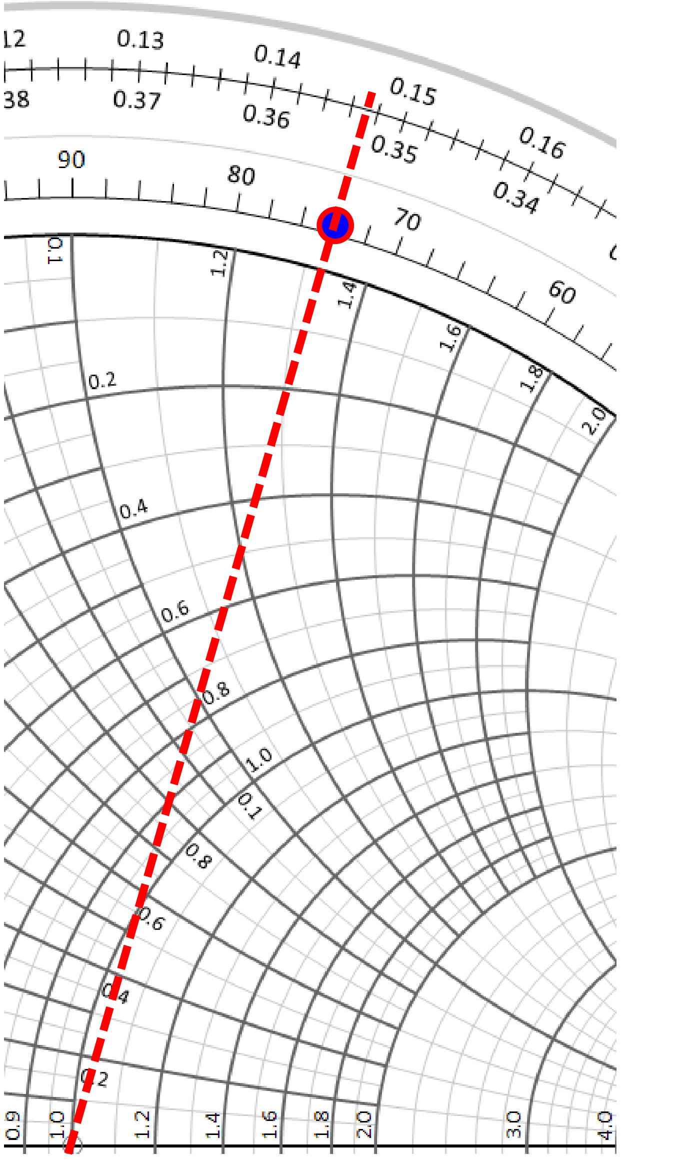
The Smith Chart – Outer Scale (contd.)

- Now, say we **move a short distance** Δz (i.e., a distance less than $\lambda/2$) along the transmission line, to a **new location** denoted as $z = z_2$ and find that the **reflection coefficient** has a value of:

$$\Gamma(z = z_2) = 0.685 e^{j74^\circ}$$

- Now finding the **phase angle** of $\theta_\Gamma = 74^o$ on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

$$0.353\lambda$$



Note: this tells us **nothing** about the location $z = z_2$. This does **not** mean that $z_1 = 0.353\lambda$, for example!

The Smith Chart – Outer Scale (contd.)

Q: So what do the values 0.160λ and 0.353λ tell us?

A: They allow us to determine the **distance between** points z_2 and z_1 on the transmission line.

$$\boxed{\Delta z = z_2 - z_1 = 0.353\lambda - 0.160\lambda = 0.193\lambda}$$

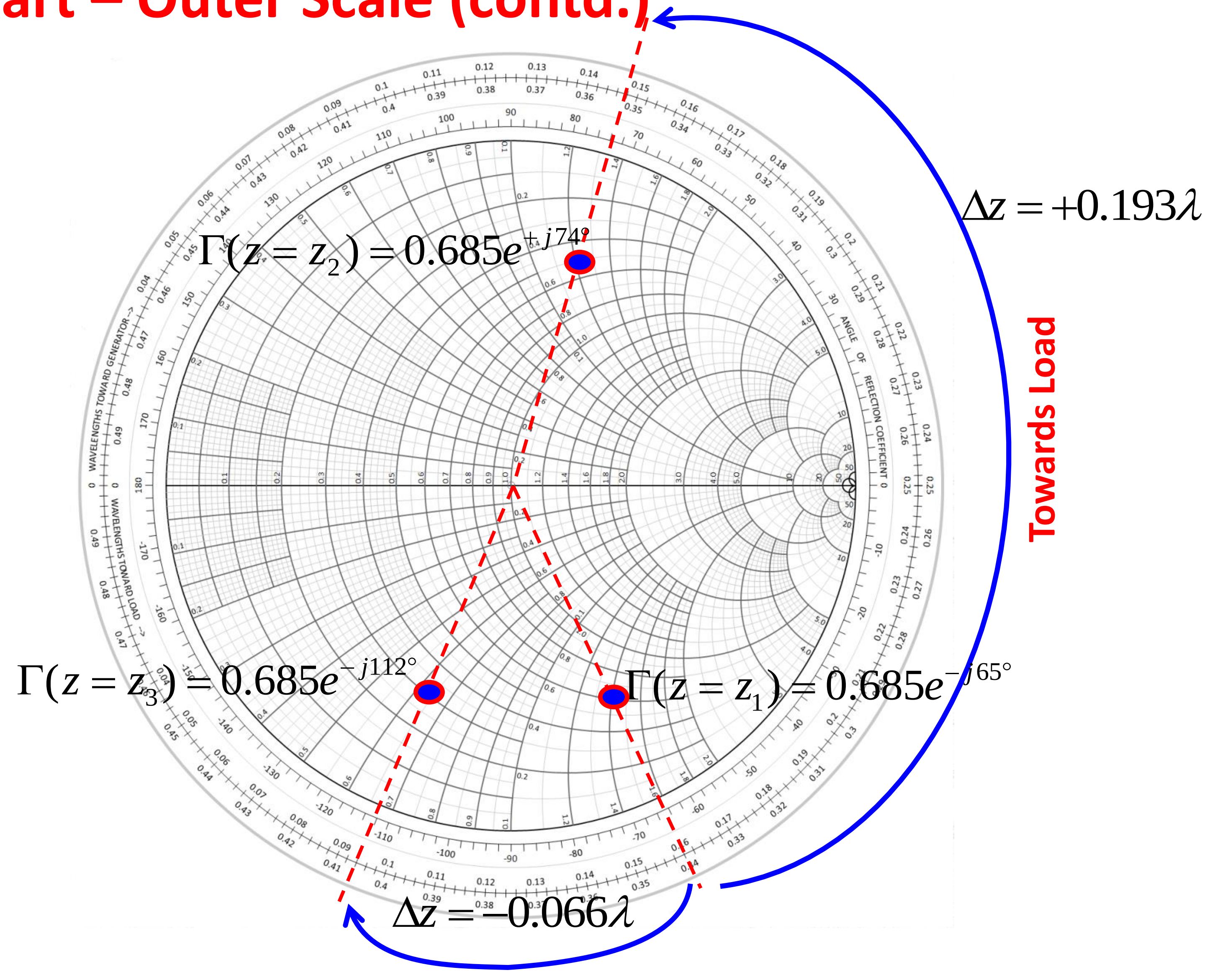
The transmission line location z_2 is a distance of 0.193λ from location z_1!

Q: But, say the reflection coefficient at some point z_3 has a phase value of $\theta_\Gamma = -112^o$, which maps to a value of 0.094λ on the outer scale of Smith chart. It gives $\Delta z = z_3 - z_1 = 0.094\lambda - 0.160\lambda = -0.066\lambda$. What does the **–ve** value mean?

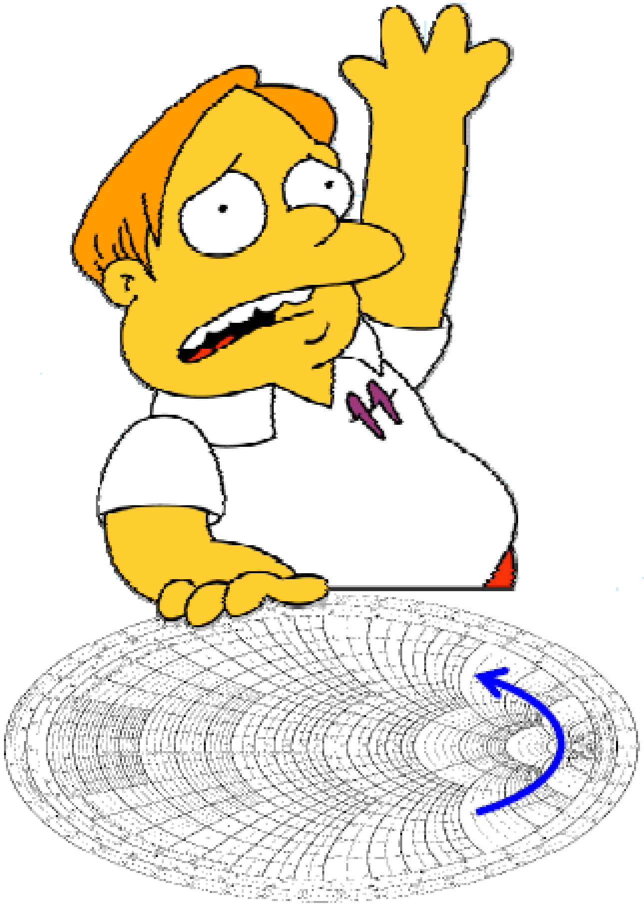
The Smith Chart – Outer Scale (contd.)

- **In the first example,** $\Delta z > 0$, meaning $z_2 > z_1$ → the location z_2 is closer to the load than is location z_1
  - the **positive** value Δz maps to a phase change of $74^o - (-65^o) = 139^o$
  - In other words, as we **move toward the load** from location z_1 to location z_2, we rotate **counter-clockwise** around the Smith chart
- **In the second example,** $\Delta z < 0$, meaning $z_3 < z_1$ → the location z_3 is closer to the beginning of the TL (i.e., farther from the load) than is location z_1
  - the **negative** value Δz maps to a phase change of $-112^o - (-65^o) = -47^o$
  - In other words, as we **move away from the load (i.e, towards the generator)** from location z_1 to location z_3, we rotate **clockwise** around the Smith chart

The Smith Chart – Outer Scale (contd.)



The Smith Chart – Outer Scale (contd.)



Q: Wait! I just used a Smith Chart to analyze a TL problem in the manner you have just explained. At one point on my transmission line the phase of the reflection coefficient is $\theta_\Gamma = +170^o$, which is denoted as 0.486λ on the **"wavelengths toward load"** scale.

I then moved a short distance along the line **toward the load**, and found that the reflection coefficient phase was $\theta_\Gamma = -144^o$, which is denoted as 0.050λ on the **"wavelengths toward load"** scale.

According to **your** "instruction", the distance between these two points is:

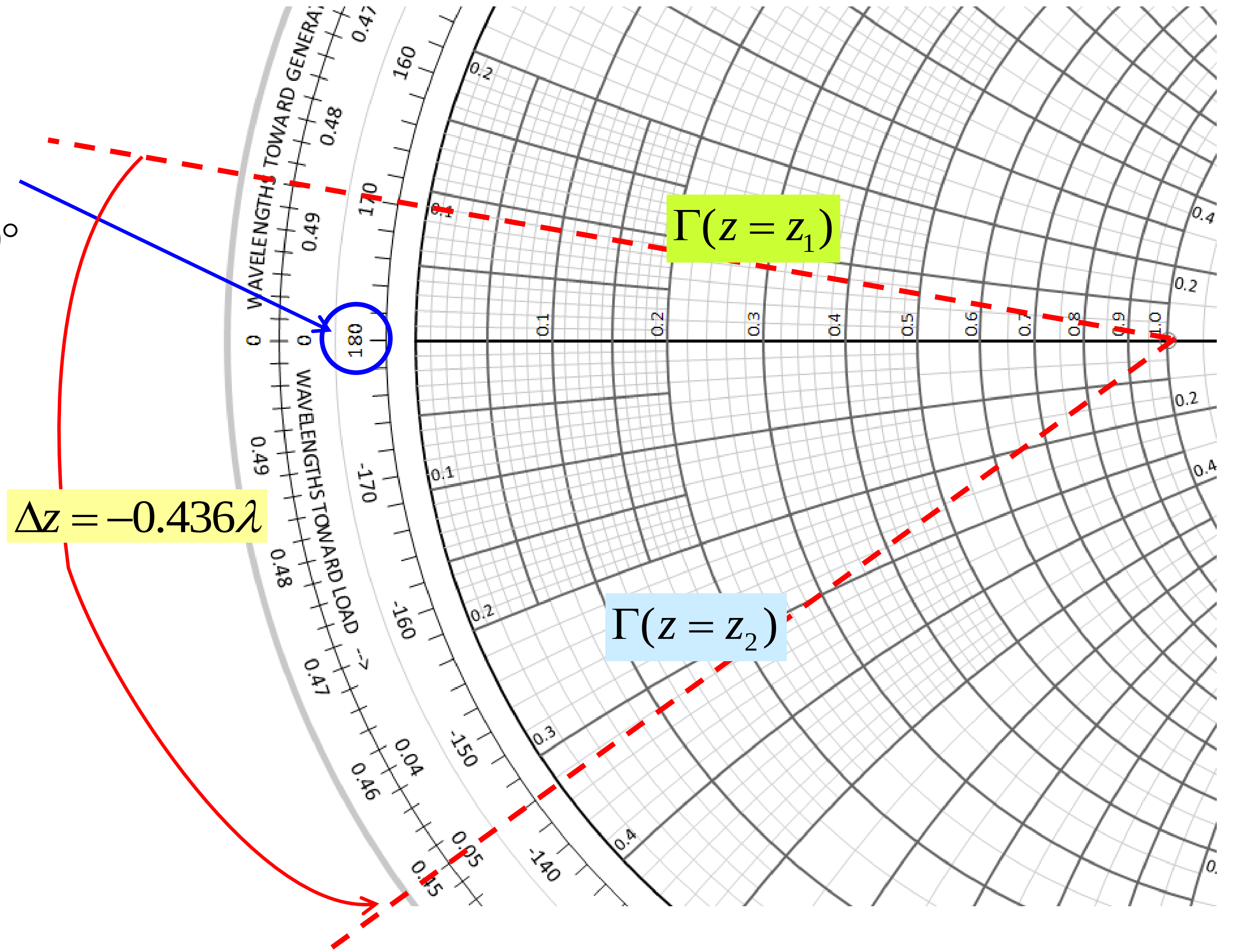
$$\boxed{\Delta z = 0.050\lambda - 0.486\lambda = -0.436\lambda}$$

A large **negative** value! This says that I moved nearly a half wavelength **away** from the load, but I know that I moved just a short distance **toward** the load! **What happened?**

The Smith Chart – Outer Scale (contd.)

The electrical length scales on the Smith chart begin and end where $\theta_\Gamma = \pm 180°$

In your example, when rotating counter-clockwise (i.e., moving toward the load) you **passed by this transition**. This makes the calculation of **Δz** a bit more problematic.



The Smith Chart – Outer Scale (contd.)

- As you rotate counter-clockwise around the Smith Chart, the "wavelengths toward load" scale increases in value, until it reaches a **maximum** value of 0.5λ (at $\theta_\Gamma = \pm\pi$)
- At that point, the scale "resets" to its **minimum** value of **zero**
- Thus, in such a situation, we must divide the problem into **two steps**:
- **Step 1:** Determine the electrical length from the **initial** point to the **"end"** of the scale at 0.5λ
- **Step 2:** Determine the electrical distance from the **"beginning"** of the scale (i.e., 0) and the **second location** on the transmission line
- **Add** the results of steps 1 and 2, and you have your answer!

For example, let's look at the case that originally gave us the erroneous result. The distance from the initial location to the **end of the scale** is:

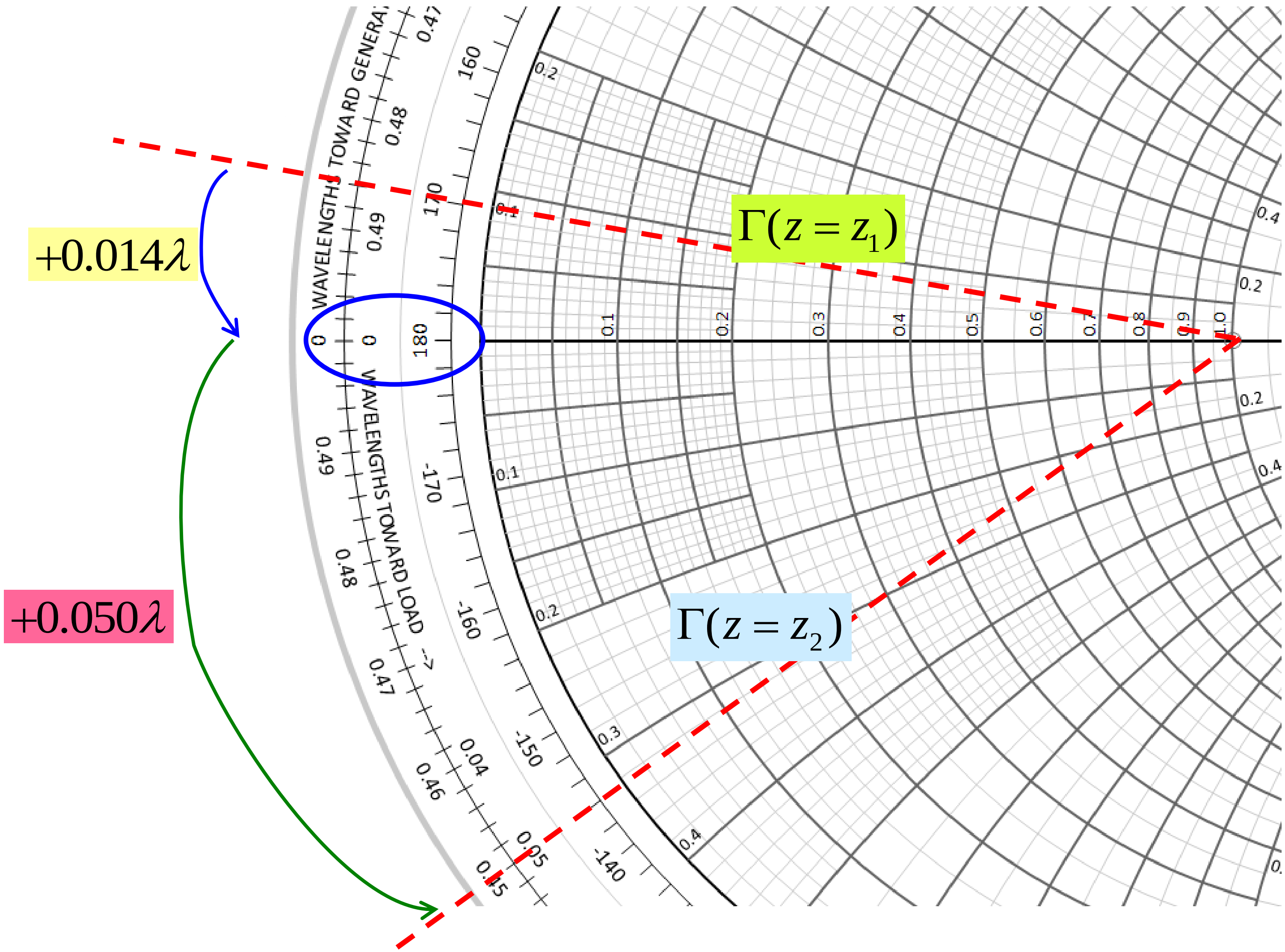
$$\boxed{0.500\lambda - 0.486\lambda = +0.014\lambda}$$

And the distance from the **beginning of the scale** to the second point is:

$$\boxed{0.050\lambda - 0.000\lambda = +0.050\lambda}$$

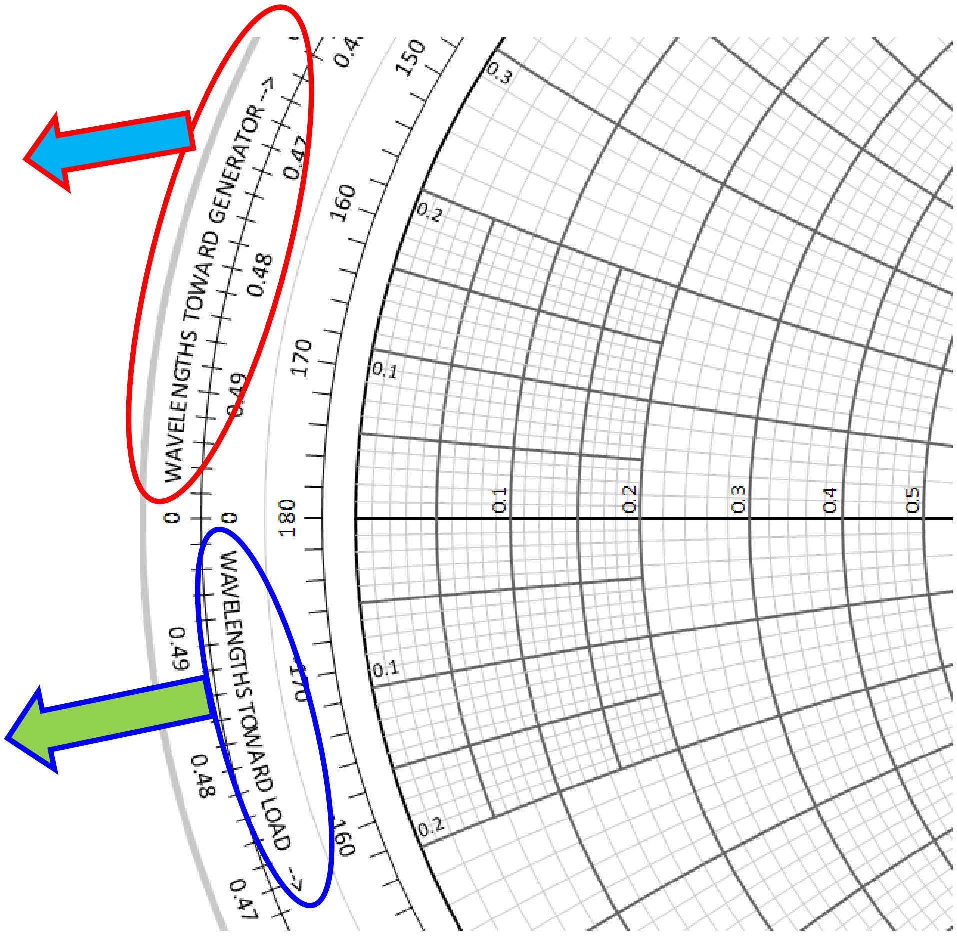
Thus the distance between the two points is: $+0.014\lambda + 0.050\lambda = +0.064\lambda$

The Smith Chart – Outer Scale (contd.)



The Smith Chart – Outer Scale (contd.)

- The Δz towards generator could also be mentioned as a +ve term if we consider the upper metric in the "Outer Scale"



Clockwise Rotation

- **gives +ve distance when moving towards generator**
- **gives –ve distance when moving towards load**

Counter-clockwise Rotation

- **gives -ve distance when moving towards generator**
- **gives +ve distance when moving towards load**