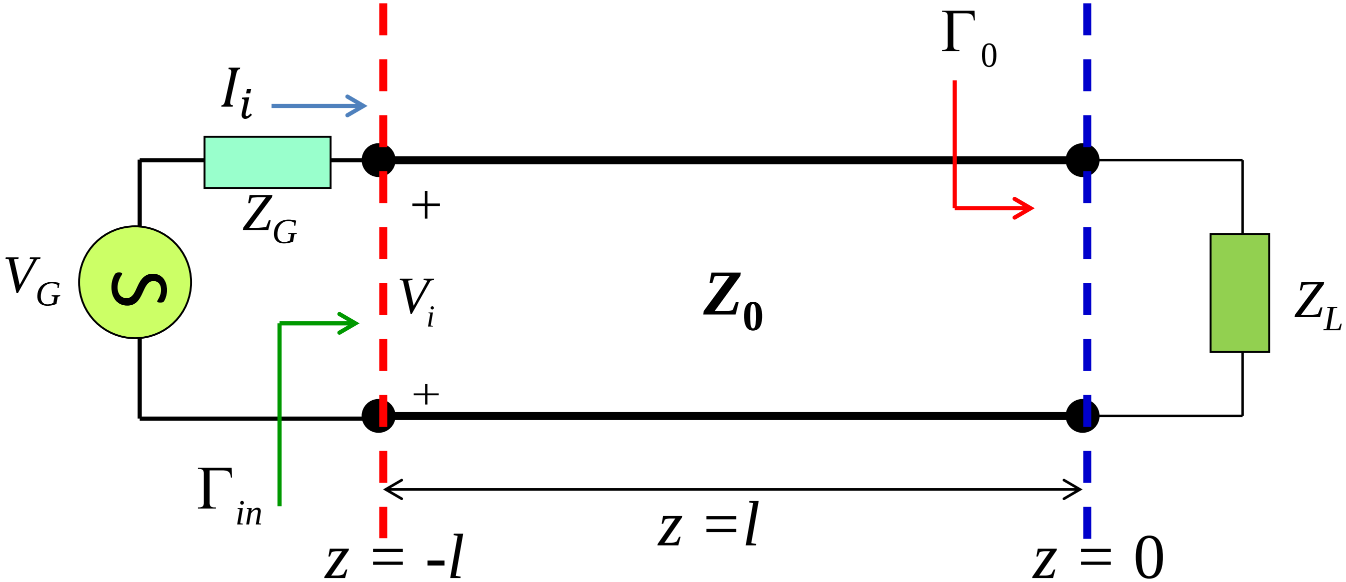
Lecture – 5

Date: 18.01.2016

- Sourced and Loaded TL
- Lossy TL

Sourced and Loaded Transmission Line

- Thus far, we have discussed a TL with terminated load impedance → Let us now consider a TL with terminated load impedance and a source at the input (with line-to-source mismatch)



- At $z = 0$

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- The current and voltage along the TL is:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_0 e^{+j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_0 e^{+j\beta z} \right]$$

V_0^+ depends on the signal **source**! To determine its exact **value**, we must now apply **boundary conditions** at $z = -l$.

Sourced and Loaded Transmission Line (contd.)

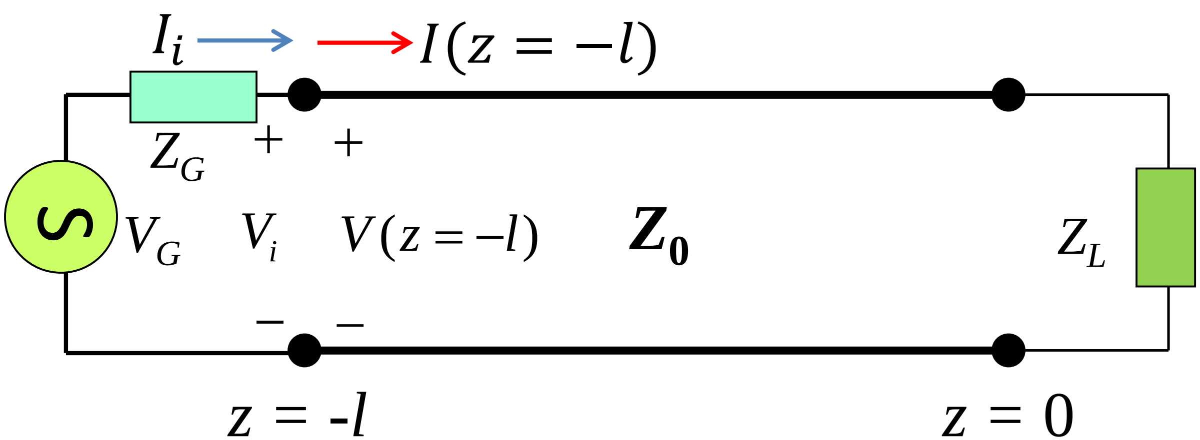
- At the **beginning** of the transmission line:

$$V(z=-l) = V_0^+\left[e^{+j\beta l} + \Gamma_0 e^{-j\beta l}\right]$$

$$I(z=-l) = \frac{V_0^+}{Z_0}\left[e^{+j\beta l} - \Gamma_0 e^{-j\beta l}\right]$$

- Likewise, we know that the **source** must satisfy:

$$V_G = V_i + Z_G I_i$$



- From **KVL** we find:

$$V_i = V(z=-l)$$

- From **KCL** we find:

$$I_i = I(z=-l)$$

- **Combining** these equations, we find:

$$V_G = V_0^+\left[e^{+j\beta l} + \Gamma_0 e^{-j\beta l}\right] + Z_G \frac{V_0^+}{Z_0}\left[e^{+j\beta l} - \Gamma_0 e^{-j\beta l}\right]$$

One equation → **one** unknown (V_0^+)!!

Sourced and Loaded Transmission Line (contd.)

- **Solving**, we find the value of V_0^+:

$$V_0^+ = V_G e^{-j\beta l} \frac{Z_0}{Z_0(1+\Gamma_{in}) + Z_G(1-\Gamma_{in})}$$

$$\Gamma_{in} = \Gamma(z=-l) = \Gamma_0 e^{-j\beta l}$$

- Note this result looks different than the equation in your book (Pozar):

$$V_0^+ = V_G \frac{Z_0}{Z_0 + Z_G} \frac{e^{-j\beta l}}{(1-\Gamma_0\Gamma_G e^{-j2\beta l})}$$

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0}$$

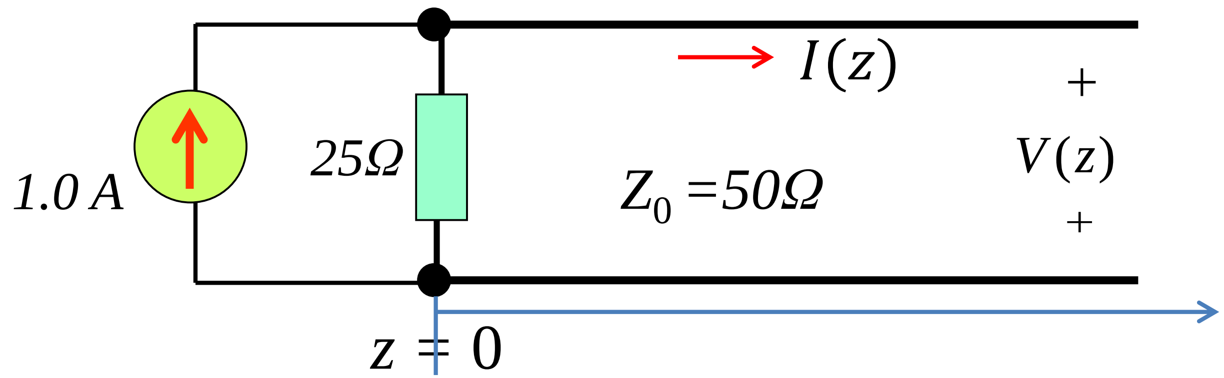
I like the first expression better.

Although the two equations are equivalent, **first** expression is explicitly written in terms of $\Gamma_{in} = \Gamma(z=-l)$ (a very **useful**, **precise**, and **unambiguous** value), while the book's expression is written in terms of this **so-called** "source reflection coefficient" Γ_G(a **misleading**, **confusing**, **ambiguous**, and mostly **useless** value).

Specifically, we might be **tempted** to equate Γ_G with the value $\Gamma_{in} = \Gamma(z=-l)$, but it is **not** $\Gamma_G \neq \Gamma(z=-l)$!

Example – 1

- Consider this circuit:



- It is known that the **current** along the transmission line is:

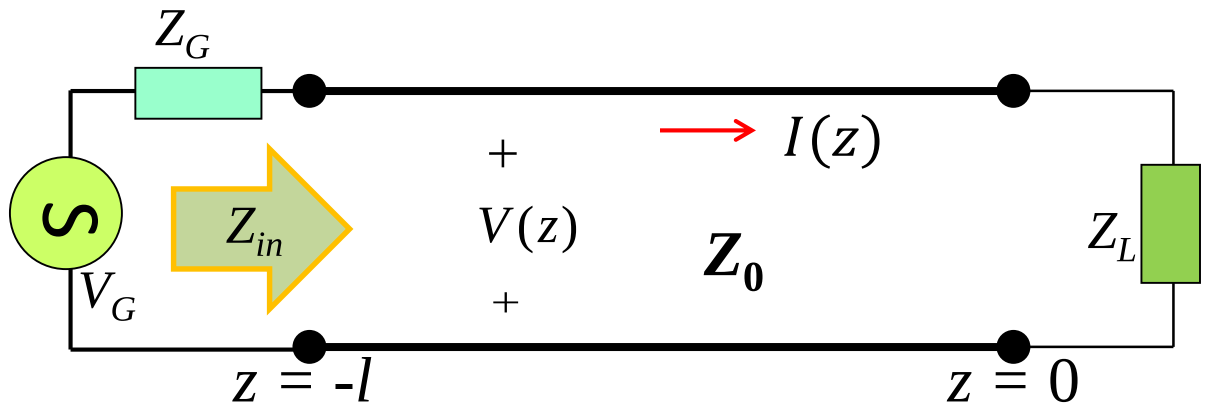
$$I(z) = 0.4e^{-j\beta z} - Be^{+j\beta z} \quad Amp \qquad for\ z > 0$$

where B is some unknown complex value.

Determine the value of B.

Sourced and Loaded Transmission Line (contd.)

Q: If the purpose of a transmission line is to transfer **power** from a source to a load, then exactly how much power is **delivered** to Z_L for this circuit??



A: We of course **could** determine V_0^+ and V_0^-, and then determine the power absorbed by the load (P_{abs}) as:

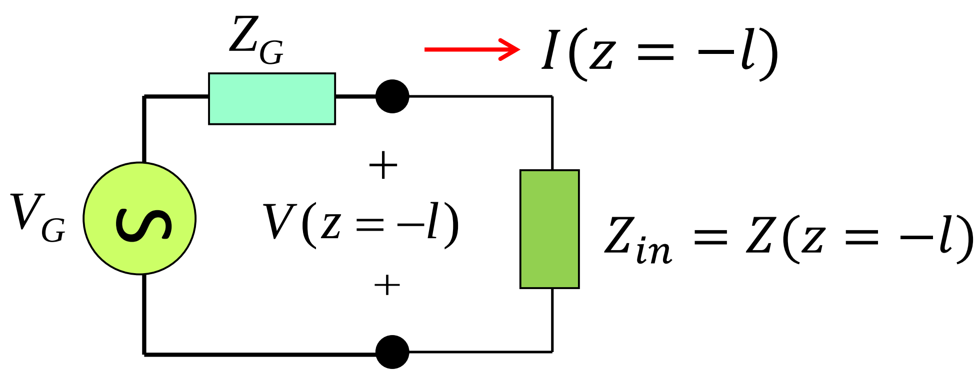
$$P_{abs} = \frac{1}{2}\text{Re}\left\{V(z=0)I^*(z=0)\right\}$$

- If the transmission line is **lossless**, then we know that the power delivered to the load must be **equal** to the power "delivered" to the **input** (P_{in}) of the transmission line:

$$P_{abs} = P_{in} = \frac{1}{2}\text{Re}\left\{V(z=-l)I^*(z=-l)\right\}$$

Sourced and Loaded Transmission Line (contd.)

- We can determine this power **without** having to solve for V_0^+ and V_0^- (i.e., V(z) and I(z)). We can simply use our knowledge of **circuit theory**!
- We can **transform** load Z_L to the beginning of the transmission line, so that we can replace the transmission line with its **input impedance** Z_{in}:



- Note by **voltage division** we can determine: $$V(z=-l)=V_G\frac{Z_{in}}{Z_G+Z_{in}}$$
- And from **Ohm's Law** we conclude: $$I(z=-l)=\frac{V_G}{Z_G+Z_{in}}$$

Sourced and Loaded Transmission Line (contd.)

- And thus, the **power** P_{in} delivered to Z_{in} (and thus the **power** P_{abs} delivered to the load Z_L) is:

$$P_{abs} = P_{in} = \frac{1}{2}\mathrm{Re}\left\{V(z=-l)I^*(z=-l)\right\} = \frac{1}{2}\mathrm{Re}\left\{V_G \frac{Z_{in}}{Z_G + Z_{in}} \frac{V_G^*}{\left(Z_G + Z_{in}\right)^*}\right\}$$

$$\Rightarrow P_{abs} = P_{in} = \frac{1}{2}\frac{\left|V_G\right|^2}{\left|Z_G + Z_{in}\right|^2}\mathrm{Re}\left\{Z_{in}\right\}$$

- Note that we could **also** determine P_{abs} from our **earlier** expression:

$$P_{abs} = \frac{1}{2}\mathrm{Re}\left\{V(z=0)I^*(z=0)\right\} = \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_0\right|^2\right)$$

But we would of course have to **first** determine V_0^+(!):

$$V_0^+ = V_G e^{-j\beta l}\frac{Z_0}{Z_0\left(1+\Gamma_{in}\right) + Z_G\left(1-\Gamma_{in}\right)}$$

Sourced and Loaded Transmission Line (contd.)

- Let's look at **specific cases** of Z_G and Z_L, and determine how they affect V_0^+ and P_{abs}.

$\boldsymbol{Z_G = Z_0}$

- For this case, we find that V_0^+ **simplifies** greatly:

$$V_0^+ = \frac{1}{2} V_G e^{-j\beta l}$$

It says that the incident wave in this case is **independent** of the load attached at the other end!

Thus, for the **one** case $\boldsymbol{Z_G = Z_0}$, we in fact can consider $V^+(z)$ as being the source wave, and then the reflected wave $V^-(z)$ as being the result of this stimulus.

- The complex value V_0^+ is the value of the incident wave evaluated at the end of the transmission line ($V_0^+ = V^+(z=0)$). We can also determine the value of the incident wave at the **beginning** of the transmission line (i.e. $V^+(z=-l)$).

$$V_0^+ = \frac{1}{2} V_G e^{-j\beta l}$$

Sourced and Loaded Transmission Line (contd.)

- Likewise, we find that the delivered power for this case can be simply stated as:

$$P_{abs} = \frac{|V_0^+|^2}{2Z_0}\left(1-|\Gamma_0|^2\right) = \frac{|V_G|^2}{8Z_0}\left(1-|\Gamma_0|^2\right)$$

$\boldsymbol{Z_L = Z_0}$

- In this case, we find that $\Gamma_0 = 0$, and thus $\Gamma_{in} = 0$. As a result:

$$V_0^+ = V_G e^{-j\beta l}\frac{Z_0}{Z_0 + Z_G}$$

- Likewise, we find that:

$$P_{abs} = \frac{|V_0^+|^2}{2Z_0}\left(1-|\Gamma_0|^2\right) = \frac{|V_0^+|^2}{2Z_0}$$

Here the delivered power P_{abs} is simply that of the incident wave (P^+), as the matched condition causes the reflected power to be zero ($P^- = 0$)!

- Inserting the value of $V_0{}^+$, we find:

$$V_0^+ = V_G e^{-j\beta l}\frac{Z_0}{Z_0 + Z_G}$$

this result can also be found by recognizing that $Z_{in} = Z_0$ when $Z_L = Z_0$.

Sourced and Loaded Transmission Line (contd.)

$Z_{in} = Z_G^*$ ← For this case, we find Z_L takes on whatever value required to make $Z_{in} = Z_G^*$. This is a **very** important case!

- First, we can express:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_G^* - Z_0}{Z_G^* + Z_0}$$

We can show that (trust me!):

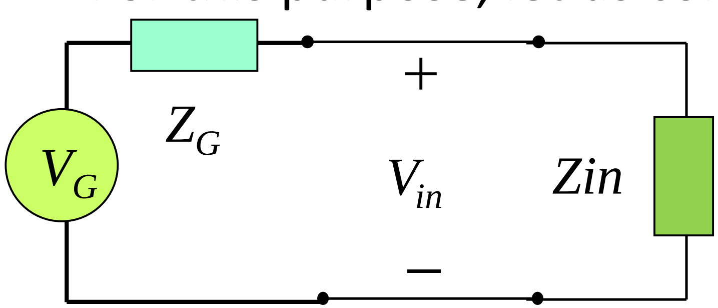
$$V_0^+ = V_G e^{-j\beta l} \frac{Z_G^* + Z_0}{4\,\mathrm{Re}\{Z_G\}}$$

- look at the absorbed power:

$$P_{abs} = \frac{1}{2} \frac{|V_G|^2}{|Z_G + Z_{in}|^2} \mathrm{Re}\{Z_{in}\}$$

It can be shown that—for a **given** V_G and Z_G—the value of input impedance Z_{in} that will absorb the **largest possible** amount of power is the value $Z_{in} = Z_G^*$.

- For this purpose, let us consider:



Power available for transfer to TL is given by:

$$P_{in} = \frac{1}{2} \mathrm{Re}\left(V_{in} \frac{V_{in}^*}{Z_{in}^*} \right) = \frac{1}{2} \frac{|V_G|^2}{\mathrm{Re}\left(Z_{in}^*\right)} \left| \frac{Z_G}{Z_G + Z_{in}} \right|^2$$

Sourced and Loaded Transmission Line (contd.)

- **If $Z_G = R_G + jX_G$** is fixed then for **complex Z_{in}** following conditions must be valid for **maximum P_{in} transferred to TL**

$$\frac{\partial P_{in}}{\partial R_{in}} = \frac{\partial P_{in}}{\partial X_{in}} = 0$$

- Elaboration of these conditions result in:

$$R_G^2 - R_{in}^2 + \left(X_G^2 + 2X_G X_{in} + X_{in}^2\right) = 0$$

$$X_{in}\left(X_G + X_{in}\right) = 0$$

Simplification gives

$$R_{in} = R_G \qquad X_{in} = -X_G \qquad \Rightarrow \qquad Z_{in} = Z_G^*$$

$$P_{abs} = \frac{1}{2}\frac{|V_G|^2}{|Z_G + Z_G^*|^2}\mathrm{Re}\{Z_G^*\} \quad \Rightarrow \quad \therefore P_{abs} = \frac{1}{2}|V_G|^2\frac{1}{4\,\mathrm{Re}\{Z_G^*\}} \doteq P_{avl}$$

This case is known as the **conjugate match**, and is essentially the goal of every transmission line problem—to deliver the largest possible power to Z_{in}, and thus to Z_L as well! → This power is known as the **available power** (P_{avl}) of the source.

Sourced and Loaded Transmission Line (contd.)

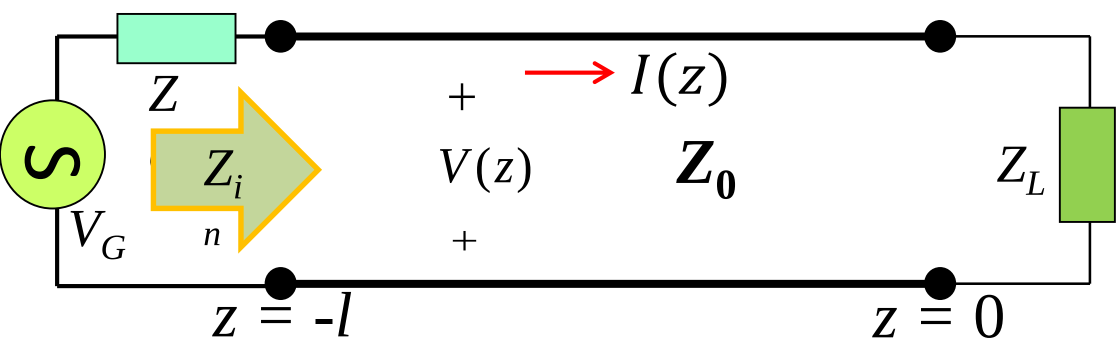
$$P_{abs} = \frac{1}{2} \frac{|V_G|^2}{|Z_G + Z_G^*|^2} \text{Re}\{Z_G^*\}$$

$$\therefore P_{abs} = \frac{1}{2}|V_G|^2 \frac{1}{4\text{Re}\{Z_G^*\}} \doteq P_{avl}$$

There are **two** very important things to understand about this result!

Very Important Thing #1

- Consider again:



- Recall that if $Z_L = Z_0$, the **reflected** wave will be **zero**, thus:

$$P_{abs} = \frac{|V_G|^2}{2} \frac{Z_0}{|Z_0 + Z_G|^2} \leq P_{avl}$$

- But note if $Z_L = Z_0$, the input impedance $Z_{in} = Z_0$ —but then $Z_{in} \neq Z_G^*$ (generally)! In other words, $Z_L = Z_0$ does **not** (generally) result in a **conjugate match**, and thus setting $Z_L = Z_0$ does **not** result in maximum power absorption!

Sourced and Loaded Transmission Line (contd.)

Q: Huh!? This makes **no** sense! A load value of $Z_L = Z_0$ will **minimize** the reflected wave $(P^- = 0)$—**all** of the incident power will be absorbed.

- Any other value of Z_L will result in **some** of the incident wave being reflected—how in the world could this **increase** absorbed power?
- After all, just **look** at the expression for absorbed power:

$$P_{abs} = \frac{|V_0^+|^2}{2Z_0}\left(1 - |\Gamma_0|^2\right)$$

Clearly, this value is maximized when $\Gamma_0 = 0$ (i.e., when $Z_L = Z_0$)

A: You are forgetting one very important fact! Although it **is** true that the load impedance Z_L affects the **reflected** wave power P^-, the value of Z_L — as we have shown— **likewise** helps determine the value of the **incident** wave (i.e., the value of P^+) as well.

Sourced and Loaded Transmission Line (contd.)

- Thus, the value of $\boldsymbol{Z_L}$ that minimizes P^- will **not** generally maximize P^+!
- **Likewise** the value of $\boldsymbol{Z_L}$ that maximizes P^+ will not generally minimize P^-.
- Instead, the value of $\boldsymbol{Z_L}$ that maximizes the **absorbed** power P_{abs} is, by definition, the value that maximizes the **difference** $P^+ - P^-$.
- We find that this impedance $\boldsymbol{Z_L}$ is the value that results in the **ideal** case of $\boldsymbol{Z_{in} = Z_G^*}$.

Q: Yes, but what about the case where $\boldsymbol{Z_G = Z_0}$? For **that** case, we determined that the incident wave **is** independent of $\boldsymbol{Z_L}$. Thus, it would seem that at least for that case, the **delivered** power would be maximized when the **reflected** power was minimized (i.e., $\boldsymbol{Z_L = Z_0}$).

A: True! But think about what the **input** impedance would be in that case—$\boldsymbol{Z_{in} = Z_0}$. Oh by the way, that provides a **conjugate match** ($\boldsymbol{Z_{in} = Z_0 = Z_G^*}$).

Sourced and Loaded Transmission Line (contd.)

- Thus, in some ways, the case $Z_G = Z_0 = Z_L$ (i.e., **both** source and load impedances are numerically equal to Z_0) is **ideal**. A **conjugate match** occurs, the incident wave is **independent** of Z_L, there is **no** reflected wave, and all the math **simplifies** quite nicely:

$$V_0^+ = \frac{1}{2} V_G e^{-j\beta l}$$

$$P_{abs} = P_{avl} = \frac{|V_G|^2}{8Z_0}$$

Very Important Thing #2

- Note the conjugate match criteria **says**: **Given** source impedance Z_G, maximum power transfer occurs when the input impedance is set at value $Z_{in} = Z_G^*$.
- It does NOT say: **Given** input impedance Z_{in}, maximum power transfer occurs when the source impedance is set at value $Z_G = Z_{in}^*$.

This last statement is in fact false!

- A **factual** statement is this: **Given** input impedance Z_{in}, maximum power transfer occurs when the source impedance is set at value $Z_G = 0 - jX_{in}$ (i.e., $R_G = 0$).

Sourced and Loaded Transmission Line (contd.)

Q: Huh??

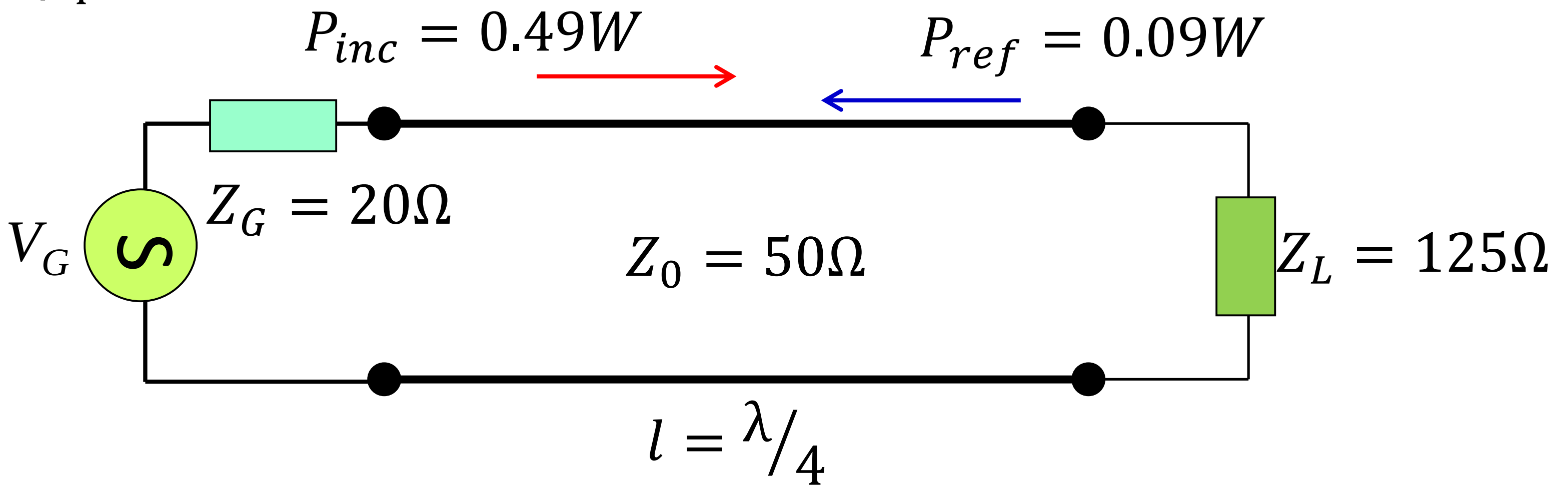
A: Remember, the value of source impedance $\boldsymbol{Z_G}$ affects the available power P_{avl} of the source. To maximize P_{avl}, the real (resistive) component of the source impedance should be as small as possible (regardless of $\boldsymbol{Z_{in}}$!), a fact that is **evident** when observing the expression for **available power**:

$$P_{avl} = \frac{1}{2}|V_G|^2 \frac{1}{4\text{Re}\{Z_G^*\}} = \frac{|V_G|^2}{8R_G}$$

- Thus, **maximizing** the power delivered **to** a load (P_{abs}), **from** a source, has **two** components:
  1. Maximize the **power available** (P_{avl}) from a source (e.g., minimize R_G).
  2. **Extract** all of this available power by setting the input impedance $\boldsymbol{Z_{in}}$ to a value $\boldsymbol{Z_{in} = Z_G{}^*}$ (thus $\boldsymbol{P_{abs} = P_{avl}}$).

Example – 2

- Consider this circuit, where the transmission line is **lossless** and has length $l = {}^{\lambda}/_{4}$:



Determine the magnitude of source voltage V_G (i.e., determine $|V_G|$).

Hint: This is **not** a boundary condition problem. Do **not** attempt to find $V(z)$ and/or $I(z)$!

Lossy Transmission Lines

- Recall that we have been **approximating** low-loss transmission lines as lossless (R =G = 0):

$$\alpha = 0 \qquad \beta = \omega\sqrt{LC}$$

- But, **long** low-loss lines require a **better** approximation:

$$\alpha = \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) \qquad \beta = \omega\sqrt{LC}$$

- Now, if we have **really long** transmission lines (e.g., long distance communications), we can apply **no** approximations at all:

$$\alpha = \text{Re}\{\gamma\} \qquad \beta = \text{Im}\{\gamma\}$$

For these **very** long transmission lines, $\beta = Im\{\gamma\}$ is a **function** of signal **frequency** ω. This results in an extremely serious problem—signal **dispersion**.

- Recall that the **phase velocity** v_p (i.e., propagation velocity) of a wave in a transmission line is:

$$v_p = \frac{\omega}{\beta}$$

$$\beta = \text{Im}\{\gamma\} = \text{Im}\left\{\sqrt{(R + j\omega L)(G + j\omega C)}\right\}$$

For a lossy line, v_p is a function of frequency ω (i.e., $v_p(\omega)$)—this is **bad**!

Lossy Transmission Lines (contd.)

- Any signal that carries significant **information** must has some non-zero **bandwidth**. In other words, the signal energy (as well as the information it carries) is **spread** across many frequencies.
- If the different frequencies that comprise a signal travel at different velocities, that signal will arrive at the end of a transmission line **distorted**. We call this phenomenon signal **dispersion.**
- Recall for **lossless** lines, however, the phase velocity is **independent** of frequency—**no** dispersion will occur!
- For lossless line: $v_p = \frac{1}{\sqrt{LC}}$

however, a perfectly lossless line is impossible, but we find phase velocity is **approximately** constant if the line is low-loss.

Lossy Transmission Lines (contd.)

Q: You say "**most** often" not a problem—that phrase seems to imply that dispersion sometimes **is** a problem!

A: for low-loss transmission lines, dispersion can be a problem **if** the lines are **very** long—just a small difference in phase velocity can result in significant differences in propagation delay **if** the line is very long!

- Modern examples of long transmission lines include phone lines and cable TV. However, the **original** long transmission line problem occurred with the **telegraph**.
- Early telegraph "engineers" discovered that if they made their telegraph lines **too long**, the dots and dashes characterizing Morse code turned into a muddled, indecipherable **mess**. Although they did not realize it, they had fallen victim to the heinous effects of **dispersion**!
- Thus, to send messages over long distances, they were forced to implement a series of intermediate "**repeater**" stations, wherein a human operator received and then **retransmitted** a message on to the next station. This **really** slowed things down!

Lossy Transmission Lines (contd.)

Q: Is there any way to **prevent** dispersion from occurring?

A: You bet! **Oliver Heaviside** figured out how in the **19th** Century!

- Heaviside found that a transmission line would be distortionless (i.e., no dispersion) **if** the line parameters exhibited the following **ratio**:

$$\frac{R}{L} = \frac{G}{C}$$

- Let's see **why** this works. Note the complex propagation constant γ can be expressed as:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{LC(R/L + j\omega)(G/C + j\omega)}$$

- For $\frac{R}{L} = \frac{G}{C}$:

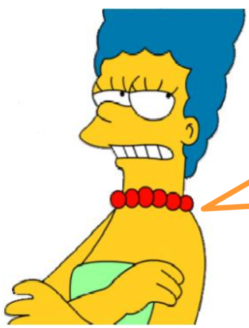
$$\gamma = \sqrt{LC(R/L + j\omega)(R/L + j\omega)} = (R/L + j\omega)\sqrt{LC} = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$$

Lossy Transmission Lines (contd.)

- Thus: $\alpha = \text{Re}\{\gamma\} = R\sqrt{\frac{C}{L}}$ $\beta = \text{Im}\{\gamma\} = \omega\sqrt{LC}$

- The propagation **velocity** of the wave is thus: $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$

The propagation velocity is **independent** of frequency! This lossy transmission line is **not** dispersive!



Q: Right. All the transmission lines I use have the property that $R/L > G/C$. I've **never** found a transmission line with this **ideal** property $R/L = G/C$!

A: It is true that typically $R/L > G/C$. But, we can reduce the ratio R/L(until it is equal to G/C) by adding series **inductors** periodically along the transmission line.

Lossy Transmission Lines (contd.)

This was **Heaviside's** solution—and it worked! **Long** distance transmission lines were made possible.

Q: Why don't we increase G instead?

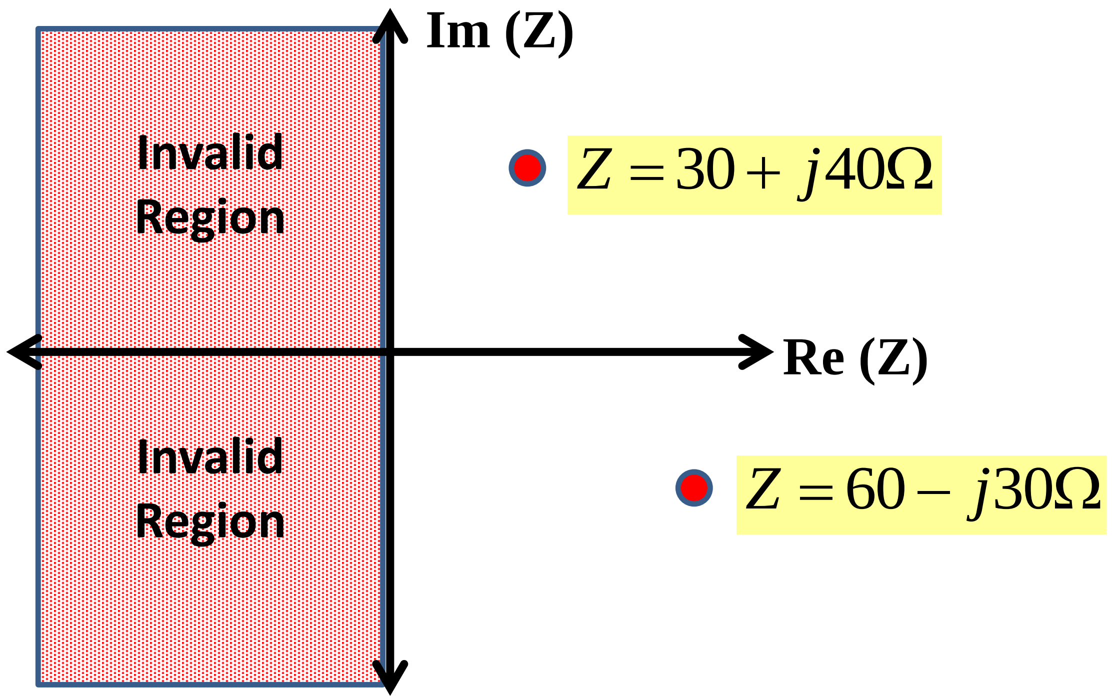
A:

Smith Chart

- Smith chart – what?
- The Smith chart is a very convenient graphical tool for analyzing TLs studying their behavior.
- It is mapping of impedance in standard complex plane into a suitable complex reflection coefficient plane.
- It provides graphical display of reflection coefficients.
- The impedances can be directly determined from the graphical display (ie, from Smith chart)
- Furthermore, Smith charts facilitate the analysis and design of complicated circuit configurations.

The Complex Γ- Plane

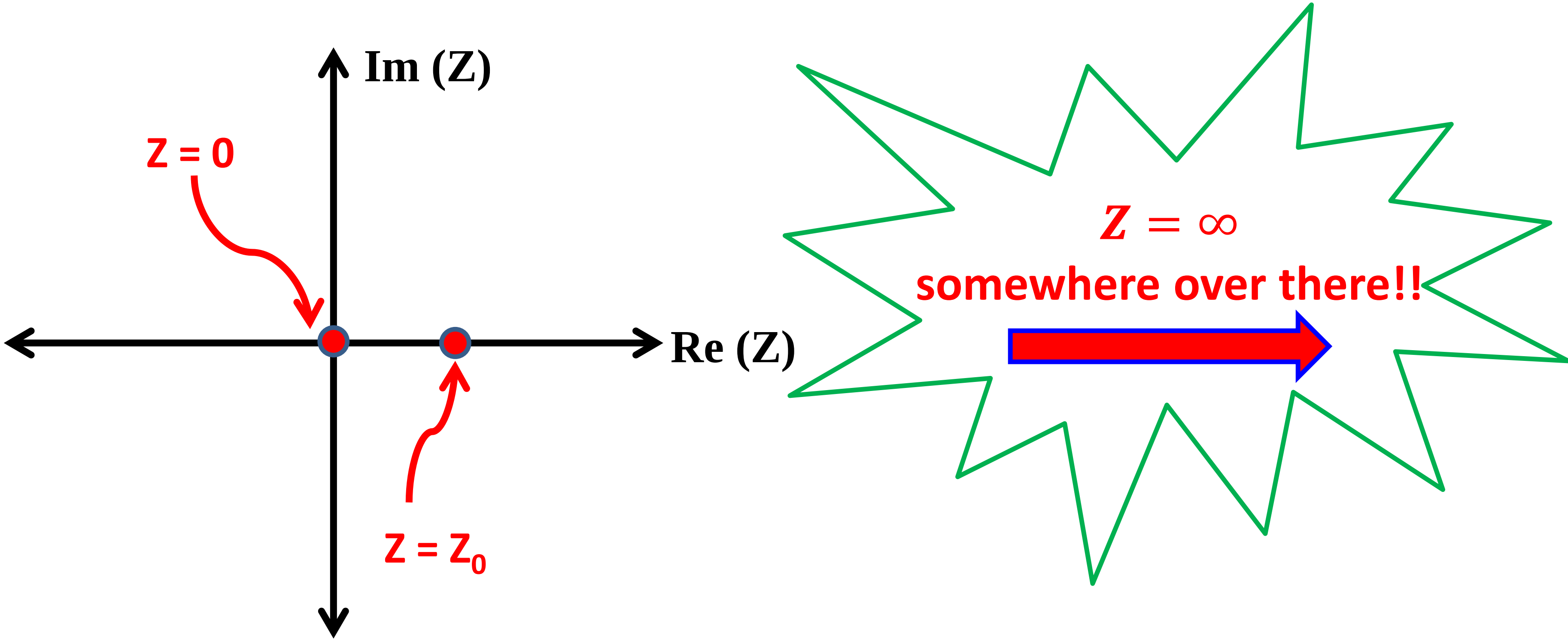
- Let us first display the impedance Z on complex Z-plane



- Note that each dimension is defined by a single real line: the horizontal line (axis) indicates the real component of Z, and the vertical line (axis) indicates the imaginary component of Z → Intersection of these lines indicate the complex impedance

The Complex Γ- Plane (contd.)

- How do we plot an **open circuit** (i.e, $Z = \infty$), **short circuit** (i.e, $Z = 0$), and **matching condition** (i.e, $Z = Z_0 = 50\Omega$) on the complex Z-plane



It is apparent that **complex** $Z - plane$ is not very useful

The Complex Γ-Plane (contd.)

- The **limitations** of complex Z-plane can be **overcome** by complex Γ-plane
- We know **Z** $\leftrightarrow \Gamma$ (i.e, if you know **one**, you know the **other**).
- We can therefore define a complex Γ-plane in the same manner that we defined a complex Z-plane.
- Let us revisit the reflection coefficient in complex form:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + \Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

Real part of Γ_0 (Γ_{0r}); Imaginary part of Γ_0 (Γ_{0i})

Where,

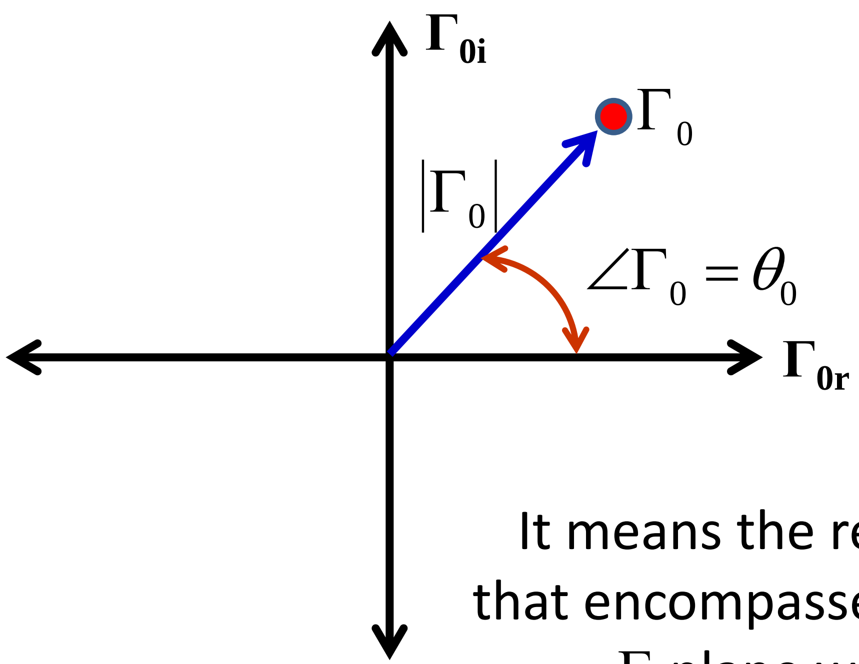
$$\theta_0 = \tan^{-1}\left(\frac{\Gamma_{0i}}{\Gamma_{0r}}\right)$$

- In the special terminated conditions of pure short-circuit and pure open-circuit conditions the corresponding Γ_0 are -1 and +1 located on the real axis in the complex Γ-plane.

The Complex Γ-Plane (contd.)

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + \Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

Representation of reflection coefficient in polar form



Observations:

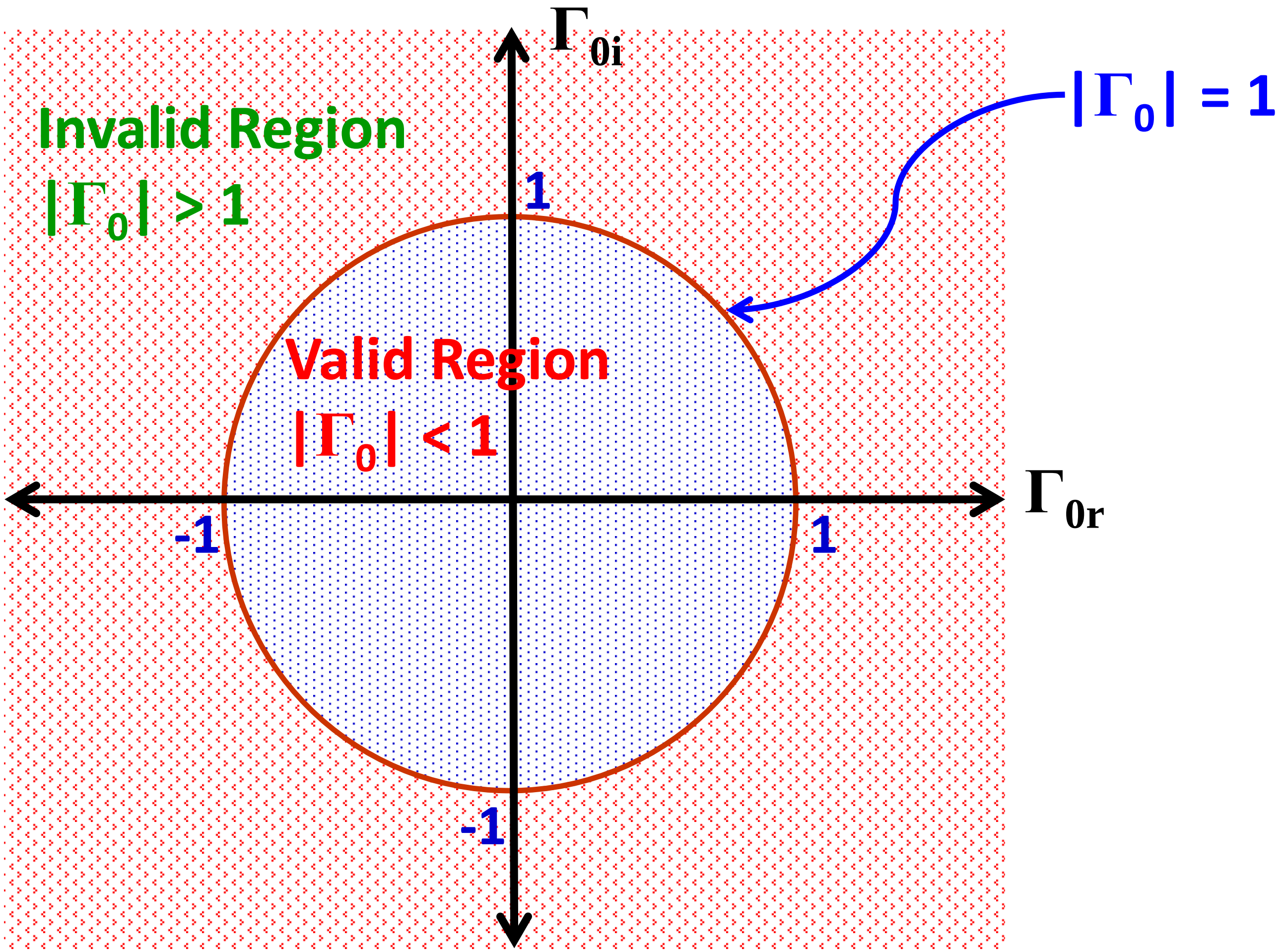
- A radial line is formed by the locus of all points whose phase is θ_0
- A circle is formed by the locus of all points whose magnitude is $|\Gamma_0|$

It means the reflection coefficient has a valid region that encompasses all the four quadrants in the complex Γ-plane within the -1 to +1 bounded region

In complex Z-plane the valid region was unbounded on the right half of the plane → as a result many important impedances could not be plotted

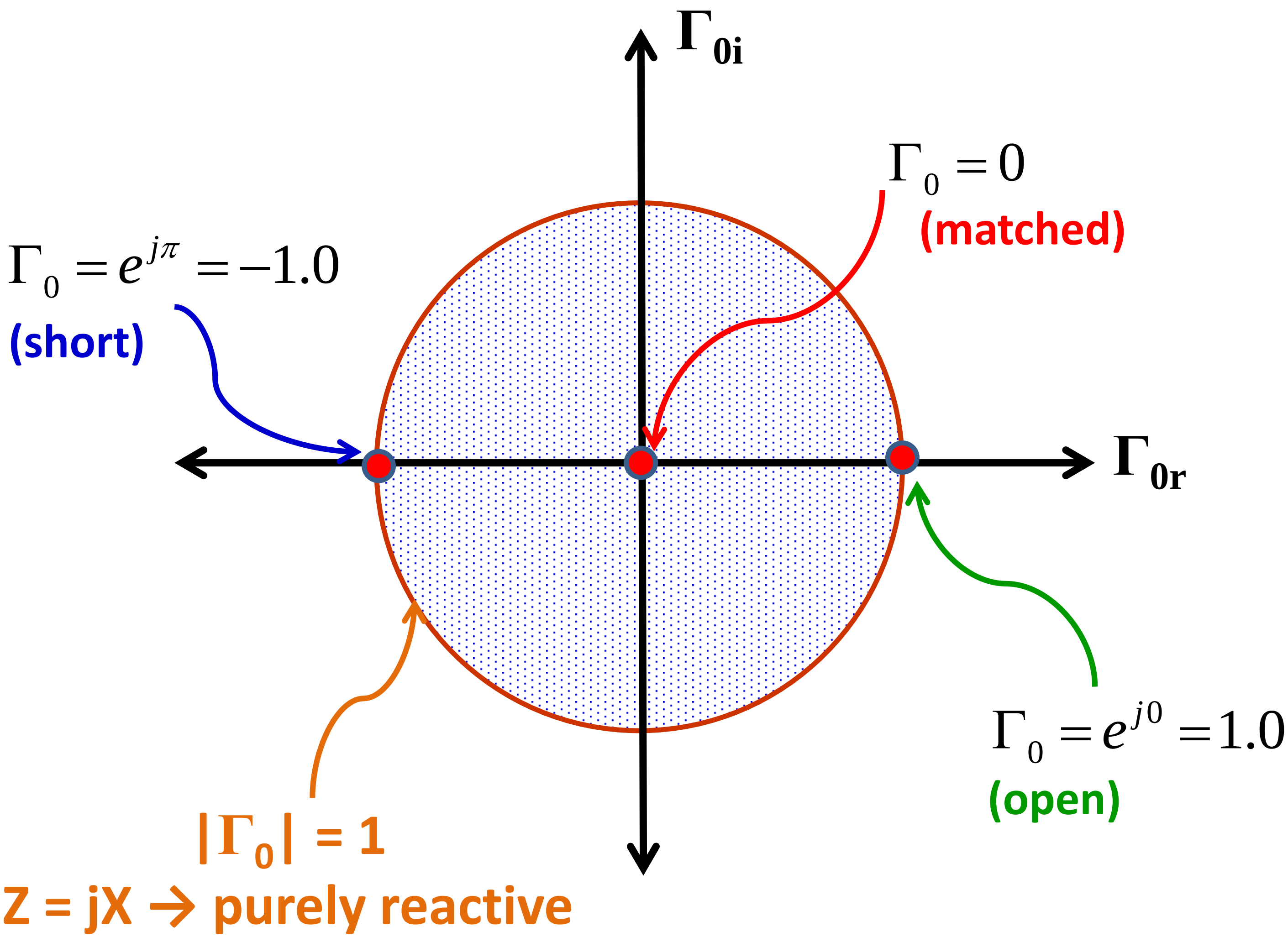
The Complex Γ-Plane (contd.)

- **Validity Region**



The Complex Γ-Plane (contd.)

- We can plot all the valid impedances (i.e R > 0) within this bounded region.



Example – 3

- A TL with a characteristic impedance of Z_0 = 50Ω is terminated into following load impedances:
  (a) $Z_L = 0$ (Short Circuit)
  (b) $Z_L \rightarrow \infty$ (Open Circuit)
  (c) $Z_L = 50\Omega$
  (d) $Z_L = (16.67 - j16.67)\Omega$
  (e) $Z_L = (50 + j50)\Omega$

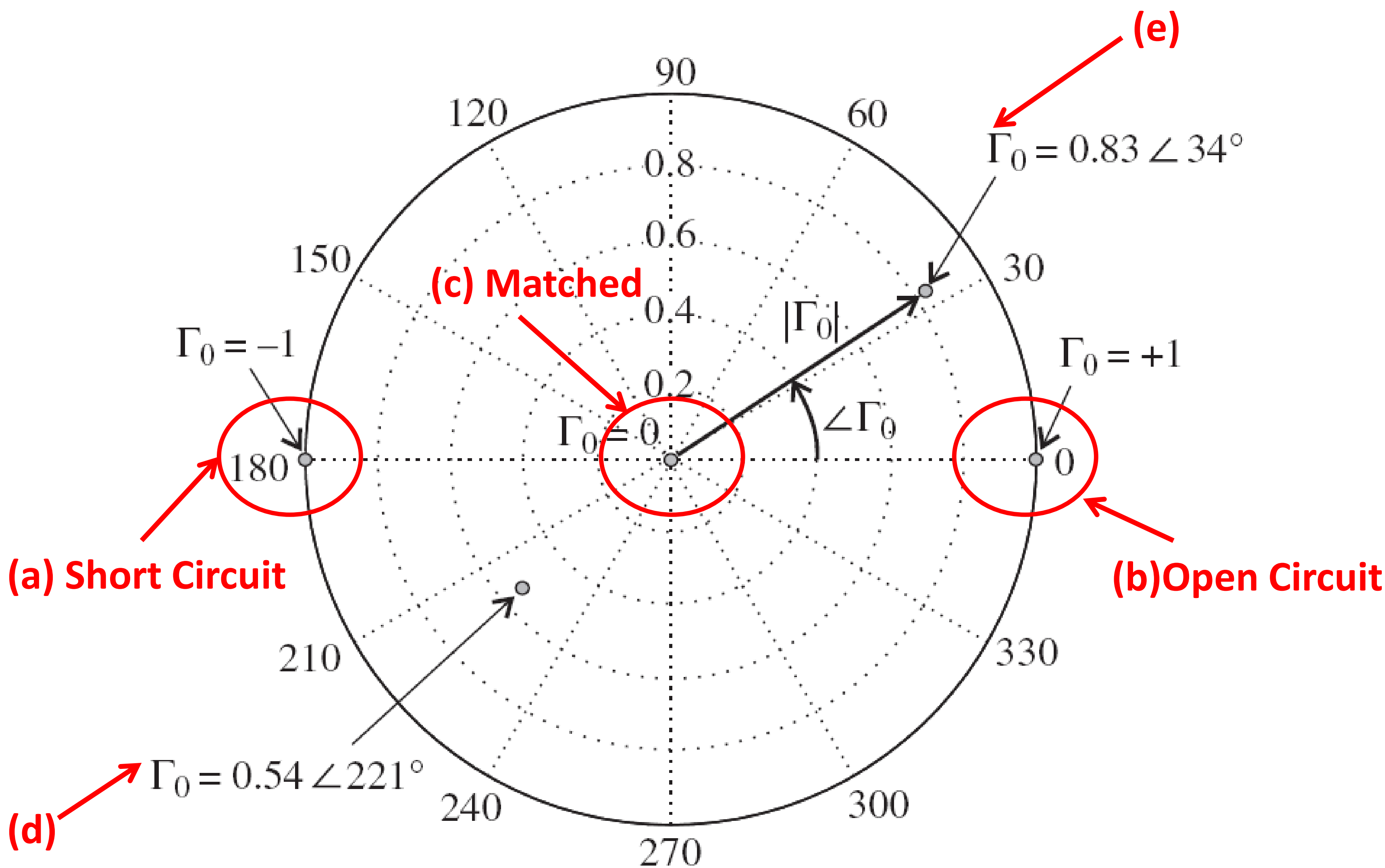
Display the respective reflection coefficients in complex Γ-plane

- **Solution:** We know the relationship between Z and Γ:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + \Gamma_{0i} = |\Gamma_0| e^{j\theta_0}$$

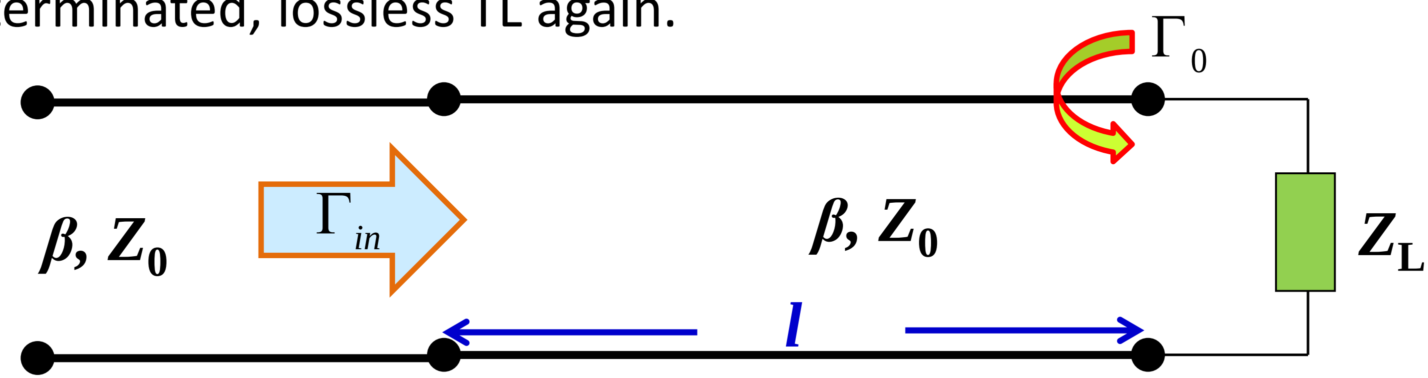
(a) $\Gamma_0 = -1$ (Short Circuit)
(b) $\Gamma_0 = 1$ (Open Circuit)
(c) $\Gamma_0 = 0$ (Matched)
(d) $\Gamma_0 = 0.54\angle 221^o$
(e) $\Gamma_0 = 0.83\angle 34^o$

Example – 3 (contd.)



Transformations on the Complex Γ-Plane

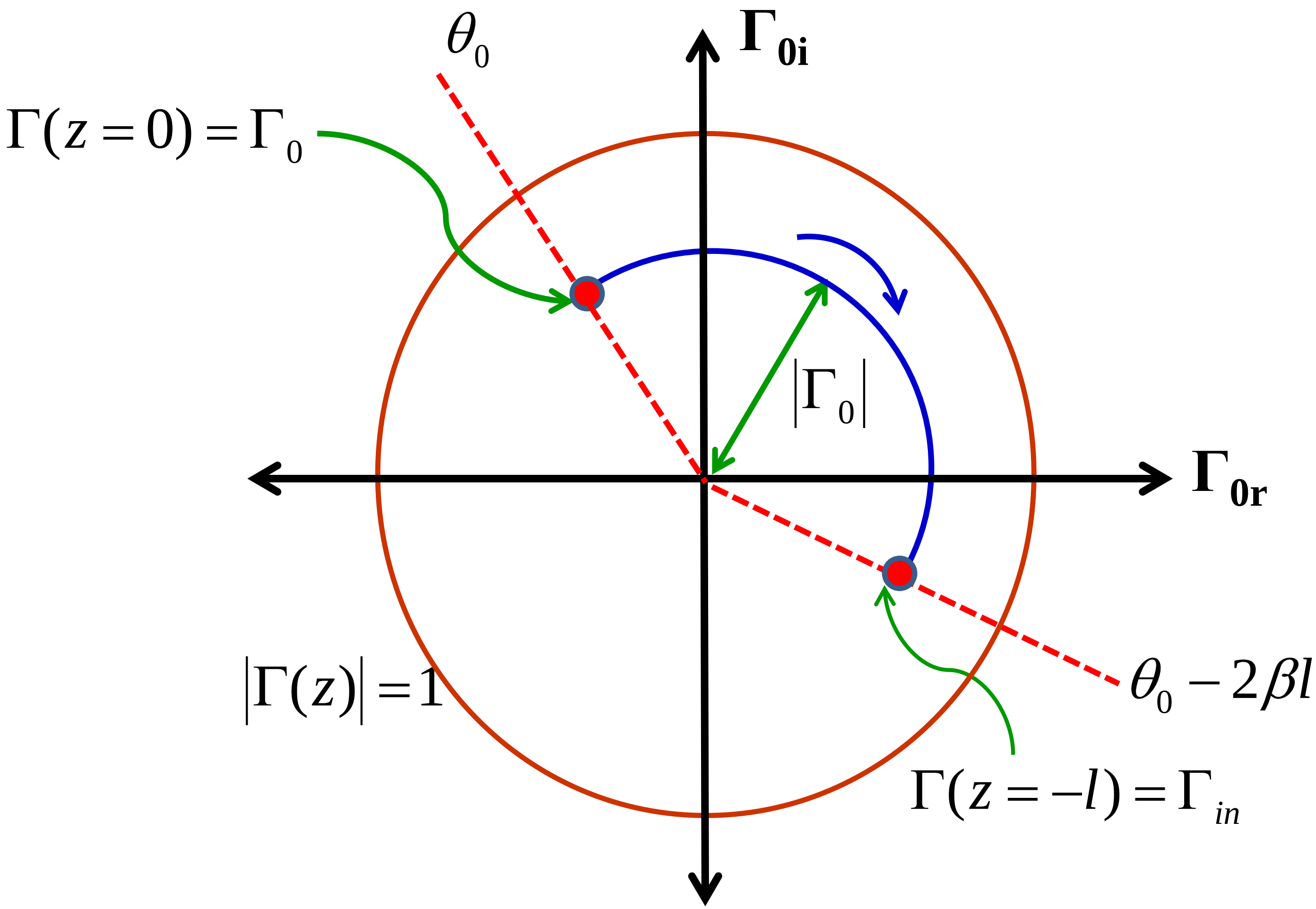
- The usefulness of the complex Γ-plane will be evident when we consider the terminated, lossless TL again.



- At $z = 0$, the reflection coefficient is called load reflection coefficient (Γ_0) → this actually describes the mismatch between the load impedance (Z_L) and the characteristic impedance (Z_0) of the TL.
- The move away from the load (or towards the input/source) in the negative z-direction (clockwise rotation) requires multiplication of Γ_0 by a factor $\exp(+j2\beta z)$ in order to explicitly define the mismatch at location 'z' known as $\Gamma(z)$.
- This transformation of Γ_0 to $\Gamma(z)$ is the key ingredient in Smith chart as a graphical design/display tool.

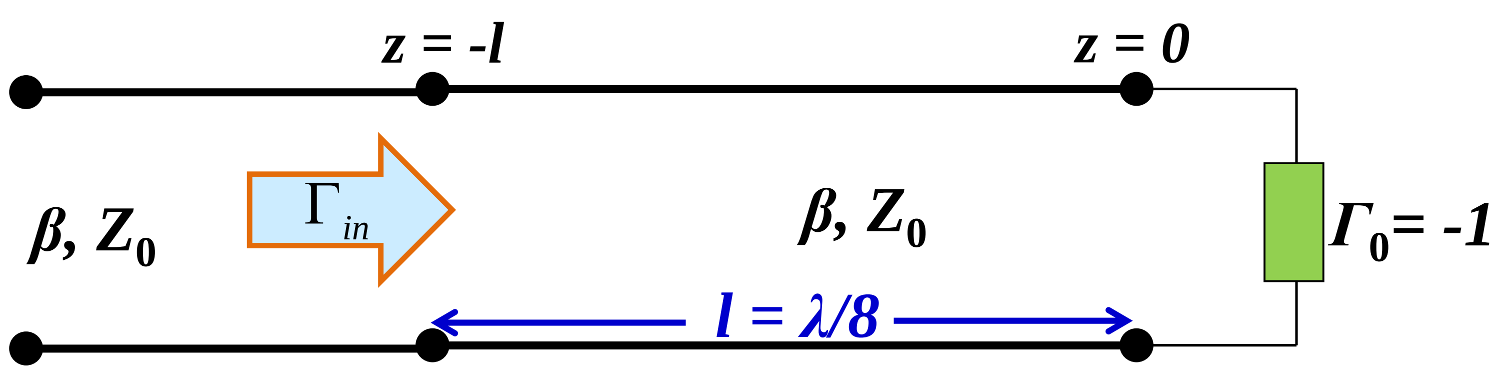
Transformations on the Complex Γ-Plane (contd.)

- Graphical interpretation of $\Gamma(z) = \Gamma_0 e^{+2j\beta z}$



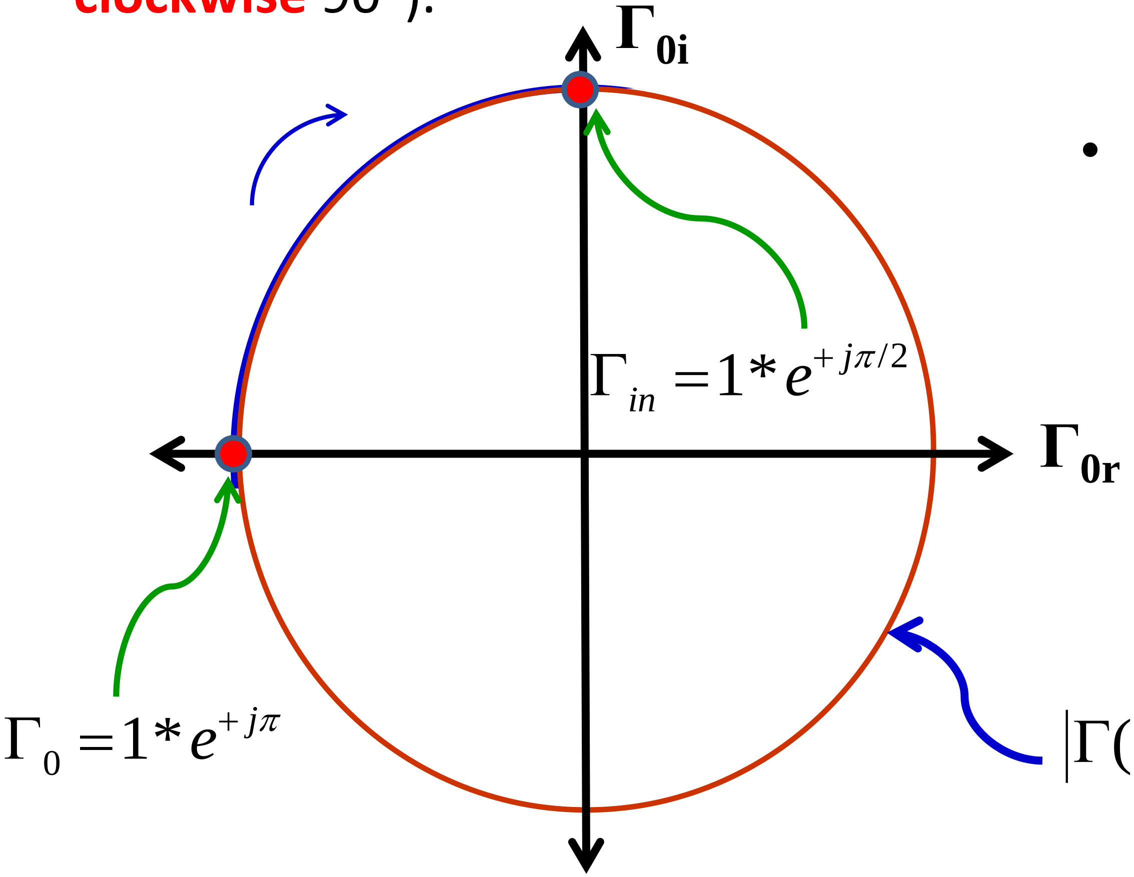
Transformations on the Complex Γ-Plane (contd.)

- It is clear from the graphical display that addition of a length of TL to a load Γ_0 **modifies** the **phase** θ_0 but **not** the **magnitude** Γ_0, we trace a **circular arc** as we parametrically plot $\Gamma(z)$! This arc has a **radius** Γ_0 and an **arc angle** $2\beta l$ radians.
- We can therefore **easily** solve many interesting TL problems **graphically**—using the complex Γ-plane! For **example,** say we wish to determine Γ_{in} for a transmission line length l = λ/8 and terminated with a **short** circuit.



Transformations on the Complex Γ-Plane (contd.)

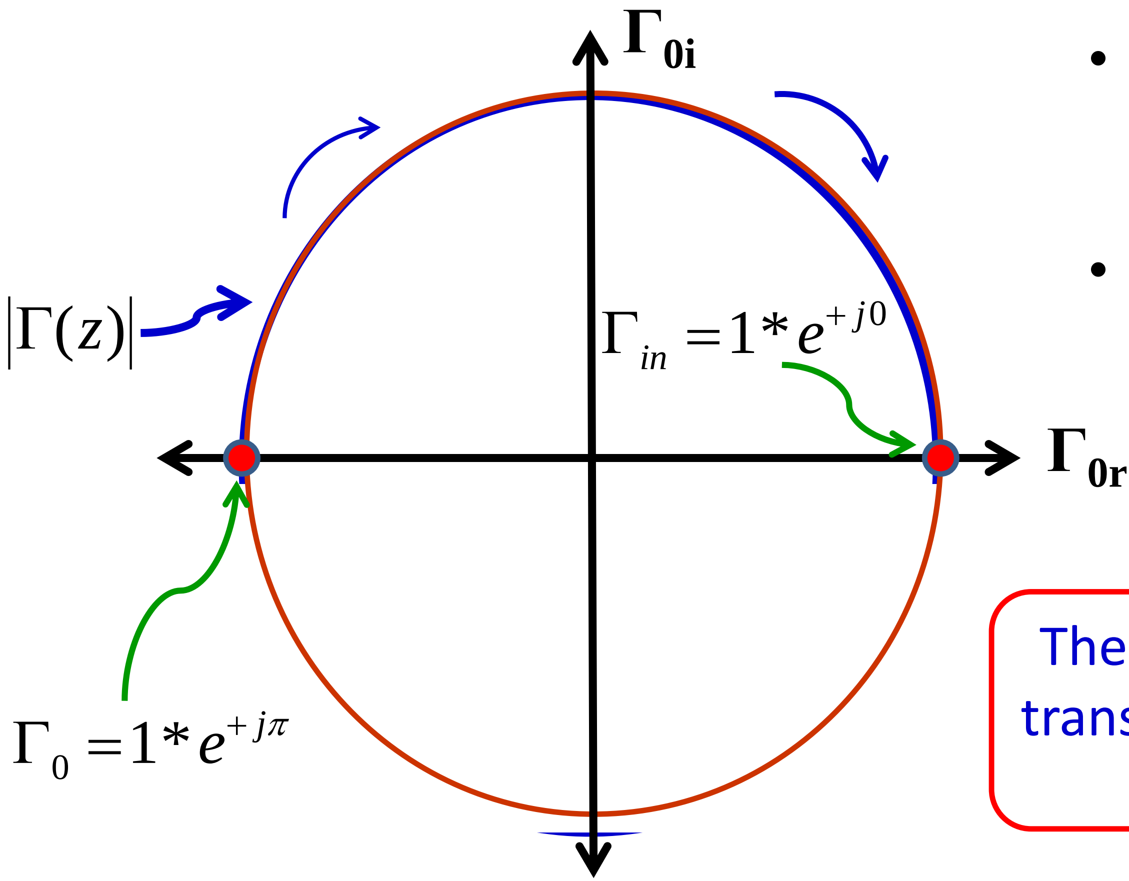
- The reflection coefficient of a **short** circuit is Γ_0 = −1 =1*e(jπ), and therefore we **begin** at the leftmost point on the complex Γ-plane. We then move along a **circular arc** −2βl = −2(π/4) = −π/2 radians (i.e., rotate **clockwise** 90^o).



- When we stop, we find we are at the point for Γ_{in}; in this case Γ_{in} = 1*e(jπ/2)

Transformations on the Complex Γ-Plane (contd.)

- Now let us consider the same problem, only with a new transmission line length $l = \lambda/4$.
- Now we rotate clockwise $2\beta l = \pi$ radians.

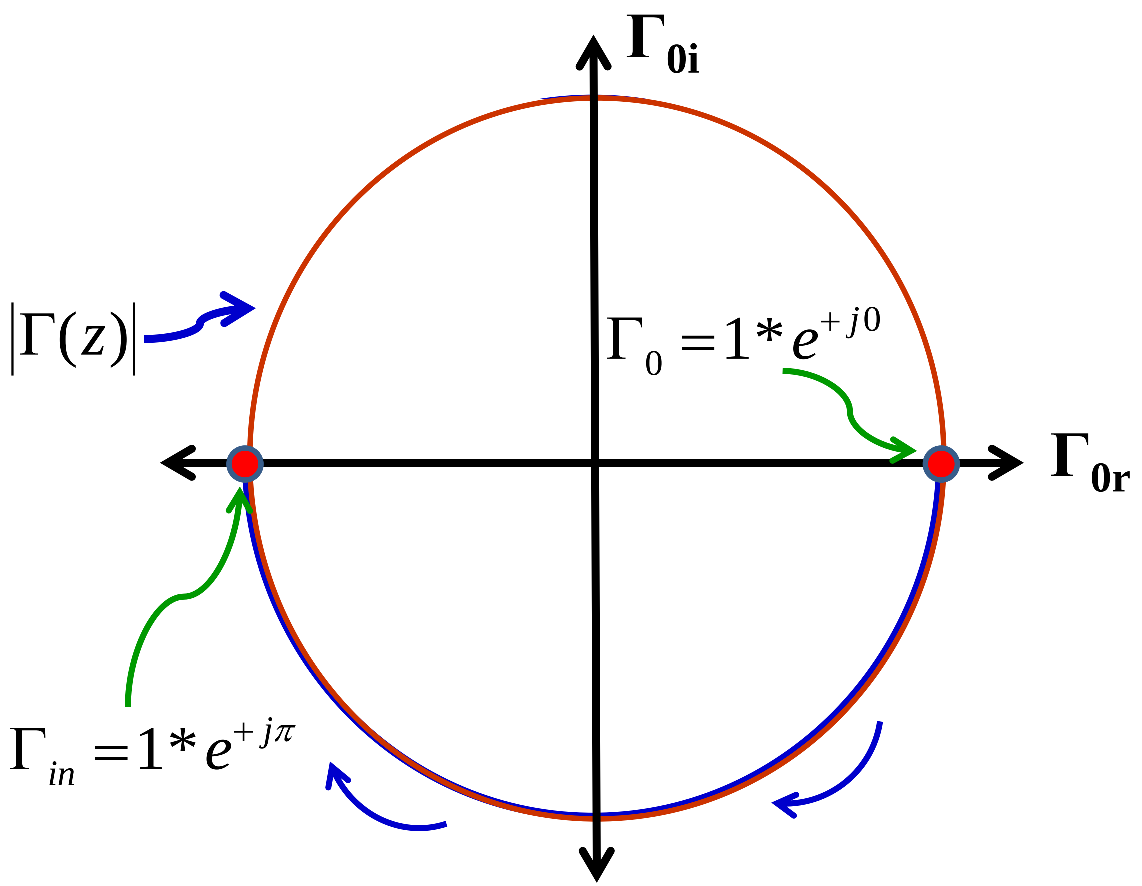


- In this case the input reflection coefficient is Γ_{in} = 1*e(j0) = 1
- The reflection coefficient of an open circuit

The short circuit load has been transformed into an open circuit with a quarter-wave TL

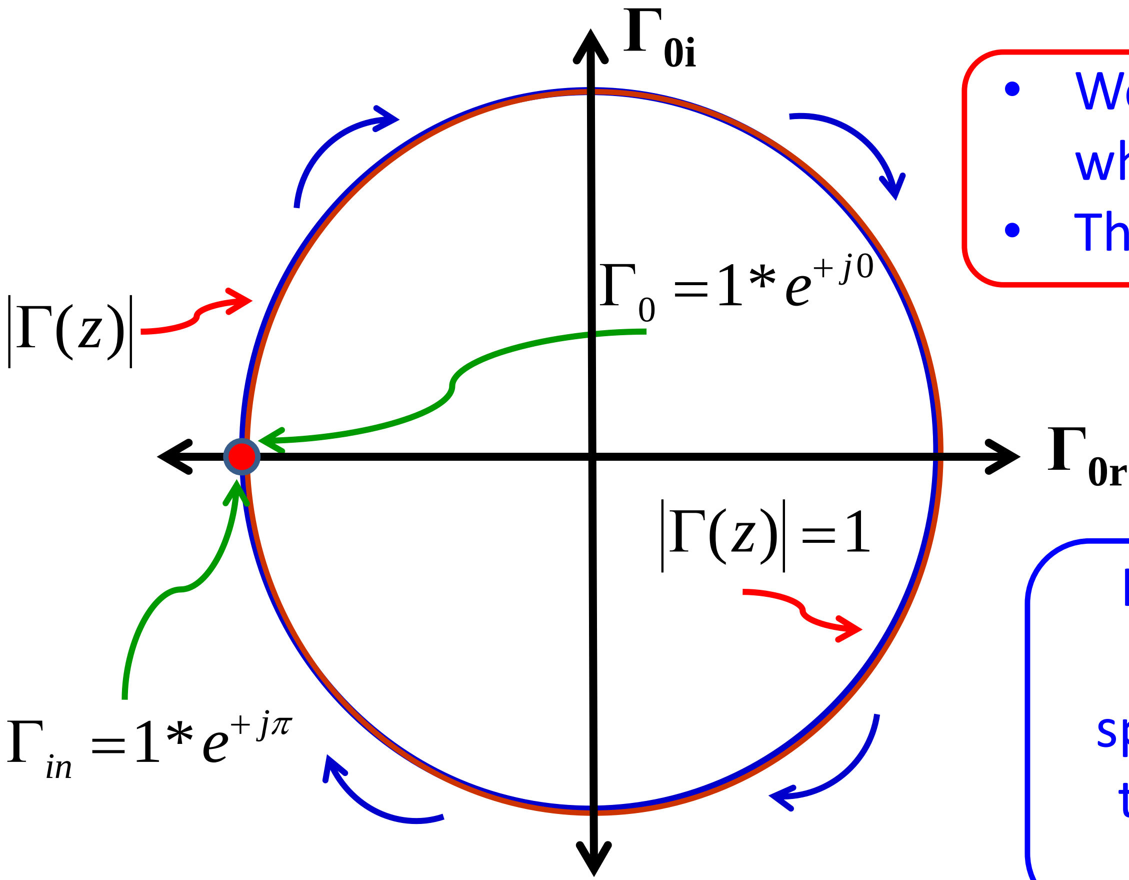
Transformations on the Complex Γ-Plane (contd.)

- We also know that a quarter-wave TL transforms an open-circuit into short-circuit → graphically it can be shown as:



Transformations on the Complex Γ-Plane (contd.)

- Now let us consider the same problem again, only with a new transmission line length $l = \lambda/2$.
- Now we rotate clockwise $2\beta l = 2\pi$ radians (360^o)

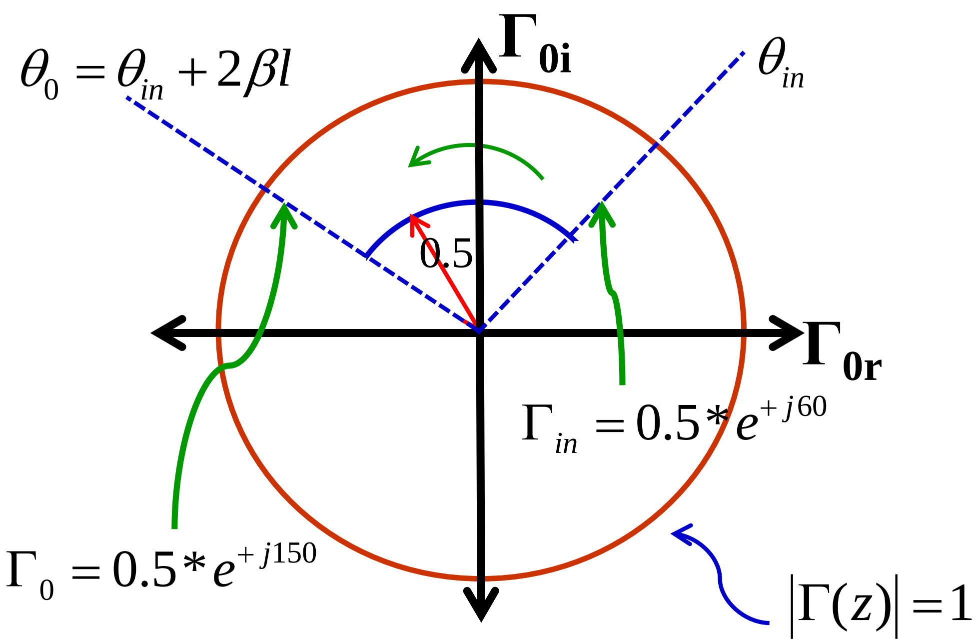


- We came clear around to where we **started!**
- Thus we conclude that $\Gamma_{in} = \Gamma_0$

It comes from the fact that **half-wavelength** TL is a special case, where we know that $Z_{in} = Z_L$ → eventually it leads to $\Gamma_{in} = \Gamma_0$

Transformations on the Complex Γ-Plane (contd.)

- Now let us consider the **opposite** problem. Say we know that the **input** reflection coefficient at the beginning of a TL with length $l = \lambda/8$ is: $\Gamma_{in} = 0.5e(j60°)$.
- What is the reflection coefficient at the **load**?
- In this case we rotate counter-clockwise along a circular arc (radius =0.5) by an amount $2\beta l = \pi/2$ radians (90^o).
- In essence, we are removing the phase associated with the TL.



The reflection coefficient at the load is:

$$\Gamma_0 = 0.5 * e^{+j150}$$

Mapping Z to Γ

- We know that the line impedance and reflection coefficient are **equivalent** – either one can be expressed in terms of the other.

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \Longleftrightarrow Z(z) = Z_0\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

- The above expressions depend on the characteristic impedance Z_0 of the TL. In order to generalize the relationship, we first define a **normalized** impedance value **z'** as:

$$z'(z) = \frac{Z(z)}{Z_0} = \frac{R(z)}{Z_0} + j\frac{X(z)}{Z_0} = r(z) + jx(z)$$

therefore

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{(Z(z)/Z_0) - 1}{(Z(z)/Z_0) + 1} = \frac{z'(z) - 1}{z'(z) + 1}$$

$$z'(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

Mapping Z to Γ (contd.)

$$\Gamma(z)=\frac{Z(z)-Z_0}{Z(z)+Z_0}=\frac{\left(Z(z)/Z_0\right)-1}{\left(Z(z)/Z_0\right)+1}=\frac{z'(z)-1}{z'(z)+1}$$

$$z'(z)=\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

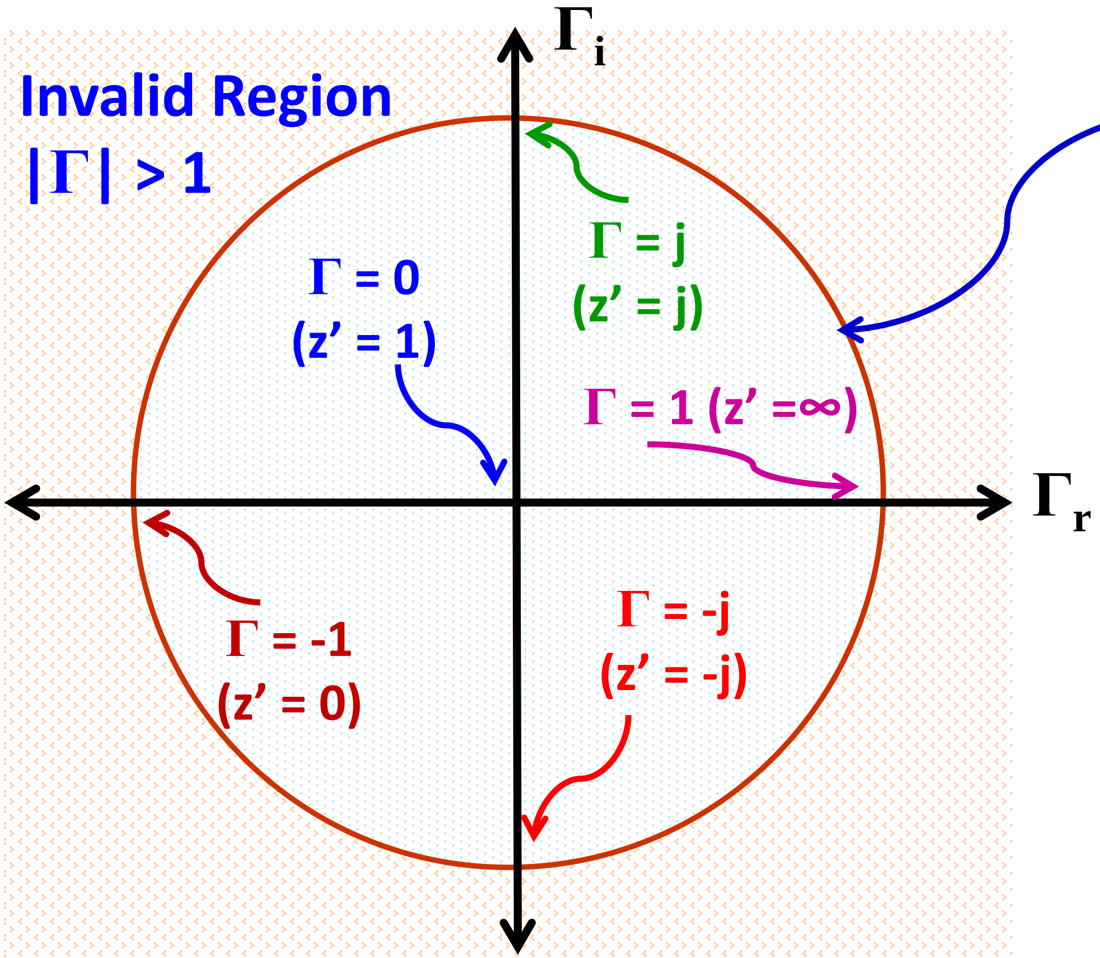
These equations describe a mapping between z' and Γ. That means that each and every normalized impedance value likewise corresponds to one specific point on the complex Γ-plane

- For example, we wish to indicate the values of some common normalized impedances (shown below) on the complex Γ-plane and vice-versa.

Case	Z	z'	Γ
1	∞	∞	1
2	0	0	-1
3	Z_0	1	0
4	jZ_0	j	j
5	$-jZ_0$	-j	-j

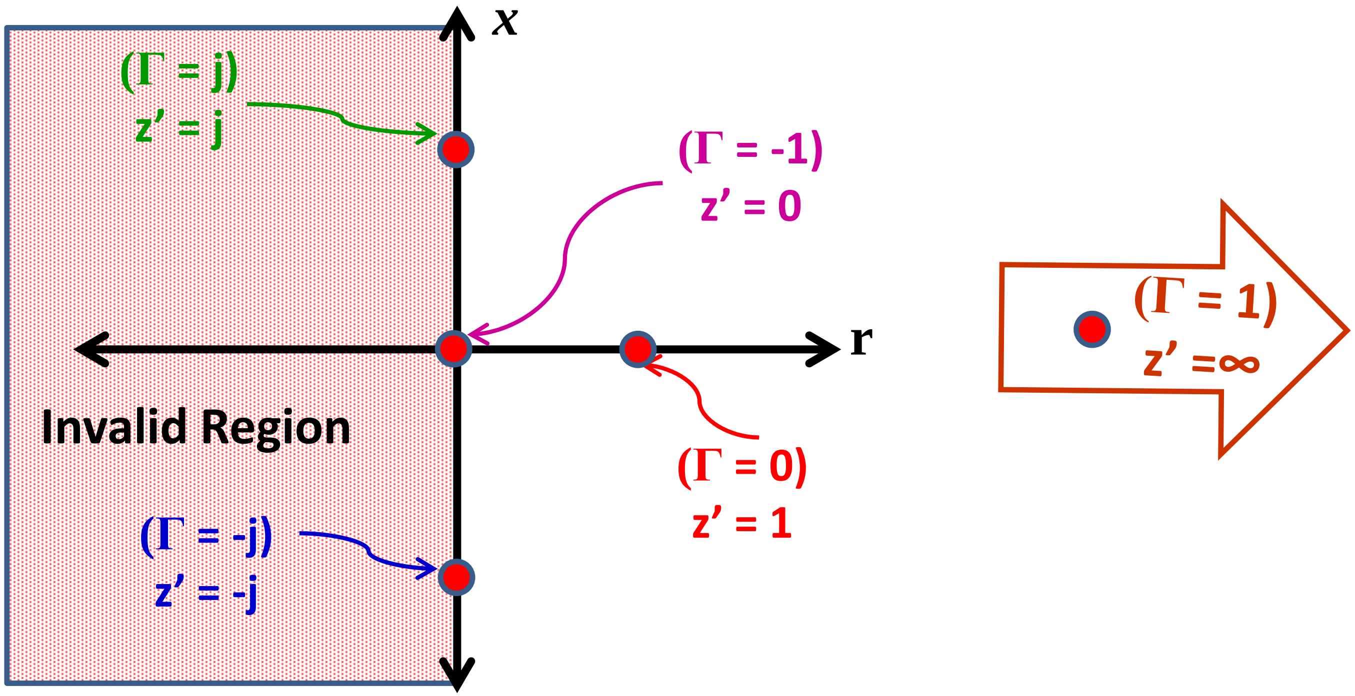
Mapping Z to Γ (contd.)

- The five normalized impedances map five specific points on the complex Γ-plane.



Mapping Z to Γ (contd.)

- The five complex-Γ map onto five points on the normalized Z-plane

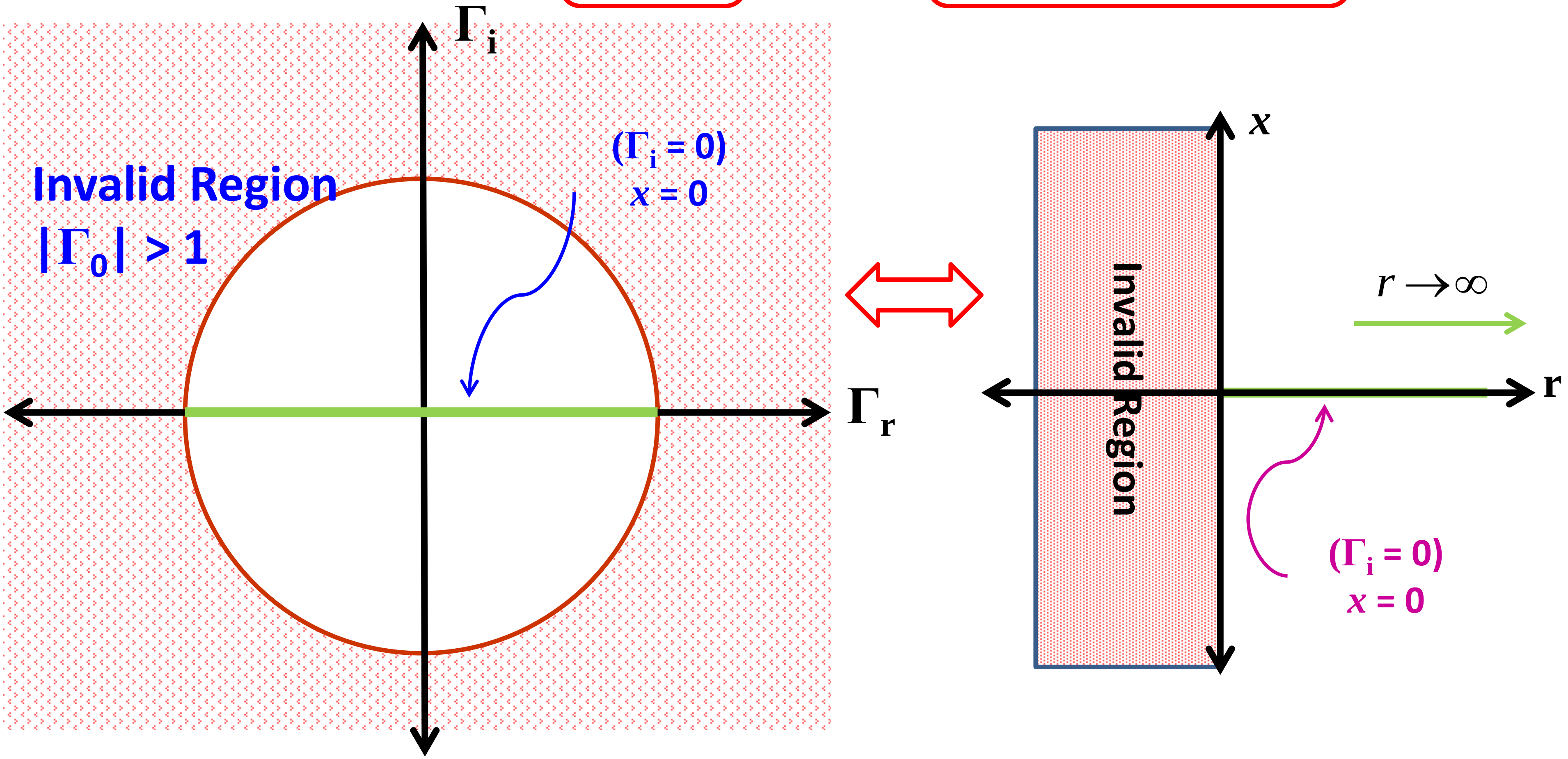


- It is apparent that the normalized impedances can be mapped on complex Γ-plane and vice versa
- It gives us a clue that whole impedance contours (i.e, set of points) can be mapped to complex Γ-plane

Mapping Z to Γ (contd.)

Case-I: Z = R → impedance is purely real

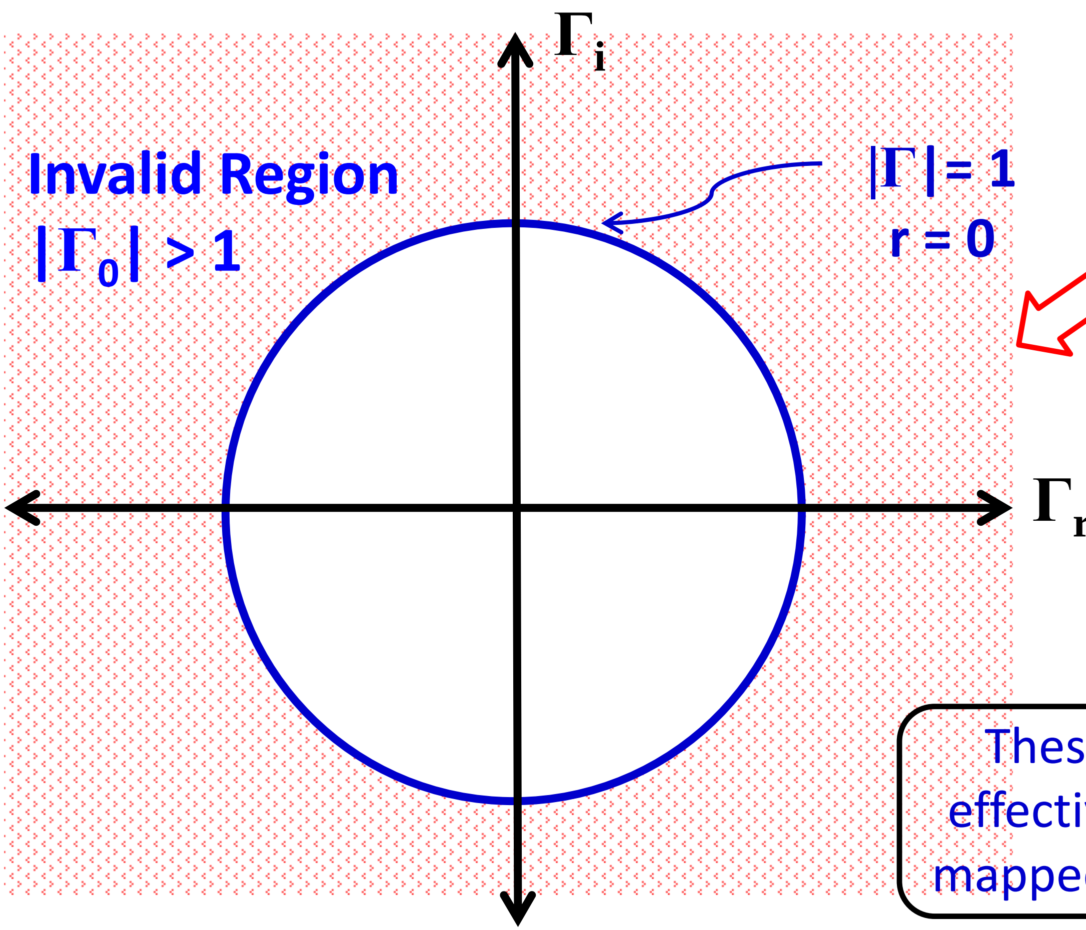
$$z' = r + j0 \;\Rightarrow\; \Gamma = \frac{r-1}{r+1} \;\Rightarrow\; \Gamma_r = \frac{r-1}{r+1} \qquad \Gamma_i = 0$$

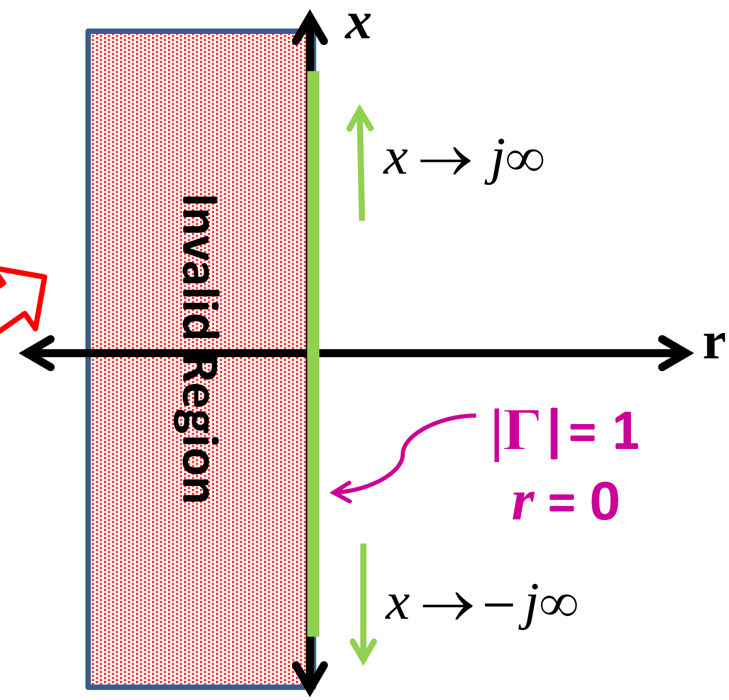


Mapping Z to Γ (contd.)

Case-II: Z = jX → impedance is purely imaginary

$z' = 0 + jx$ ← Purely reactive impedance results in a reflection coefficient with unity magnitude $|\Gamma| = 1$





These cases (I and II) demonstrate that effectively any complex impedance can be mapped to complex Γ-plane → **Smith Chart**

The Smith Chart

In summary

- A vertical line $r = 0$ on complex Z-plane maps to a circle $|\Gamma| = 1$ on the complex Γ-plane
- A horizontal line $x = 0$ on complex Z-plane maps to the line $\Gamma_i = 0$ on the complex Γ-plane

Very fascinating in an academic sense, but are not relevant considering that actual values of impedance generally have both a real and imaginary component

Mappings of more general impedance contours (e.g, $r = 0.5$ and $x = -1.5$ corresponding to normalized impedance $0.5 - j1.5$) can also be mapped

Smith Chart