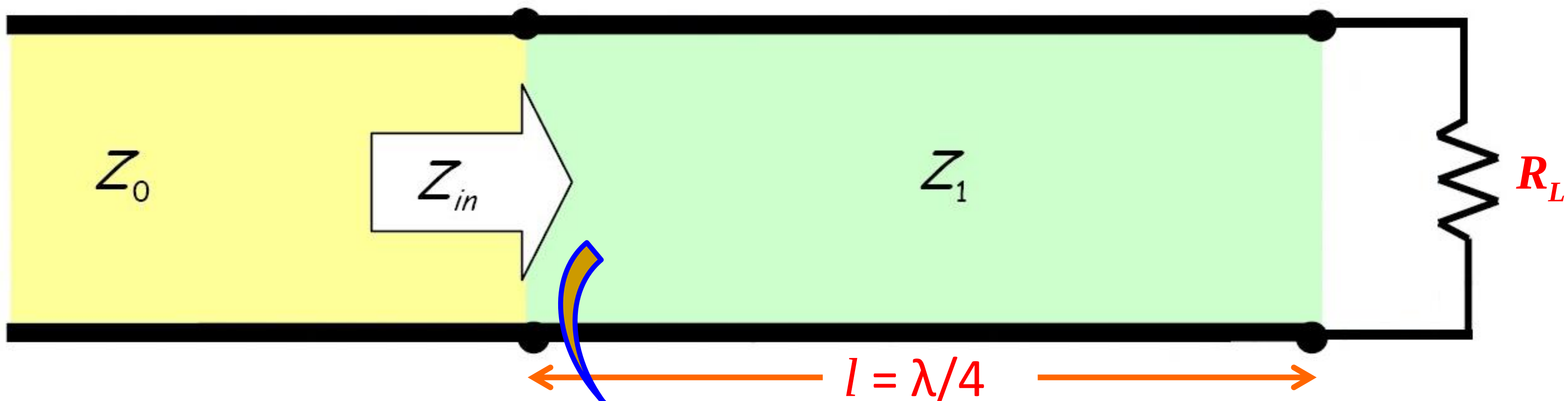
Lecture – 13

Date: 29.02.2016

- Quarter-wave Impedance Transformer
- The Theory of Small Reflections

The Quarter Wave Transformer (contd.)

- The quarter-wave transformer is simply a transmission line with characteristic impedance Z_1 and length $l = \lambda/4$ (i.e., a quarter-wave line).



This λ/4 line **is** the **matching network**!

- the quarter wavelength case is one of the **special** cases that we studied. We know that the **input** impedance of the quarter wavelength line is:

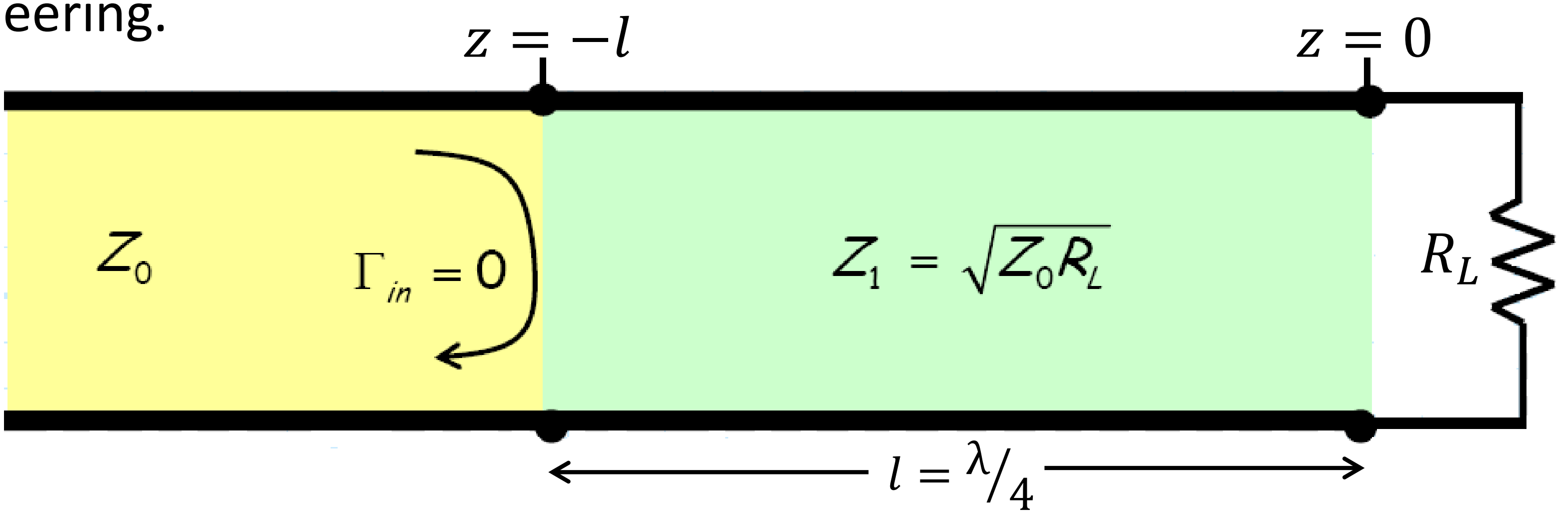
$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

- Solving for Z_1, we find its **required** value to be:

$$Z_1 = \sqrt{Z_0 R_L}$$

Multiple Reflection Viewpoint

- The **quarter-wave** transformer brings up an interesting question in μ-wave engineering.

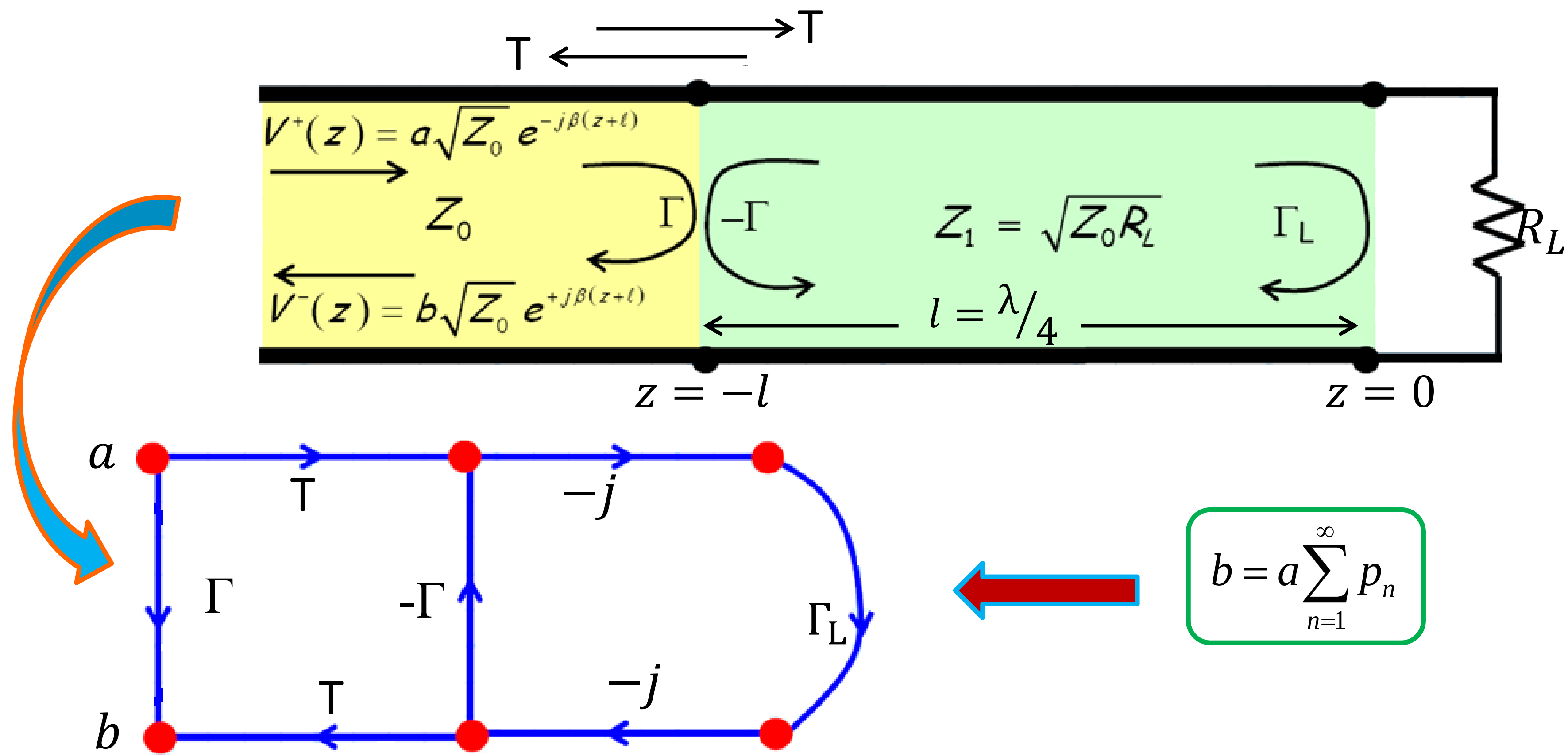


Q: **Why** is there **no** reflection at $z = -l$? It appears that the line is **mismatched** at both $z = 0$ and $z = -l$.

A: In fact there **are** reflections at these mismatched interfaces—an **infinite** number of them!

We can use **signal flow graph** to determine the propagation series, once we determine all the **propagation paths** through the quarter-wave transformer.

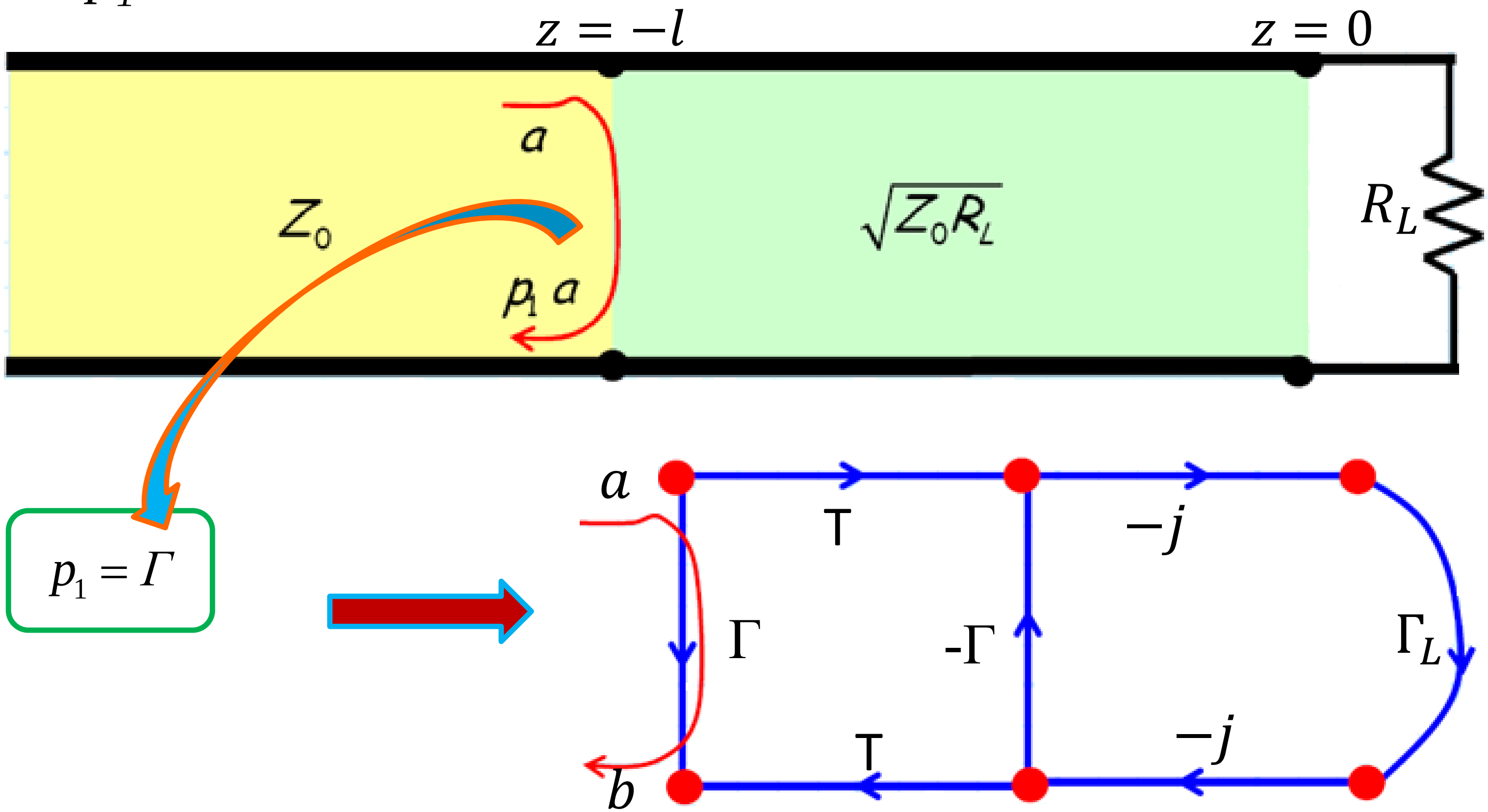
Multiple Reflection Viewpoint (contd.)



- Now, let's try to interpret what **physically** happens when the **incident** voltage wave reaches the interface at $z = -l$.
- We find that there are **two forward paths** through the quarter-wave transformer signal flow graph.

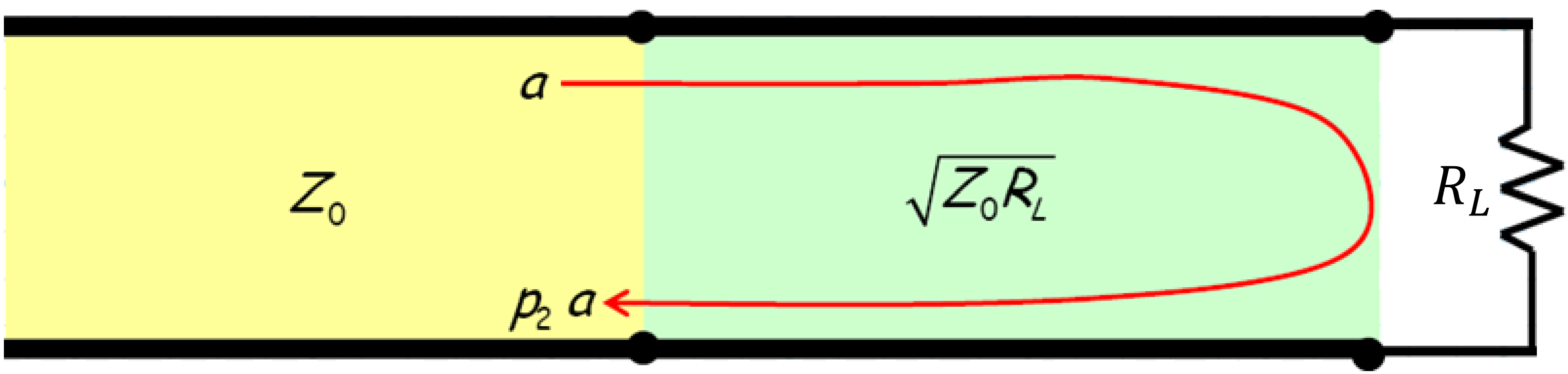
Multiple Reflection Viewpoint (contd.)

Path 1. At $z = -l$, the characteristic impedance of the transmission line changes from Z_0 to Z_1. This mismatch creates a **reflected** wave, with complex amplitude $p_1 a$:



Multiple Reflection Viewpoint (contd.)

Path 2. However, a **portion** of the incident wave is transmitted (T) across the interface at $z = -l$, this wave travels a distance of $\beta l = 90^{\circ}$ to the load at $z = 0$, where a portion of it is reflected (Γ_L). This wave travels back $\beta l = 90^{\circ}$ to the interface at $z = -l$, where a portion is again transmitted (T) across into the Z_0 transmission line—**another** reflected wave !

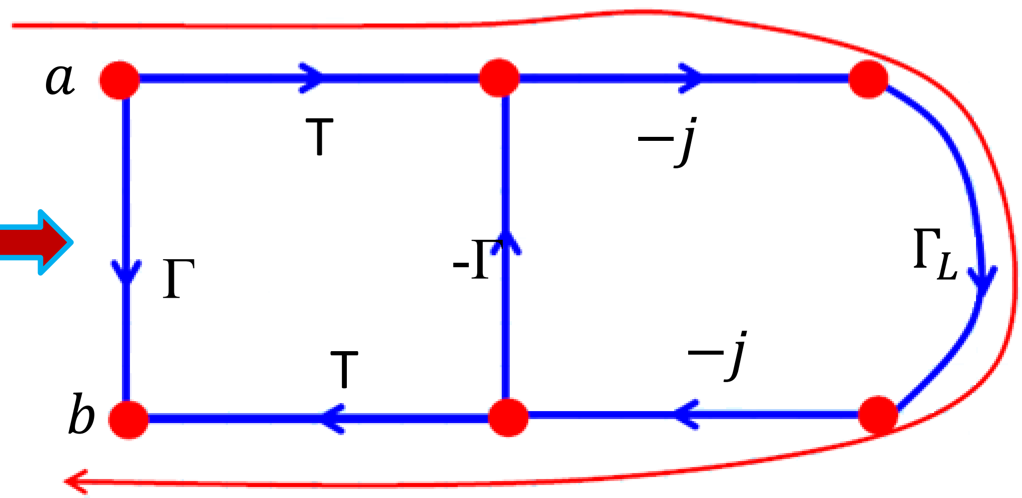


- So the **second direct path** is: $p_2 = Te^{-j90^{\circ}}\Gamma_L e^{-j90^{\circ}}T = -T^2\Gamma_L$

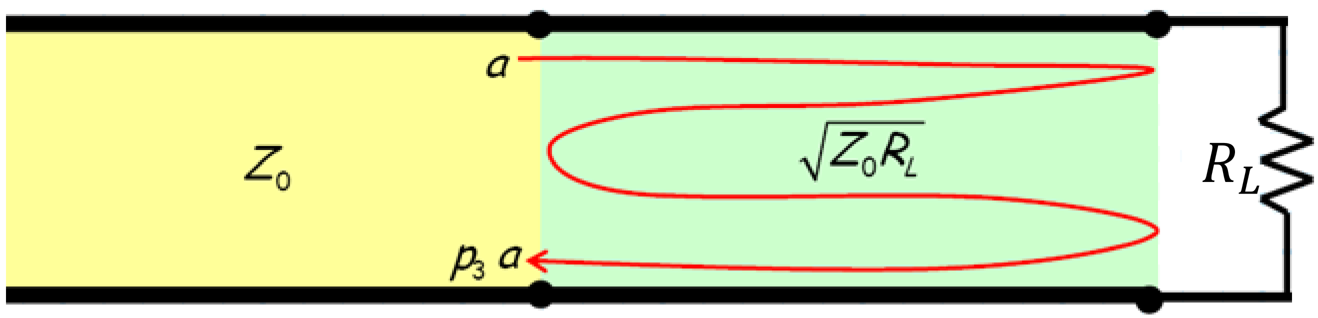
note that traveling $2\beta l = 180^{\circ}$ has produced a **minus** sign in the result.

Multiple Reflection Viewpoint (contd.)

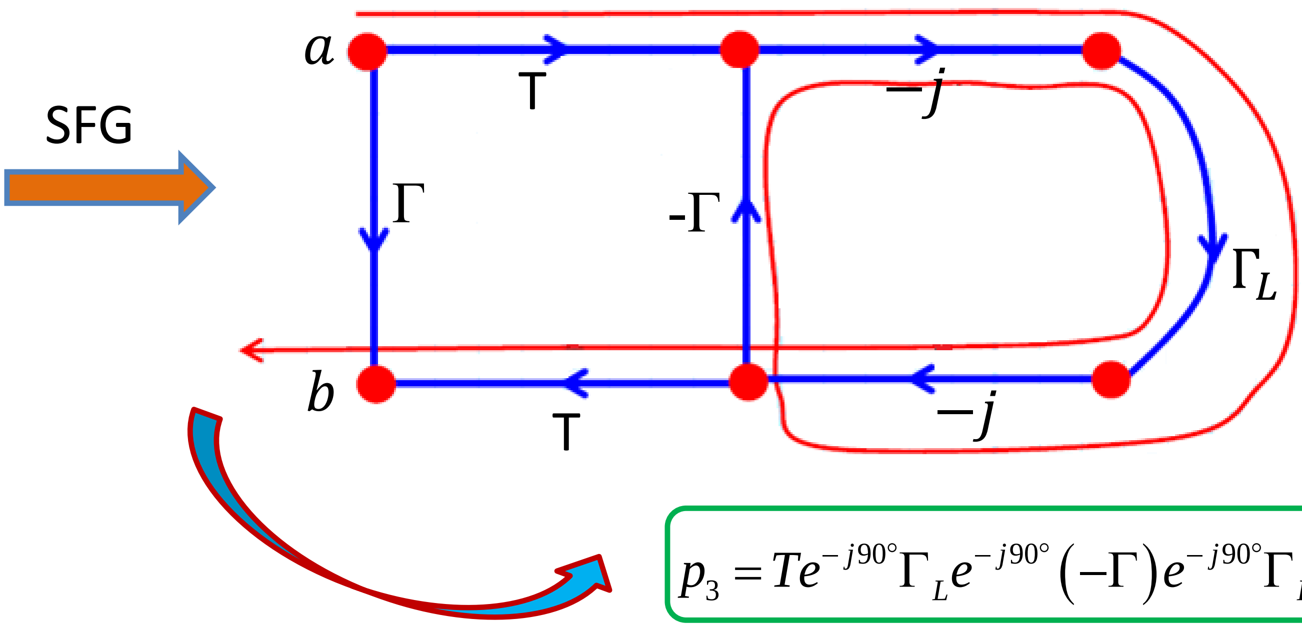
$b = p_2 a$



Path 3. However, a **portion** of this **second** wave is also **reflected** (Γ) back into the Z_1 transmission line at $z = -l$, where it again travels to $\beta l = 90^{\circ}$ the load, is partially reflected (Γ_L), travels $\beta l = 90^{\circ}$ back to $z = -l$, and is partially transmitted into Z_0(T)—our **third** reflected wave!



Multiple Reflection Viewpoint (contd.)



Note that path 3 is **not** a direct path!

$$p_3 = Te^{-j90^\circ}\Gamma_L e^{-j90^\circ}(-\Gamma)e^{-j90^\circ}\Gamma_L e^{-j90^\circ}T = -T^2(\Gamma_L)^2\Gamma$$

Path n. We can see that this "bouncing" back and forth can go on **forever**, with each trip launching a **new** reflected wave into the Z_0 transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

Multiple Reflection Viewpoint (contd.)

Q: But, why then is $\Gamma = 0$?

A: Each reflected wave is a **coherent** wave. That is, they all oscillate at same frequency ω; the reflected waves differ only in terms of their **magnitude** and **phase**.

- Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—this summation results in our **propagation series**, a series that must converge for passive devices.

$$b = a\sum_{n=1}^{\infty} p_n$$

- It can be shown that the infinite propagation series for **this** quarter-wavelength structure **converges** to the closed-form expression:

$$\frac{b}{a} = \sum_{n=1}^{\infty} p_n = \frac{\Gamma - \Gamma^2\Gamma_L - T^2\Gamma_L}{1-\Gamma^2}$$

- Thus, the **input** reflection coefficient is:

$$\Gamma_{in} = \frac{b}{a} = \frac{\Gamma - \Gamma^2\Gamma_L - T^2\Gamma_L}{1-\Gamma^2}$$

- Using our definitions, it can be shown that the **numerator** of this expression is:

$$\Gamma - \Gamma^2\Gamma_L - T^2\Gamma_L = \frac{2\left(Z_1^2 - Z_0 R_L\right)}{\left(Z_1 + Z_0\right)\left(R_L + Z_1\right)}$$

Multiple Reflection Viewpoint (contd.)

- It is evident that the numerator (and therefore Γ_{in}) will be **zero if**:

$$Z_1^2 - Z_0 R_L = 0 \Rightarrow Z_1 = \sqrt{Z_0 R_L}$$

Just as we expected!

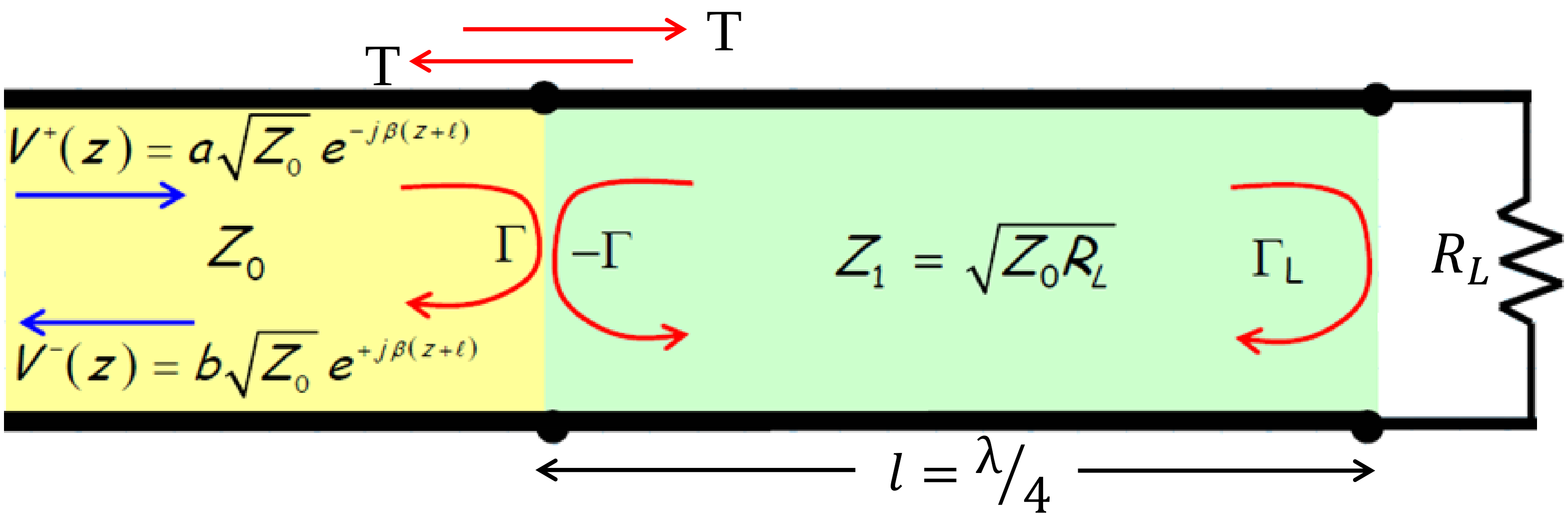
Physically, this result ensures that all the reflected waves add coherently together to produce a **zero value**!

Note **all** of our transmission line analysis has been **steady-state** analysis. We assume our signals are **sinusoidal**, of the form $\exp(j\omega t)$. This signal exists for **all time** t—the signal is assumed to have been "on" **forever**, and assumed to continue on forever.

In other words, in steady-state analysis, **all** the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is **zero**!

The Theory of Small Reflections

- Recall that we analysed a **quarter-wave** transformer using the multiple reflection view point.



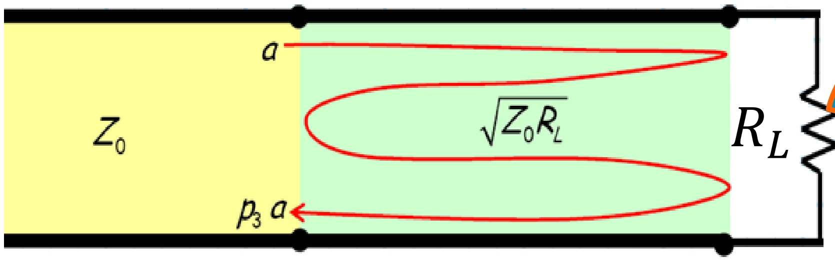
- We found that the solution could be written as an **infinite** summation of terms (the **propagation series**):

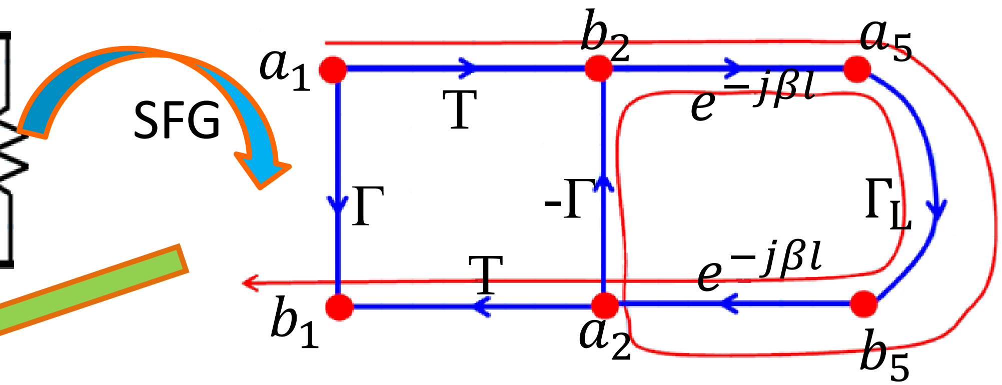
$$b = a\sum_{n=1}^{\infty} p_n$$

where each term had a specific **physical** interpretation, in terms of reflections, transmissions, and propagations.

The Theory of Small Reflections (contd.)

- For example, the **third** term was **path**:





$$p_3 = T^2 (\Gamma_L)^2 \Gamma e^{-j2\beta l}$$

- Now let's consider the **magnitude** of this path:

$$|p_3| = |T|^2 |\Gamma_L|^2 |\Gamma| |e^{-j2\beta l}| \Rightarrow |p_3| = |T|^2 |\Gamma_L|^2 |\Gamma|$$

- Recall that $\Gamma = \Gamma_L$ for a **properly designed** quarter-wave transformer:

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \Rightarrow |p_3| = |T|^2 |\Gamma_L|^3$$

- For the case where values R_L and Z_1 are numerically **"close",** $|R_L - Z_1| \ll |R_L + Z_1|$, the magnitude of the reflection coefficient will be **very** small:

$$|\Gamma_L| = \left|\frac{R_L - Z_1}{R_L + Z_1}\right| \ll 1.0$$

The Theory of Small Reflections (contd.)

- As a result, the value $|\Gamma_L|^3$ will be **very**, **very**, **very** small.
- Moreover, we know (since the connector is **lossless**) that:

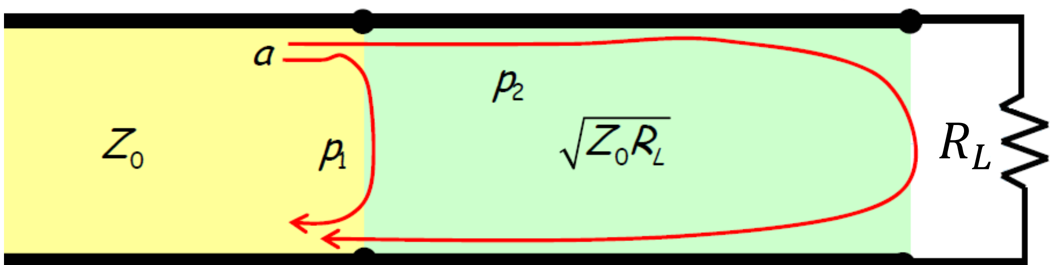
$$|\Gamma|^2 + |T|^2 = |\Gamma_L|^2 + |T|^2 = 1$$

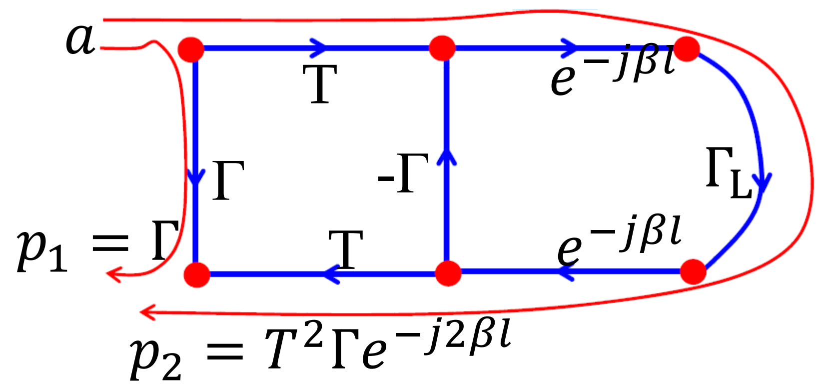
- We can thus conclude that the **magnitude** of path *p_3* is likewise **very**, **very**, **very** small:

$$|p_3| = |T|^2 |\Gamma_L|^3 \approx |\Gamma_L|^3 \ll 1$$

This is a **classic case** where we can approximate the propagation series using only the **forward paths**!!

- Recall there are **two** forward paths:





The Theory of Small Reflections (contd.)

- Therefore **if** Z_0 and R_L are very **close** in value, the **approximate** reflected wave using only the **direct paths** of the infinite series can be found from the SFG:

$$b \simeq (p_1 + p_2)a = \left(\Gamma + T^2\Gamma_L e^{j2\beta l}\right)a$$

- Now, if we likewise apply the **approximation** that $|T| \cong 1.0$, we conclude for this quarter wave transformer (at the design frequency):

$$b \simeq (p_1 + p_2)a = \left(\Gamma + \Gamma_L e^{j2\beta l}\right)a$$

This **approximation**, where we:

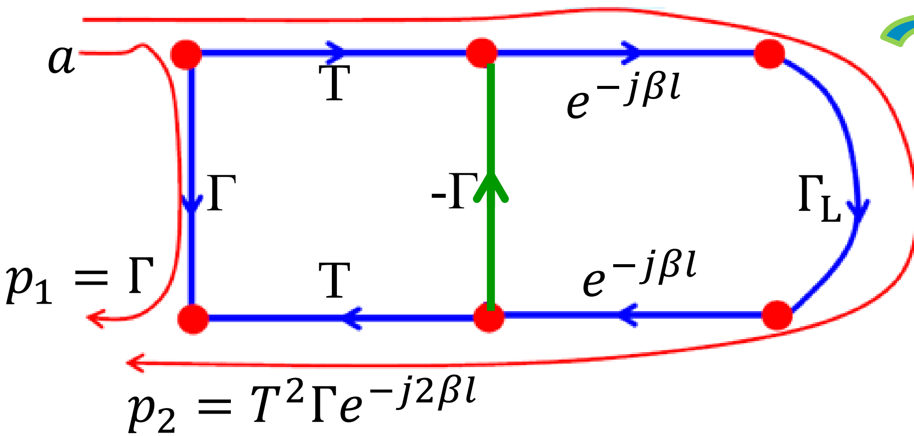
1. use only the **direct paths** to calculate the propagation series,

2. approximate the **transmission** coefficients as **one** (i.e., $|T| = 1.0$).

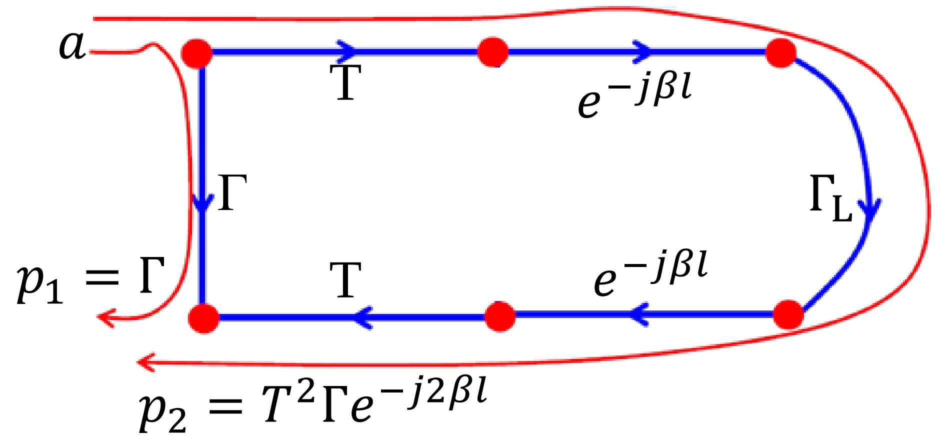
is known as the **Theory of Small Reflections**, and allows us to use the propagation series as an **analysis** tool (we don't have to consider an **infinite** number of terms!).

The Theory of Small Reflections (contd.)

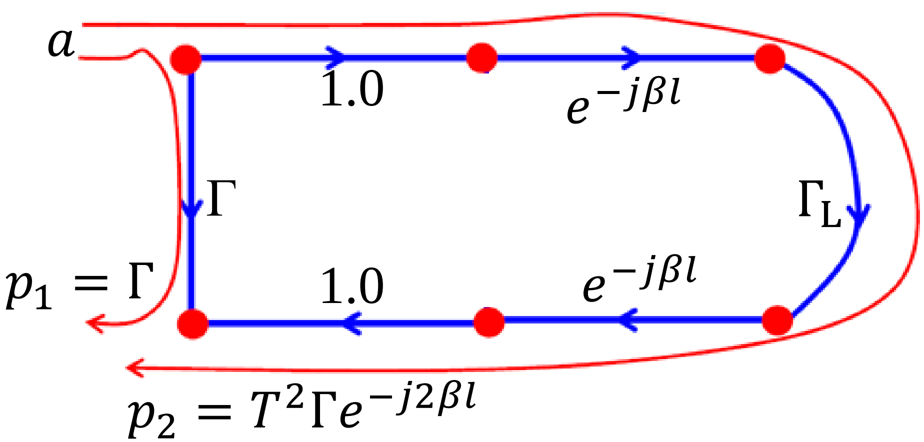
- Consider again the quarter-wave matching network SFG. Note there is **one branch** ($-\Gamma = S_{22}$ of the connector), that is **not included** in either **direct path**.



With respect to the theory of small reflections (where **only** direct paths are considered), this branch can be **removed** from the SFG **without affect**.



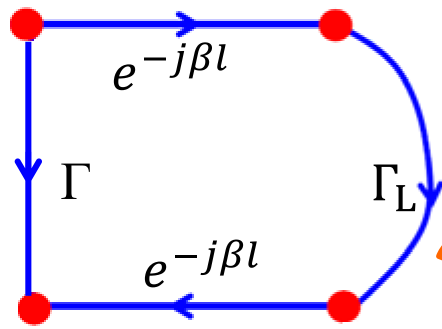
Moreover, the theory of small reflections implements the **approximation**, $|T| = 1.0$, so that the SFG becomes:



The Theory of Small Reflections (contd.)

- **Reducing** this SFG by combining the 1.0 branch and the $e^{-j\beta l}$ branch via the **series rule**, we get the following **approximate** SFG:

The approximate SFG when applying the theory of small reflections !



$$\Gamma_{in} = \frac{b}{a} \approx \Gamma + \Gamma_L e^{-j2\beta l}$$

Note this **approximate** SFG provides **precisely** the results of the theory of small reflections!

Q: But wait! The quarter-wave transformer is a **matching** network, therefore $\Gamma_{in} = 0$. The **theory of small reflections**, however, provides the **approximate result**:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta l}$$

Is this **approximation** very **accurate**? How **close** is this **approximate** value to the correct answer of $\Gamma_{in} = 0$?

The Theory of Small Reflections (contd.)

A: Let's find out!

- Recall that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, and so:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{j2\beta l} = \Gamma_L \left(1 + e^{-j2\beta l}\right)$$

- Likewise, $l = \lambda/4$ (but **only** at the design frequency!) so that:

$$2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where **you** of course recall that $\beta = {}^{2\pi}/_{\lambda}$!

- Thus: $\Gamma_{in} \approx \Gamma + \Gamma_L e^{j2\beta l} = \Gamma_L \left(1 + e^{-j\pi}\right) = \Gamma_L (1-1) = 0$

Q: Wow! The theory of small reflections appears to be a **perfect** approximation—**no error** at all!?!

A: Not so fast.

The Theory of Small Reflections (contd.)

The **theory of small reflections** most definitely provides an **approximate** solution (e.g., it **ignores** most of the terms of the propagation series, and it **approximates** connector transmission as T = 1, when in fact T ≠ 1).

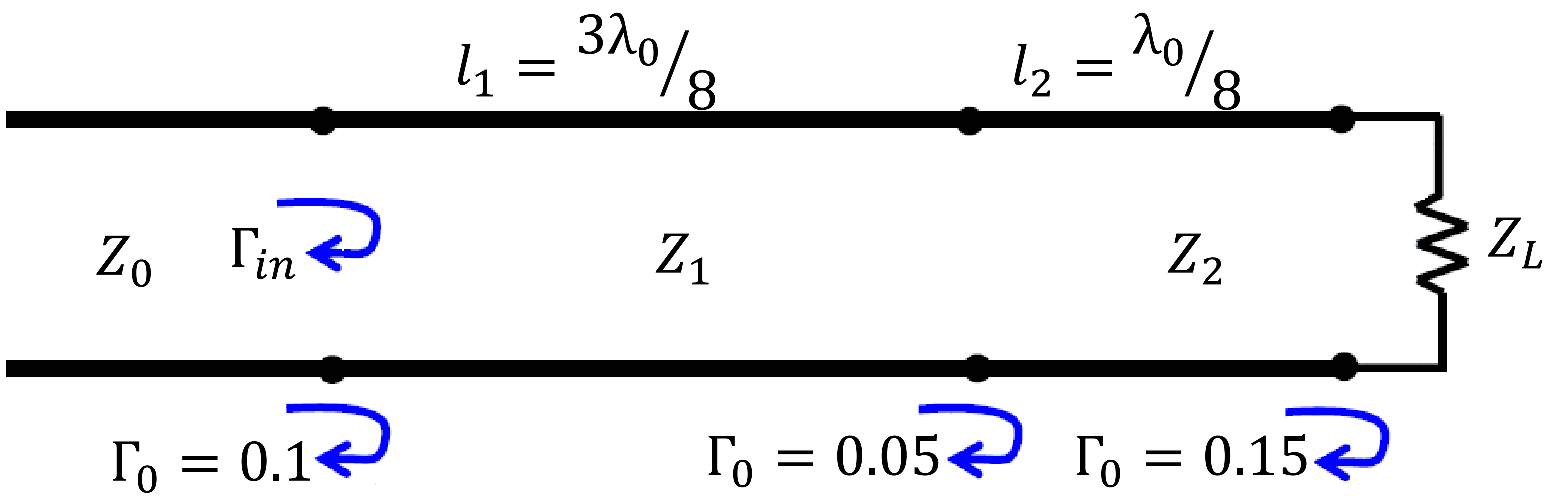
As a result, the solutions derived using the **theory of small reflections** will—generally speaking—exhibit **some** (hopefully small) **error**.

We just got a bit **"lucky"** for the quarter-wave matching network; the "approximate" result $\Gamma_{in} = 0$ was exact for this one case!

➡ The **theory of small reflections** is an **approximate** analysis tool!

Example – 1

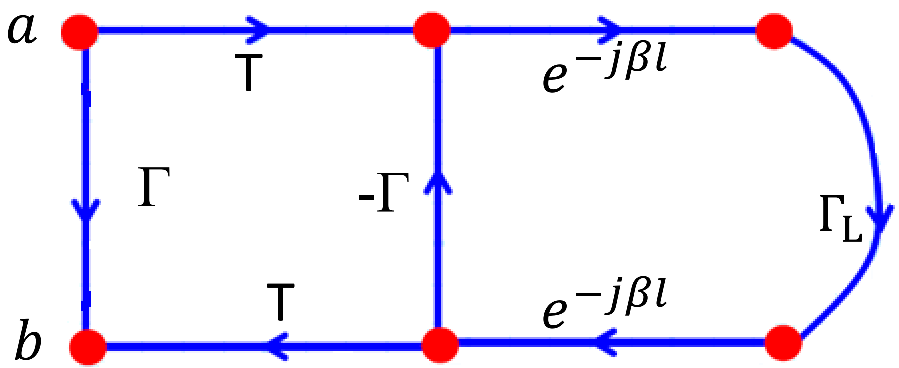
- Use the **theory of small reflections** to determine a **numeric** value for the **input** reflection coefficient Γ_{in}, at the design frequency ω_0.



Note that the transmission line sections have **different lengths**!

Frequency Response of a λ/4 Matching Network

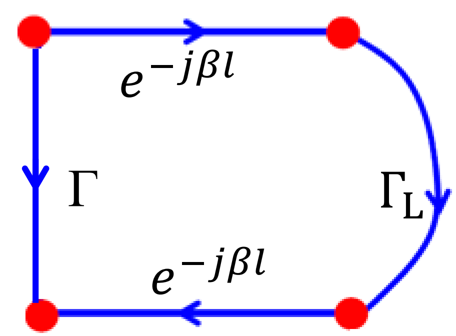
Q: You have once again provided us with **confusing** and perhaps useless information. The quarter-wave matching network has an **exact** SFG of:



Using our **reduction rules**, we can **quickly** conclude that:

$$\Gamma_{in} \doteq \frac{b}{a} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta l}}{1 - \Gamma \Gamma_L}$$

- You could have left this **simple** and **precise** analysis **alone**— BUT **NOOO!!**
- **You** had to foist upon us a long, **rambling** discussion of "the propagation series" and "direct paths" and "the theory of small reflections", culminating with the **approximate** (i.e., less accurate!) SFG:



Freq. Response of a λ/4 Matching Network (contd.)

- From the approximate SFG we were able to conclude the **approximate** (i.e., less accurate!) result:

$$\Gamma_{in} \doteq \frac{b}{a} = \Gamma + \Gamma_L e^{-j2\beta l}$$

The **exact** result was **simple**—and **exact**! **Why** did you make us determine this **approximate** result?

A: In a word: frequency response*.

* OK, **two** words.

the **mathematical form** of the result is much simpler to **analyze** and/or **evaluate** (e.g., no **fractional** terms!).

Q: What exactly would we be analysing and/or evaluating?

A: The **frequency response** of the matching network, for one thing.

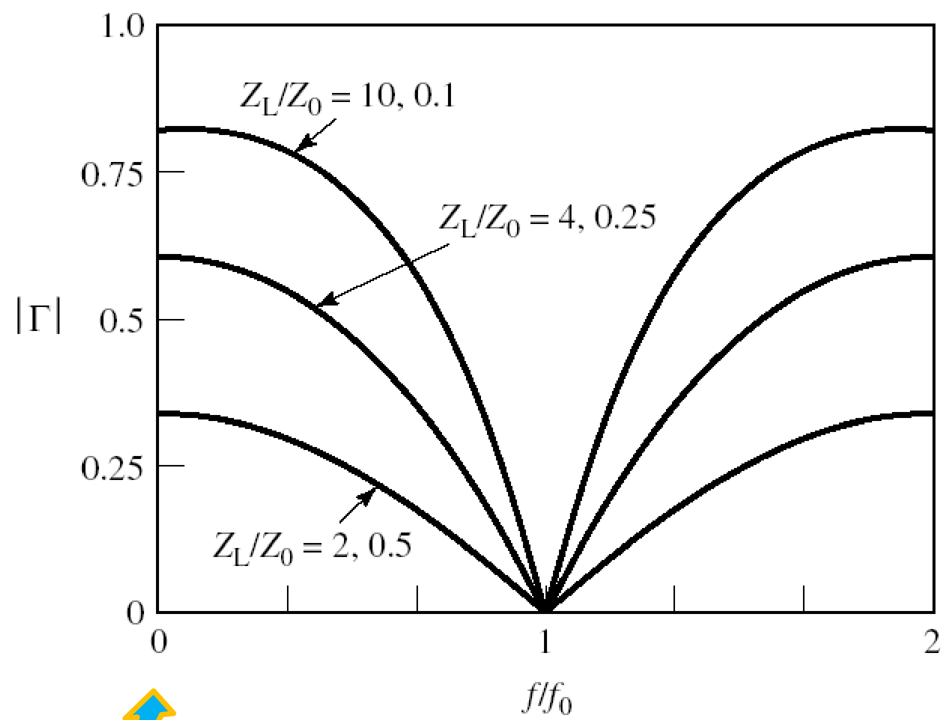
Remember, all matching networks must be **lossless**, and so must be made of **reactive** elements (e.g., lossless transmission lines). The impedance of every reactive element is a **function of frequency**, and so too then is Γ_{in}.

Freq. Response of a λ/4 Matching Network (contd.)

Say we wish to determine **function** $\Gamma_{in}(\omega)$.

Q: Isn't $\Gamma_{in}(\omega) = 0$ for a quarter wave matching network?

A: Oh my gosh **no**! A properly designed matching network will typically result in a perfect match (i.e., $\Gamma_{in}(\omega) = 0$) at **one frequency** (i.e., the design frequency). However, if the signal frequency is **different** from this design frequency, then no match will occur (i.e., $\Gamma_{in}(\omega) \neq 0$).



Recall we discussed this behavior **before**:

Freq. Response of a λ/4 Matching Network (contd.)

Q: But **why** is the result: $$\Gamma_{in} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta l}}{1 - \Gamma \Gamma_L}$$ or its approx form: $$\Gamma_{in} = \Gamma + \Gamma_L e^{-j2\beta l}$$ dependent on **frequency**? I don't **see** frequency variable ***ω*** anywhere in these results!

A: Look **closer**!

- Remember that the value of spatial frequency ***β*** (in radians/meter) is dependent on the frequency ***ω*** of our eigen function (aka "the signal"):

$$\beta = \left(\frac{1}{v_p}\right)\omega$$

where you will recall that v_p is the propagation velocity of a wave moving along a transmission line.

- This velocity is a constant (i.e., $v_p = {}^{1}/_{\sqrt{LC}}$), and so the spatial frequency β is directly proportional to the temporal frequency ω.
- Thus, we can rewrite:

$$\beta l = \frac{\omega l}{v_p} = \omega T$$

Where $T = {}^{l}/_{v_p}$ is the **time** required for the wave to **propagate** a distance l down a transmission line.

Freq. Response of a λ/4 Matching Network (contd.)

- As a result, we can write the input reflection coefficient as a function of **spatial frequency *β***:

$$\Gamma_{in}(\beta) = \Gamma + \Gamma_L e^{-j2\beta l}$$

- Or equivalently as a function of **temporal frequency *ω***:

$$\Gamma_{in}(\omega) = \Gamma + \Gamma_L e^{-j2\omega T}$$

- **Frequently**, the reflection coefficient is simply written in terms of the **electrical length *θ*** of the transmission line, which is simply the **difference in relative phase** between the wave at the beginning and end of the length l of the TL.

$$\beta l = \theta = \omega T$$

- So that:

$$\Gamma_{in}(\theta) = \Gamma + \Gamma_L e^{-j2\theta}$$

Note we can simply insert the value $\theta = \beta l$ into this expression to get $\Gamma_{in}(\beta)$, or insert $\theta = \omega T$ into the expression to get $\Gamma_{in}(\omega)$.

- Now, we know that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, so the reflection coefficient function can be written as:

$$\Gamma_{in}(\theta) = \Gamma_L \left(1 + e^{-j2\theta}\right)$$

Freq. Response of a λ/4 Matching Network (contd.)

- Note that: $1 = e^{j0} = e^{-j(\theta-\theta)} = e^{-j\theta}e^{+j\theta}$
- And that: $e^{-j2\theta} = e^{-j(\theta+\theta)} = e^{-j\theta}e^{-j\theta}$
- And so: $\Gamma_{in}(\theta) = \Gamma_L\left(1 + e^{-j2\theta}\right)$ → $= \Gamma_L\left(e^{-j\theta}e^{+j\theta} + e^{-j\theta}e^{-j\theta}\right)$

  $= \Gamma_L e^{-j\theta}\left(e^{+j\theta} + e^{-j\theta}\right)$ → $= \Gamma_L e^{-j\theta}\left(2\cos\theta\right)$

- Now, **magnitude** of our result is: $|\Gamma_{in}(\theta)| = |\Gamma_L||e^{-j\theta}||2||\cos\theta| = 2|\Gamma_L||\cos\theta|$
- Note: $|\Gamma_{in}(\theta)|$ is **zero-valued** only when $cos\theta = 0$. This of course occurs when $\theta = {}^{\pi}/_{2}$.

$$|\Gamma_{in}(\theta)|_{\theta=\pi/2} = 2|\Gamma_L|\left|\cos\frac{\pi}{2}\right| = 0$$

Q: What the heck does this mean?

A: Remember, $\theta = \beta l$. Thus if $\theta = {}^{\pi}/_{2}$: $l = \frac{\theta}{\beta} = \frac{\pi/2}{2\pi/\lambda} = \frac{\lambda}{4}$

As we (should have) suspected, the match occurs at the frequency whose wavelength is equal to **four times** the matching (Z_1) transmission line length, i.e. $\lambda = 4l$.

Freq. Response of a λ/4 Matching Network (contd.)

In other words, a perfect match occurs at the **frequency** where $l = {}^{\lambda}/_{4}$.

- Note the **physical** length l of the transmission line does **not** change with frequency, but the signal **wavelength** does:

$$\lambda = \frac{v_p}{f}$$

Q: So, at precisely what **frequency** does a quarter-wave transformer with length l provide a **perfect** match?

A: Recall that $\theta = \omega T$, where $T = {}^{l}/_{v_p}$. Thus, for $\theta = {}^{\pi}/_{2}$:

$$\theta = \frac{\pi}{2} = \omega T \quad \Rightarrow \quad \omega = \frac{\pi}{2}\frac{1}{T} = \frac{\pi}{2}\frac{v_p}{l}$$

- This frequency is called the **design frequency** of the matching network—it's the frequency where a **perfect** match occurs. We denote this as frequency ω_0, which has wavelength λ_0, i.e.:

$$\omega_0 = \frac{\pi}{2T} = \pi\frac{v_p}{2l} \quad \Rightarrow \quad \omega_0 = \frac{\pi}{2T} = \pi\frac{v_p}{2l} \quad \Rightarrow \quad \omega_0 = \frac{\pi}{2T} = \pi\frac{v_p}{2l}$$

Freq. Response of a λ/4 Matching Network (contd.)

- Given this, yet **another way** of expressing $\theta = \beta l$ is:

$$\theta = \beta l = \frac{\omega}{v_p}\left(\pi \frac{v_p}{2\omega_0}\right) = \pi \frac{\omega}{2\omega_0} = \pi \frac{f}{2 f_0}$$

- Thus, we conclude:

$$\left|\Gamma_{in}(\mathrm{f})\right| = 2\left|\Gamma_L\right|\left|\cos\left(\pi \frac{f}{2 f_0}\right)\right|$$

This expression helps in the determination (approximately) of the **bandwidth** of the quarter-wave transformer!

- First, we must **define** what we mean by bandwidth. Say the **maximum** acceptable level of the reflection coefficient is value Γ_m. This is an arbitrary value, set by **you** the microwave engineer (typical values of Γ_m range from 0.05 to 0.2).
- Let us denote the frequencies where this maximum value Γ_m occurs f_m. In other words:

$$\left|\Gamma_{in}(\mathrm{f} = \mathrm{f}_m)\right| = \Gamma_m = 2\left|\Gamma_L\right|\left|\cos\left(\pi \frac{f_m}{2 f_0}\right)\right|$$

Freq. Response of a λ/4 Matching Network (contd.)

- There are **two solutions** to this equation, the first is:

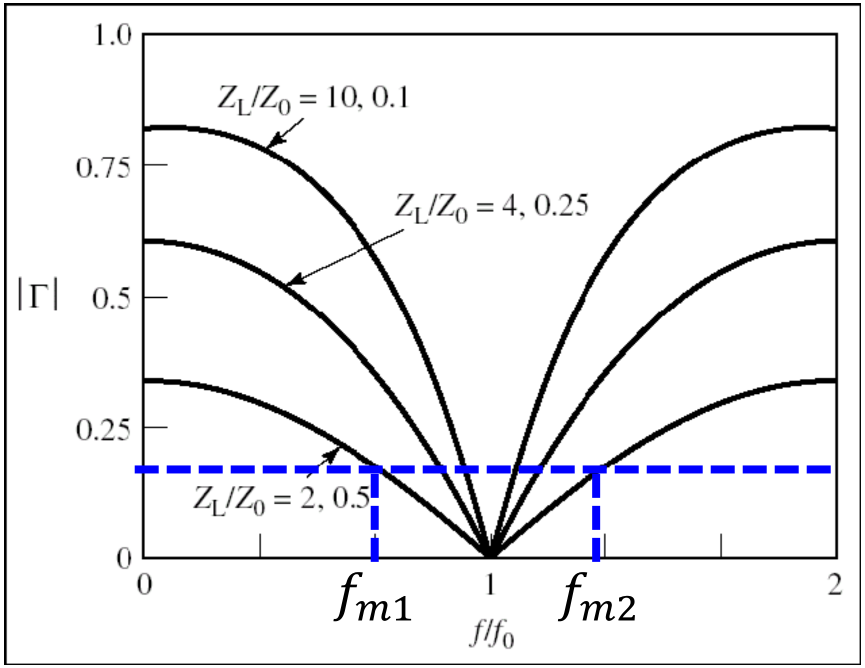
$$f_{m1} = \frac{2f_0}{\pi}\cos^{-1}\left(\frac{\Gamma_m}{2|\Gamma_L|}\right)$$

- And the second:

$$f_{m2} = \frac{2f_0}{\pi}\cos^{-1}\left(-\frac{\Gamma_m}{2|\Gamma_L|}\right)$$

Important note! Make sure $cos^{-1}x$ is expressed in **radians**!

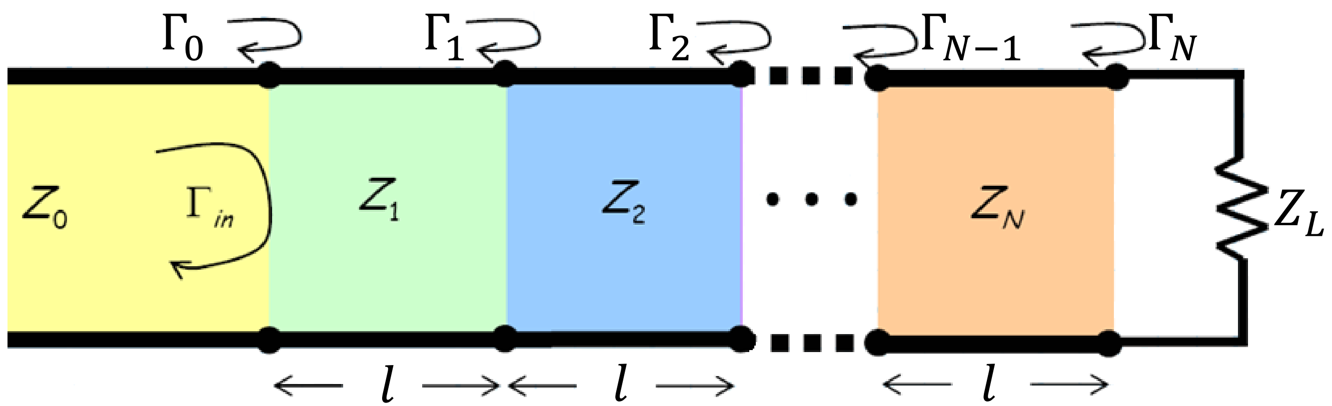
- You will find that $f_{m1} < f_0 < f_{m2}$. So the values f_{m1} and f_{m2} define the **lower** and **upper** limits on matching network **bandwidth**.



All this analysis was brought to you by the **"simple" mathematical form** of $\Gamma_{in}(f)$ that resulted from the theory of small reflections!

The Multi-section Transformer

- Consider a sequence of N transmission line **sections**; each section has **equal length** l, but **dissimilar** characteristic impedances:



- Where the marginal reflection coefficients are:

$$\Gamma_0 \doteq \frac{Z_1 - Z_0}{Z_1 + Z_0} \qquad \Gamma_n \doteq \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \qquad \Gamma_N \doteq \frac{Z_L - Z_N}{Z_L + Z_N}$$

- If the load resistance R_L is **less** than Z_0, then we should design the transformer such that:

$$Z_0 > Z_1 > Z_2 > Z_3 > \cdots > Z_N > R_L$$

- Conversely, if R_L is **greater** than Z_0, then we will design the transformer such that:

$$Z_0 < Z_1 < Z_2 < Z_3 < \cdots < Z_N < R_L$$

The Multi-section Transformer (contd.)

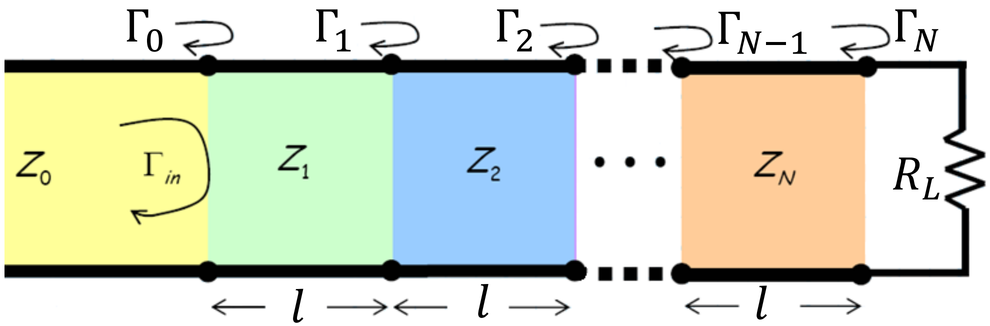
In other words, we **gradually transition** from Z_0 to R_L!

Note that since R_L is **real**, and since we assume **lossless** transmission lines, all Γ_n will be **real** (this is important!).

- Likewise, since we **gradually** transition from one section to another, each value: $Z_{n+1} - Z_n$ will be **small**.
- As a result, each marginal reflection coefficient Γ_n will be **real** and have a **small** magnitude.

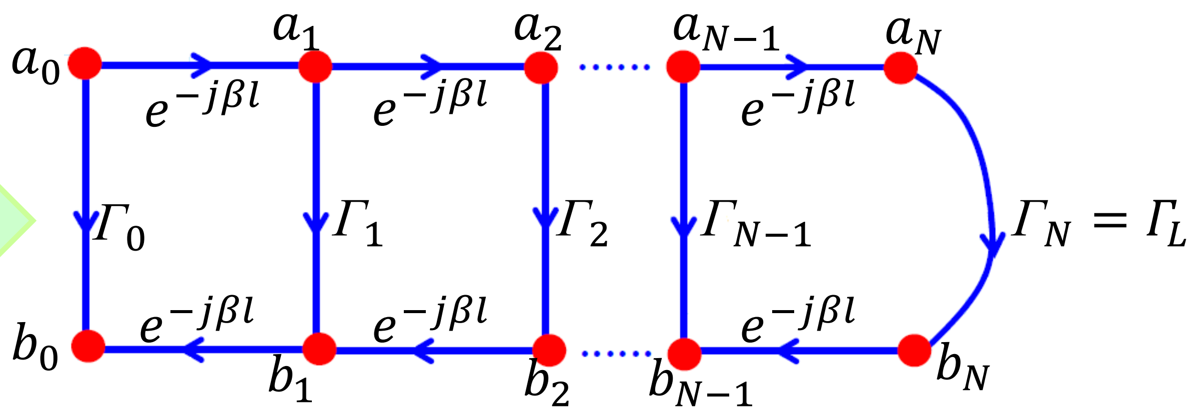
This is also **important**, as it means that we can apply the "**theory of small reflections**" to analyse this multi-section transformer!

- The theory of small reflections allows us to **approximate** the input reflection coefficient of the transformer as:



The Multi-section Transformer (contd.)

The approximate SFG when applying the theory of small reflections!



$$\frac{b_0}{a_0} = \Gamma_{in}(\beta) \Rightarrow \simeq \Gamma_0 + \Gamma_1 e^{-j2\beta l} + \Gamma_2 e^{-j4\beta l} + \ldots + \Gamma_N e^{-j2N\beta l} \Rightarrow = \sum_{n=0}^{N} \Gamma_n e^{-j2n\beta l}$$

- We can alternatively express the input reflection coefficient as a function of **frequency** ($\beta l = \omega T$):

$$\Gamma_{in}(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \ldots + \Gamma_N e^{-j2N\omega T} \Rightarrow = \sum_{n=0}^{N} \Gamma_n e^{-j(2nT)\omega}$$

where: $T = \dfrac{l}{v_p} \leftarrow propagation\ time\ through\ 1\ section$

The Multi-section Transformer (contd.)

- We see that the function $\Gamma_{in}(\omega)$ is expressed as a **weighted** set of N **basis functions**! i.e.,

$$\Gamma_{in}(\omega) = \sum_{n=0}^{N} c_n \Psi(\omega)$$

$$c_n = \Gamma_n$$

$$\Psi(\omega) = e^{-j(2nT)\omega}$$

- We find, therefore, that by **selecting** the proper values of basis weights c_n(i.e., the proper values of reflection coefficients Γ_n), we can **synthesize** any function $\Gamma_{in}(\omega)$ of frequency ω, provided that:
  1. $\Gamma_{in}(\omega)$ is **periodic** in $\omega = {}^{1}/_{2T}$.
  2. we have sufficient **number** of sections N.

Q: What function **should** we synthesize?

A: Ideally, we would want to make $\Gamma_{in}(\omega) = 0$ (i.e., the reflection coefficient is zero for all frequencies).

Bad News: this **ideal** function $\Gamma_{in}(\omega) = 0$ would require an **infinite** number of sections (i.e., $N = \infty$)!

The Multi-section Transformer (contd.)

Therefore, we seek to find an "**optimal**" function for $\Gamma_{\text{in}}(\omega)$, given a **finite** number of N elements.

Once we determine these optimal functions, we can find the values of coefficients Γ_n (or equivalently, Z_n) that will result in a matching transformer that exhibits this **optimal** frequency response.

- To **simplify** this process, we can make the transformer **symmetrical**, such that:

$$\Gamma_0 = \Gamma_N, \qquad \Gamma_1 = \Gamma_{N-1}, \qquad \Gamma_2 = \Gamma_{N-2}, \qquad \ldots\ldots$$

Note: this **does NOT** mean that:

$$Z_0 = Z_N, \qquad Z_1 = Z_{N-1}, \qquad Z_2 = Z_{N-2}, \qquad \ldots\ldots$$

- We then find that:

$$\boldsymbol{\Gamma(\omega) = e^{-jN\omega T}\left[\Gamma_0\left(e^{jN\omega T} + e^{-jN\omega T}\right) + \Gamma_1\left(e^{j(N-2)\omega T} + e^{-j(N-2)\omega T}\right) + \Gamma_2\left(e^{j(N-4)\omega T} + e^{-j(N-4)\omega T}\right) + \cdots\right]}$$

The Multi-section Transformer (contd.)

- and since: $e^{jx} + e^{-jx} = 2\cos(x)$
- we can write for N **even**: $\Gamma(\omega) = 2e^{-jN\omega T}\left[\Gamma_0 cos N\omega T + \Gamma_1 \cos(N-2)\omega T + \cdots + \Gamma_n \cos(N\right.$
- whereas for N **odd**: $\Gamma(\omega) = 2e^{-jN\omega T}\left[\Gamma_0 cos N\omega T + \Gamma_1 \cos(N-2)\omega T + \cdots + \Gamma_n \cos(N\right.$

The remaining **question** then is this: given an optimal and realizable function $\Gamma_{in}(\omega)$, **how** do we determine the necessary number of **sections** N, and **how** do we determine the **values** of all reflection coefficients Γ_n??

Multi-section transformer is often used to maximize the bandwidth of transformer.

The Multi-section Transformer (contd.)

Alternatively, we can say that one way to **maximize bandwidth** is to construct a multi-section matching network with a function $\Gamma(f)$ that is either **maximally flat** or can be considered flat **albeit with pass-band ripple**.

Binomial Function satisfies the condition of maximum flatness

Chebyshev Polynomial can be considered fla**t with pass-band ripple**

The Binomial Multi-Section Transformer

- Recall that a **multi-section matching network** can be described using the theory of small reflections as:

$$\Gamma_{in}(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \ldots + \Gamma_N e^{-j2N\omega T} = \sum_{n=0}^{N} \Gamma_n e^{-j(2nT)\omega}$$

where: $T = \dfrac{l}{v_p} \leftarrow propagation\ time\ through\ 1\ section$

Note that for a multi-section transformer, we have N **degrees of design freedom**, corresponding to the N characteristic impedance values Z_n.

Q: What should the values of Γ_n (i.e., Z_n) be?

A: We need to define N independent **design equations**, which we can then use to solve for the N values of **characteristic impedance** Z_n.

- First, we start with a single **design frequency** ω_0, where we wish to achieve a **perfect** match:

$$\Gamma_{in}(\omega = \omega_0) = 0$$

That's just one design equation: we need N -1 more!

- These addition equations can be selected using **many** criteria—one such is to make the function $\Gamma_{in}(\omega)$ **maximally flat** at the point $\omega = \omega_0$.

The Binomial Multi-Section Transformer (contd.)

- To accomplish this, we first consider the **Binomial Function**:

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

- This function has the desirable **properties** that:

$$\Gamma\left(\theta = \frac{\pi}{2}\right) = A\left(1 + e^{-j\pi}\right)^N = A(1-1)^N = 0$$

- and that:

$$\left.\frac{d^n\Gamma(\theta)}{d\theta^n}\right|_{\theta = \pi/2} = 0 \qquad for \quad n = 1,2,3,\ldots,N-1$$

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

In other words, this Binomial Function is **maximally flat** at the point $\theta = \pi/2$, where it has a value of $\Gamma(\theta = \pi/2) = 0$.

Q: So? What does **this** have to do with our multi-section matching network?

A: Let's **expand** (multiply out the N identical product terms) the Function:

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N \Rightarrow = A\left(C_0^N + C_1^N e^{-j2\theta} + C_2^N e^{-j4\theta} + C_3^N e^{-j6\theta} + \ldots + C_N^N e^{-j2N\theta}\right)$$

where: $C_n^N \doteq \dfrac{N!}{(N-n)!\,n!}$

The Binomial Multi-Section Transformer (contd.)

- it is obvious the two functions have **identical** forms, **provided** that:

$$\Gamma_n = AC_n^N \qquad \omega T = \theta$$

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

Moreover, we find that this function is very **desirable** from the standpoint of the a matching network. Recall that $\Gamma(\theta) = 0$ at $\theta = \pi/2$ —a **perfect** match!

Additionally, function is **maximally flat** at $\theta = \pi/2$, therefore $\Gamma(\theta) \approx 0$ over a wide range around $\theta = \pi/2$ — a **wide bandwidth**!

Q: But how does $\theta = \pi/2$ relate to frequency ω?

A: Remember that $\omega T = \theta$, so $\theta = \pi/2$ corresponds to the frequency:

$$\omega_0 = \frac{1}{T}\frac{\pi}{2} = \frac{v_p}{l}\frac{\pi}{2}$$

This frequency (ω_0) is therefore our **design** frequency—the frequency where we have a **perfect** match.

- Note that the length l has an interesting **relationship** with this frequency:

$$l = \frac{v_p}{\omega_0}\frac{\pi}{2} = \frac{1}{\beta_0}\frac{\pi}{2} = \frac{\lambda_0}{2\pi}\frac{\pi}{2} = \frac{\lambda_0}{4}$$

The Binomial Multi-Section Transformer (contd.)

- **Binomial** Multi-section matching network will have a **perfect** match at the frequency where the section lengths l are a **quarter wavelength**!

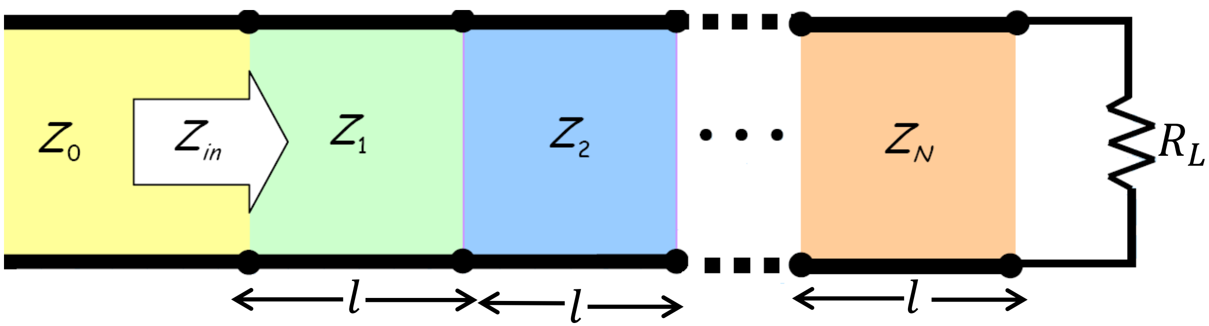
Thus, we have our **first design rule**:

Set section lengths l so that they are a **quarter-wavelength** $\left(\lambda_0/_4\right)$ at the design frequency ω_0.

Q: I see! And then we select all the values Z_n such that $\Gamma_n = AC_n^N$. But wait! **What** is the value of **A** ??

A: We can determine this value by evaluating a **boundary condition!**

- Specifically, we can **easily** determine the value of $\Gamma(\omega)$ at ω = 0.



- As ω approaches **zero**, the electrical length βl of each section will **likewise** approach zero. Thus, the input impedance Z_{in} will simply be equal to $\boldsymbol{R_L}$ as $\omega \rightarrow 0$.

The Binomial Multi-Section Transformer (contd.)

- As a result, the input reflection coefficient $\Gamma(\omega = 0)$ **must** be:

$$\Gamma(\omega=0) = \frac{Z_{in}(\omega=0) - Z_0}{Z_{in}(\omega=0) + Z_0} = \frac{R_L - Z_0}{R_L + Z_0}$$

- However, we **likewise** know that:

$$\Gamma(0) = A\left(1 + e^{-j2(0)}\right)^N = A(1+1)^N = A2^N$$

- **Equating** the two expressions:

$$A2^N = \frac{R_L - Z_0}{R_L + Z_0}$$

- therefore:

$$A = 2^{-N}\frac{R_L - Z_0}{R_L + Z_0}$$

(A can be negative!)

- We now have a formulation to calculate the **required marginal reflection coefficients** Γ_n:

$$\Gamma_n = AC_n^N = \frac{AN!}{(N-n)!n!} = 2^{-N}\frac{R_L - Z_0}{R_L + Z_0}\frac{N!}{(N-n)!n!}$$

we **also** know that these marginal reflection coefficients are physically related to the characteristic impedances of each section as:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$$

The Binomial Multi-Section Transformer (contd.)

- Equating the two and solving, we find that that the section characteristic impedances must satisfy:

$$Z_{n+1} = Z_n \frac{1+\Gamma_n}{1-\Gamma_n} = Z_n \frac{1+AC_n^N}{1-AC_n^N}$$

Note this is an **iterative** procedure—we determine Z_1 from Z_0, Z_2 from Z_1, and so forth.

Q: This result **appears** to be our second design equation.

A: Alas, there is a **big problem** with this result.

- Note that there are N+1 coefficients Γ_n (i.e., n∈{0,1,…,N}) in the Binomial series, yet there are only N design degrees of freedom (i.e., there are only N transmission line sections!).
- Thus, our design is a bit **over constrained**, a result that manifests itself the finally marginal reflection coefficient Γ_N.

- Note from this iterative solution, the **last** transmission line impedance Z_N is selected to satisfy the **mathematical** requirement of the **penultimate** reflection coefficient Γ_{N-1}.

$$\Gamma_{N-1} = \frac{Z_N - Z_{N-1}}{Z_N + Z_{N-1}} = AC_{N-1}^N$$

The Binomial Multi-Section Transformer (contd.)

- Therefore the last impedance must be: $$Z_N = Z_{N-1}\frac{1+AC_{N-1}^N}{1-AC_{N-1}^N}$$
- But there is **one more** mathematical requirement! The last marginal reflection coefficient **must** likewise satisfy: $$\Gamma_N = AC_N^N = 2^{-N}\frac{R_L - Z_0}{R_L + Z_0}$$

where we use the fact that ${C_N}^N = 1$.

But, we **selected** Z_N to satisfy the requirement for Γ_{N-1},—we have no **physical** design parameter to satisfy this last **mathematical** requirement for Γ_N!

- As a result, we find to our great consternation that the last requirement is not satisfied: $$\Gamma_N = \frac{R_L - Z_N}{R_L + Z_N} \neq AC_N^N$$

Q: Yikes! Does this mean that the resulting matching network will **not** have the desired Binomial frequency response?

A: That's **exactly** what it means!

The Binomial Multi-Section Transformer (contd.)

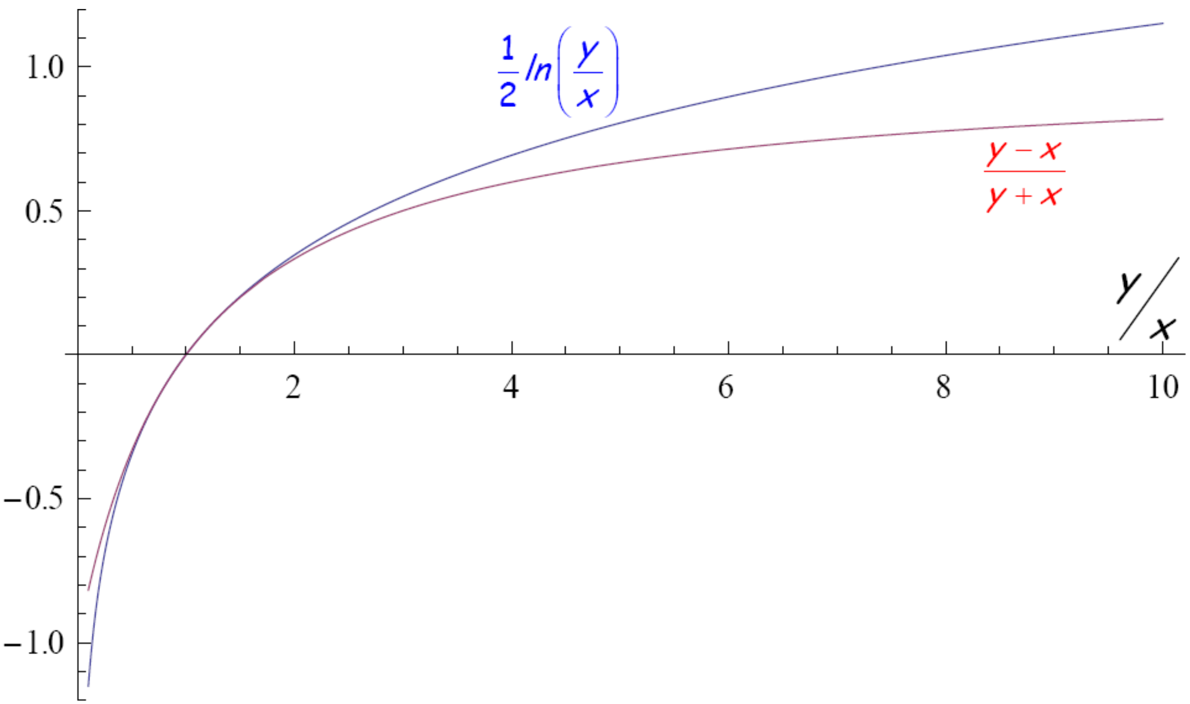
Q: You big #%@#$%&!!!! **Why** did you **waste** all my time discussing an over-constrained design problem that can't be built?

A: Relax; there is a **solution** to our dilemma—albeit an **approximate** one.

- **You** undoubtedly have previously used the **approximation**:

$$\frac{y-x}{y+x} \approx \frac{1}{2}\ln\left(\frac{y}{x}\right)$$

This approximation is especially **accurate** when y−x is small (i.e., when $y/x \approx 1$).



The Binomial Multi-Section Transformer (contd.)

- Now, we know that the values of Z_{n+1} and Z_n in a multi-section matching network are typically **very close**, such that $|Z_{n+1} - Z_n|$ **is** small.
- Thus, we use the approximation:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \approx \frac{1}{2}\ln\left(\frac{Z_{n+1}}{Z_n}\right)$$

- Likewise, we can **also** apply this approximation (although not as accurately) to the value of A:

$$A = 2^{-N}\frac{R_L - Z_0}{R_L + Z_0} \approx 2^{-(N+1)}\ln\left(\frac{R_L}{Z_0}\right)$$

- let's **start over**, this time we'll use these **approximations.** First, determine A:

$$A \approx 2^{-(N+1)}\ln\left(\frac{R_L}{Z_0}\right)$$

(A can be negative!)

- Now use this result to calculate the **mathematically required** marginal reflection coefficients Γ_n:

$$\Gamma_n = AC_n^N = \frac{AN!}{(N-n)!n!}$$

The Binomial Multi-Section Transformer (contd.)

- Of course, we **also** know that these marginal reflection coefficients are physically related to the characteristic impedances of each section as:

$$\Gamma_n \approx \frac{1}{2}\ln\left(\frac{Z_{n+1}}{Z_n}\right)$$

- Equating the two and solving, we find that that the section characteristic impedances **must** satisfy:

$$Z_{n+1} = Z_n \exp\left[2\Gamma_n\right]$$

This is our **second design rule.** Note it is an **iterative** rule—we determine Z_1 from Z_0, Z_2 from Z_1, and so forth.

Q: Huh? How is this any better? How does applying **approximate** math lead to a **better** design result??

A: Applying these approximations help resolve our over constrained problem. Recall that the over-constraint resulted in:

$$\Gamma_N = \frac{R_L - Z_N}{R_L + Z_N} \neq A C_N^N$$

The Binomial Multi-Section Transformer (contd.)

- But, as it turns out, the approximations leads to the happy situation where:

$$\Gamma_N \approx \frac{1}{2}\ln\left(\frac{R_L}{Z_N}\right) = AC_N^N$$

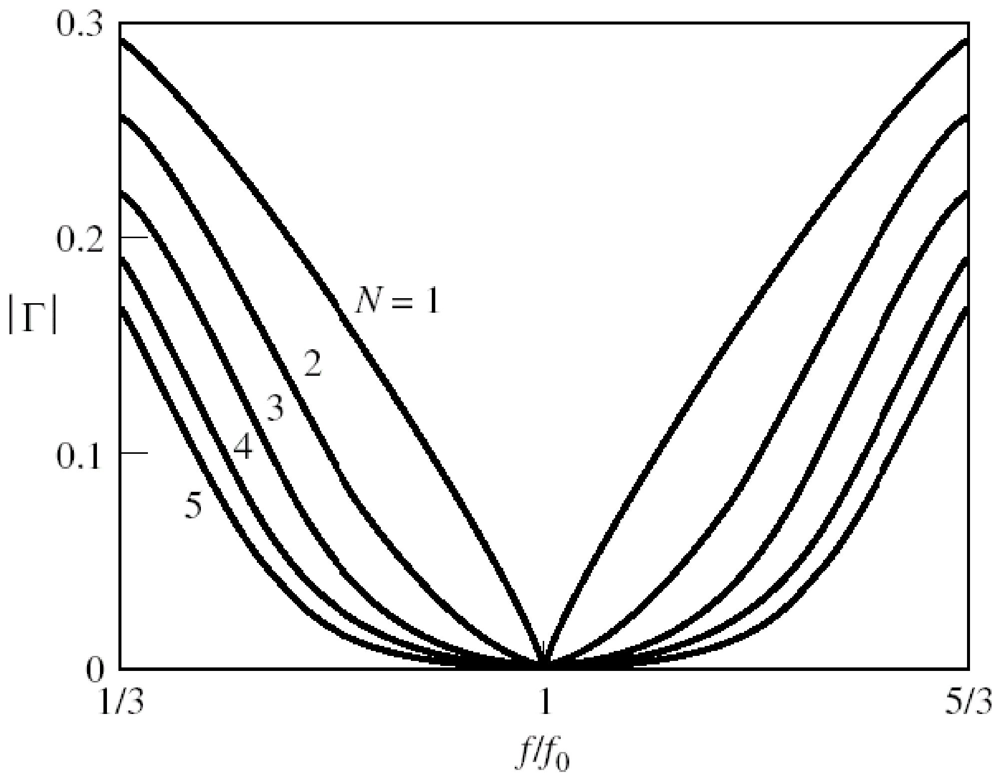
provided that the value A is the approximation as well.

- Effectively, these approximations couple the results, such that each value of characteristic impedance Z_n **approximately** satisfies both Γ_n and Γ_{n+1}. Summarizing:

  a. If you use the **"exact"** design equations to determine the characteristic impedances Z_n, the last value Γ_n will exhibit a significant numeric error, and your design **will not** appear to be maximally flat.

  b. If you instead use the **"approximate"** design equations to determine the characteristic impedances Z_n, all values Γ_n will exhibit a slight error, but the resulting design **will** appear to be **maximally flat, Binomial** reflection coefficient function $\Gamma(\omega)$!

The Binomial Multi-Section Transformer (contd.)

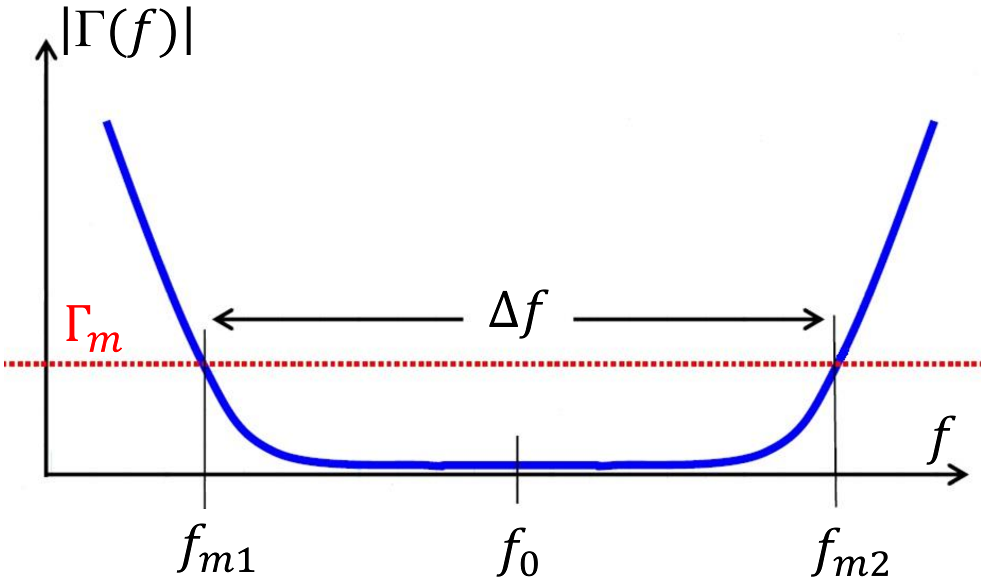


Note that as we **increase** the number of **sections**, the matching **bandwidth** increases.

Q: Can we determine the **value** of this bandwidth?

A: Sure! But we first must **define** what we mean by bandwidth.

The Binomial Multi-Section Transformer (contd.)



- As we move from the design (perfect match) frequency f_0 the value $|\Gamma(f)|$ will **increase**. At some frequency (say, f_m) the magnitude of the reflection coefficient will increase to some **unacceptably** high value (say, Γ_m). At that point, we **no longer** consider the device to be matched.

- Note there are **two** values of frequency f_m —one value **less** than design frequency f_0, and one value **greater** than design frequency f_0. These two values define the **bandwidth** Δf of the matching network:

$$\Delta f = f_{m2} - f_{m1} = 2\left(f_0 - f_{m1}\right) = 2\left(f_{m2} - f_0\right)$$

The Binomial Multi-Section Transformer (contd.)

Q: So what is the **numerical** value of Γ_m?

A: I don't know—it's up to **you** to decide!

Every engineer must determine what **they** consider to be an acceptable match (i.e., decide Γ_m). This decision depends on the **application** involved, and the **specifications** of the overall microwave system being designed.

However, we **typically** set Γ_m to be 0.2 or less.

Q: OK, after we have selected Γ_m, can we determine the **two** frequencies f_m?

A: Sure! We just have to do a little **algebra**.

- We start by **rewriting** the Binomial function:

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N \rightarrow = Ae^{-jN\theta}\left(e^{+j\theta} + e^{-j\theta}\right)^N \rightarrow = Ae^{-jN\theta}\left(2\cos\theta\right)^N$$

- Now, we take the **magnitude** of this function:

$$\left|\Gamma(\theta)\right| = 2^N\left|A\right|\left|e^{-jN\theta}\right|\left|\cos\theta\right|^N \rightarrow \left|\Gamma(\theta)\right| = 2^N\left|A\right|\left|\cos\theta\right|^N$$

The Binomial Multi-Section Transformer (contd.)

- Now, we **define** the values θ where $|\Gamma(\theta)| = \Gamma_m$ as θ_m. i.e., :

$$\Gamma_m = \left|\Gamma\left(\theta = \theta_m\right)\right| = 2^N |A| |\cos\theta_m|^N$$

- We can now solve for θ_m(in **radians**!) in terms of Γ_m:

$$\theta_{m1} = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

$$\theta_{m2} = \cos^{-1}\left[-\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

Note that there are **two solutions** (one **less** than $\pi/_2$ and one **greater** than $\pi/_2$)!

- Now, we can convert the values of θ_m into specific frequencies.
- Recall that ωT =θ, therefore:

$$\omega_m = \frac{1}{T}\theta_m = \frac{v_p}{l}\theta_m$$

The Binomial Multi-Section Transformer (contd.)

- But recall also that $l = {}^{\lambda_0}/_{4}$, where λ_0 is the wavelength at the **design frequency** f_0(not f_m!), and where $\lambda_0 = {}^{v_p}/_{f_0}$.
- Thus we can conclude:

$$\omega_m = \frac{v_p}{l}\theta_m = \frac{4v_p}{\lambda_0}\theta_m = (4f_0)\theta_m \Rightarrow f_m = \frac{\omega_m}{2\pi} = \frac{(2f_0)\theta_m}{\pi}$$

where θ_m is expressed in **radians.**

- Therefore:

$$f_{m1} = \frac{2f_0}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right] \qquad f_{m2} = \frac{2f_0}{\pi}\cos^{-1}\left[-\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

- Thus, the **bandwidth** of the binomial matching network can be determined as:

$$\Delta f = 2(f_0 - f_{m1}) = 2f_0 - \frac{4f_0}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

Note that this equation can be used to determine the **bandwidth** of a binomial matching network, given Γ_m and number of sections N.

The Binomial Multi-Section Transformer (contd.)

$$\Delta f = 2\left(f_0 - f_{m1}\right) = 2f_0 - \frac{4f_0}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

It can also be used to determine the **number of sections** N required to meet a specific bandwidth requirement!

- Finally, we can list the **design steps** for a binomial matching network:

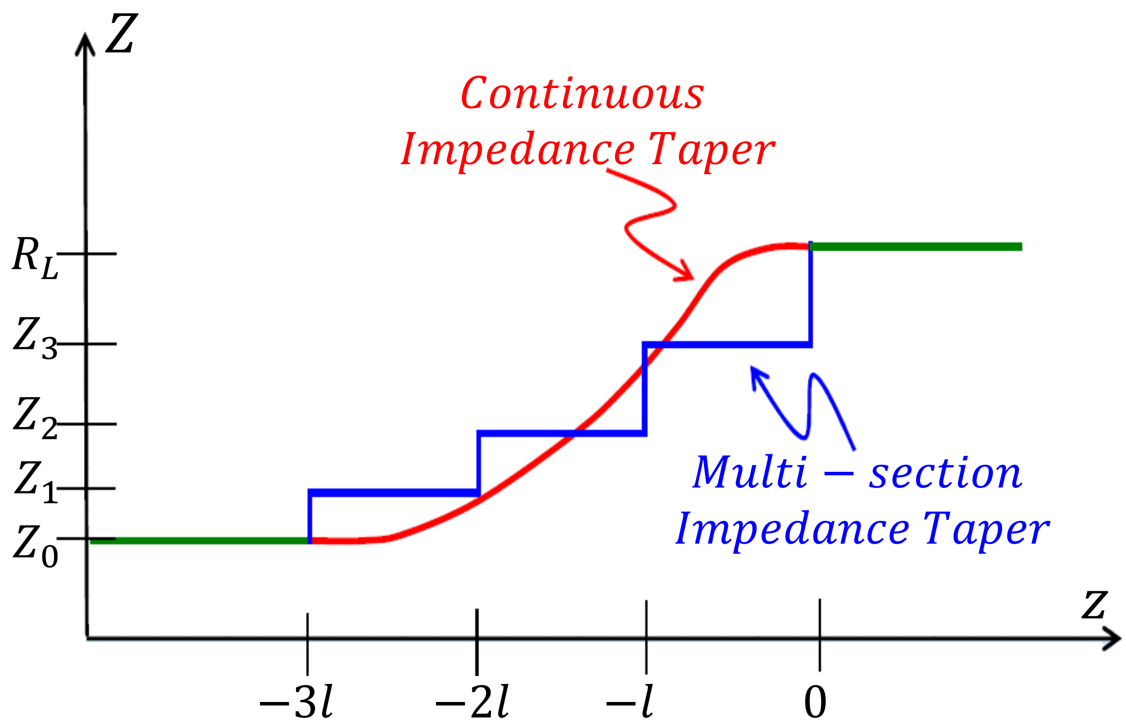
1. **Determine** the value N required to meet the bandwidth (Δf and Γ_m) requirements.
2. Determine the **approximate** value A from Z_0, R_L and N.
3. Determine the **marginal reflection coefficients** $\Gamma_n = AC_n{}^N$ required by the **binomial** function.
4. Determine the characteristic impedance of each section using the **iterative approximation**: $Z_{n+1} = Z_n exp[2\Gamma_n]$.
5. Perform the **sanity check**: $\Gamma_N \approx \frac{1}{2} ln\left(\frac{R_L}{Z_n}\right) = AC_n{}^N$.
6. Determine section **length** $l = {}^{\lambda_0}/_4$ for design frequency f_0.

Chebyshev Multi-section Matching Transformer

Self Study

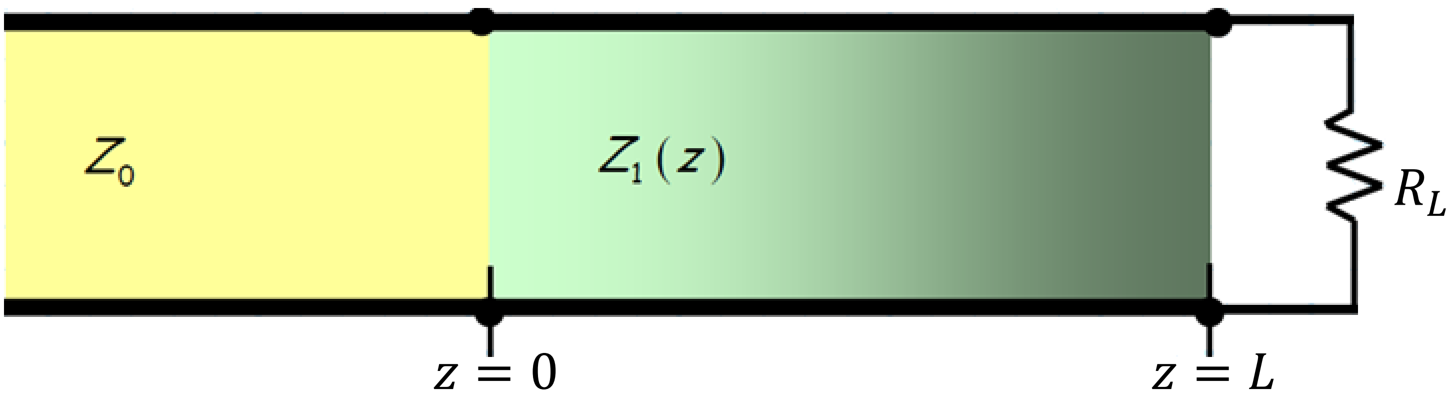
Tapered Lines

- We can also build matching networks where the characteristic impedance of a transmission line changes **continuously** with position z.
- We call these matching networks **tapered lines**.
- Note **all** our multi-section transformer designs have involved a **monotonic** change in characteristic impedance, from Z_0 to R_L (e.g., $Z_0 < Z_1 < Z_2 < \cdots < R_L$).
- Now, instead of having a **stepped** change in characteristic impedance as a function of position z (i.e., a multi-section transformer), we can also design matching networks with **continuous tapers.**



Tapered Lines (contd.)

- A tapered impedance matching network is defined by **two** characteristics—its **length** L and its taper **function** $Z_1(z)$.



There are of course an **infinite** number of possible functions $Z_1(z)$. Your book discusses **three**: the **exponential** taper, the **triangular** taper, and the **Klopfenstein** taper.

Tapered Lines (contd.)

- For example, the **exponential** taper has the form:

$$Z_1(z) = Z_0 e^{az} \qquad 0 < z < L$$

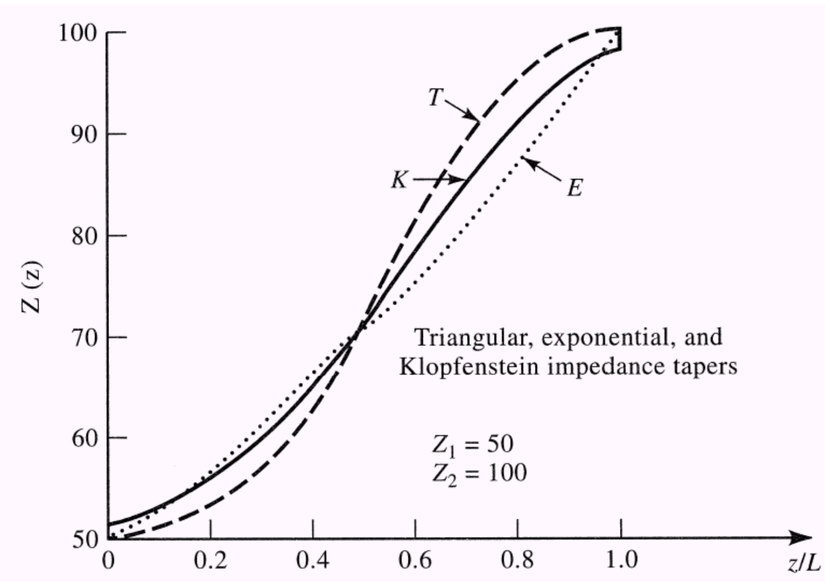
- where:

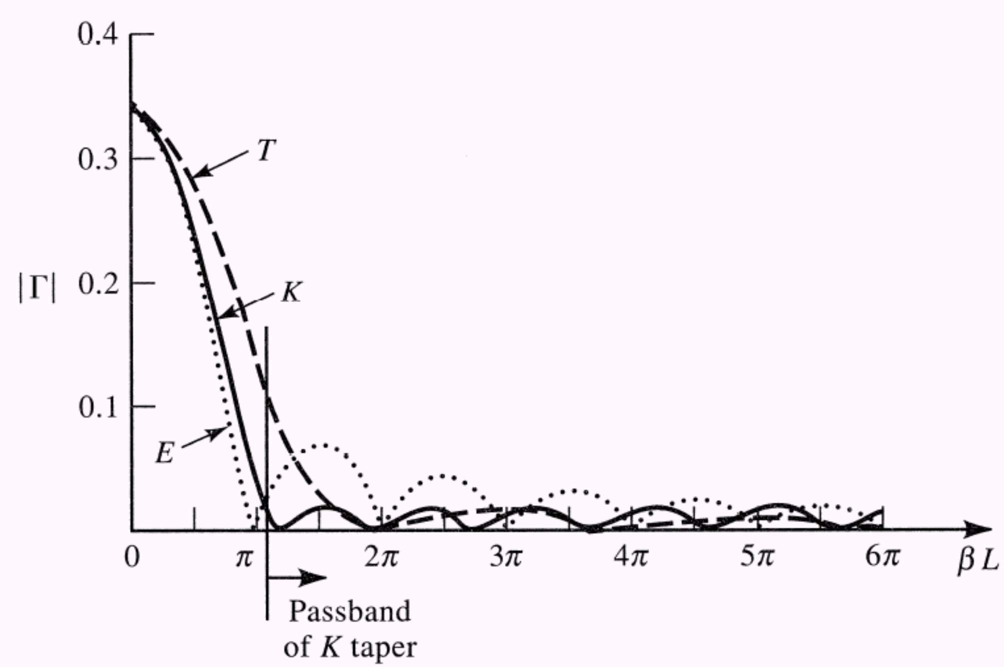
$$a = \frac{1}{L}\ln\left(\frac{Z_L}{Z_0}\right)$$

Note for the exponential taper, we get the **expected** result that $Z_1(z=0) = Z_0$ and $Z_1(z=L) = R_L$.

Recall the **bandwidth** of a multi-section matching transformer **increases** with the **number** of sections. Similarly, the bandwidth of a tapered line will typically **increase** as the **length** L is increased.

Tapered Lines (contd.)





Impedance variations for the triangular, exponential, and Klopfenstein tapers.

Resulting reflection coefficient magnitude versus frequency for the tapers

Tapered Lines (contd.)

Q: But how can we **physically** taper the characteristic impedance of a transmission line?

A: Most tapered lines are implemented in **stripline** or **microstrip**. As a result, we can modify the characteristic impedance of the transmission line by simply tapering the **width** W of the conductor (i.e., $W(z)$).

In other words, we can **continuously** increase or decrease the **width** of the microstrip or stripline to create the **desired** impedance taper $Z_1(z)$.