Lecture – 12

Date: 11.02.2016

- Nodal Quality Factor
- T- and Pi- Matching Networks
- Microstrip Matching Networks
- Series- and Shunt-stub Matching
- Quarter Wave Impedance Transformer

Forbidden Region, Frequency Response, and Quality Factor

Self Study - Section 8.1.2 in the Text Book

- The L-type matching networks can be considered as resonance circuits with f_0 being the resonance frequency.
- These networks can be described by a loaded quality factor, Q_L, given by:

$$Q_L = \frac{f_0}{BW}$$

Similarity to bandpass filter

- However, analysis of matching circuit based on bandpass filter concept is complex $\rightarrow$ In addition, it only allows approximate estimation of the bandwidth.
- More simpler and accurate method is design and analysis through the use of nodal quality factor, Q_n.

Nodal Quality Factor

- During L-type matching network analysis it was apparent that at each node the impedance can be expressed in terms of equivalent series impedance $Z_s = R_s + jX_s$ or admittance $Y_P = G_P + jB_P$.
- Therefore, at each node we can define Q_n as the ratio of the absolute value of reactance X_s to the corresponding resistance R_s.

$$Q_n = \frac{|X_S|}{R_S}$$

- Similarly,

$$Q_n = \frac{|B_P|}{G_P}$$

- The "nodal quality factor" and loaded quality factor are related as:

$$Q_L = \frac{Q_n}{2}$$

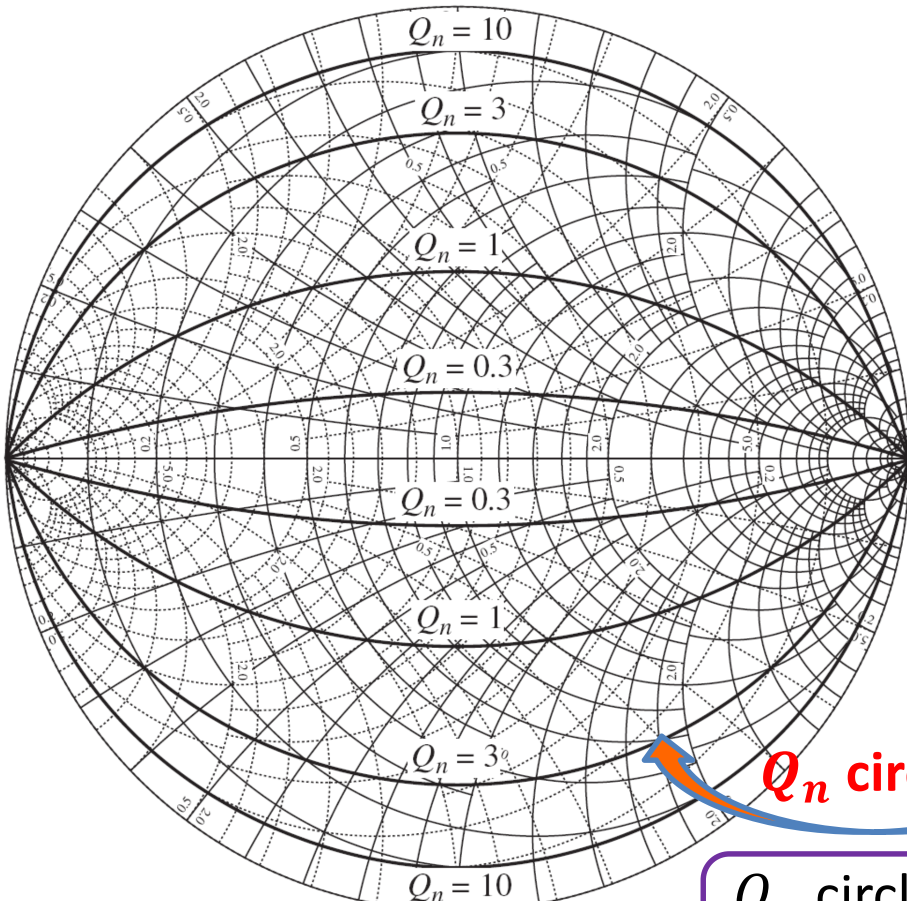
True for any L-type matching network

For more complicated networks, $Q_L = Q_n$

- Bandwidth of the matching network can be easily estimated once the "nodal quality factor" is known.

Nodal Quality Factor (contd.)

- To simplify the matching network design process even further, we can draw constant Q_n contours in the Smith chart.



We know the Smith chart expression:

$$z = r + jx = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} + j\frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

- The expression for Q_n:

$$Q_n = \frac{|x|}{r} = \frac{2|\Gamma_i|}{1-\Gamma_r^2-\Gamma_i^2}$$

$$\Gamma_i^2 + \left(\Gamma_r \pm \frac{1}{Q_n}\right)^2 = 1 + \frac{1}{Q_n^2}$$

Q_n circles!

Q_n circles in the Smith chart help in direct determination of Q_L.

Nodal Quality Factor (contd.)

- Once you go through section 8.1.2, it will be apparent that quality factor of matching network is extremely important.
- For example, broadband amplifier requires matching circuit with low-Q. Whereas oscillators require high-Q networks to eliminate undesired harmonics in the output signal.
- It will also be apparent that L-type matching networks have no control over the values of Q_n → **Limitation!!!**
- To gain more **freedom in choosing the values of Q or Q_n**, another element in the matching network is incorporated → results in T- or Pi-network

T- and Pi- Matching Networks

- The knowledge of nodal quality factor (Q_n) of a network enables estimation of loaded quality factor → hence the Band Width (BW).
- The addition of third element into the matching network allows control of Q_L by choosing an appropriate intermediate impedance.

Example – 1

- Design a T-type matching network that transforms a load impedance Z_L = (60 – j30)Ω into a Z_{in} = (10 + j20)Ω input impedance and that has a maximum **Q_n of 3**. Compute the values for the matching network components, assuming that matching is required at f = 1GHz.

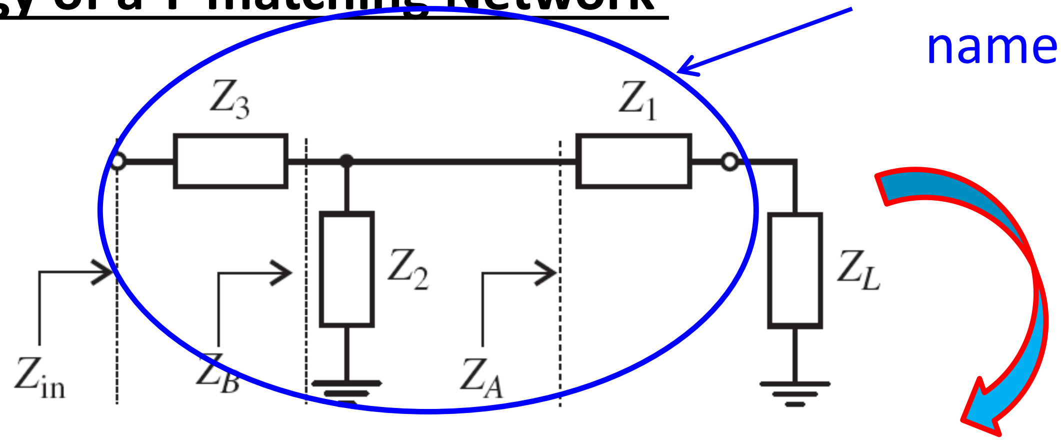
Solution

- Several possible configurations! Let us focus on just one!

Example – 1 (contd.)

General Topology of a T-matching Network

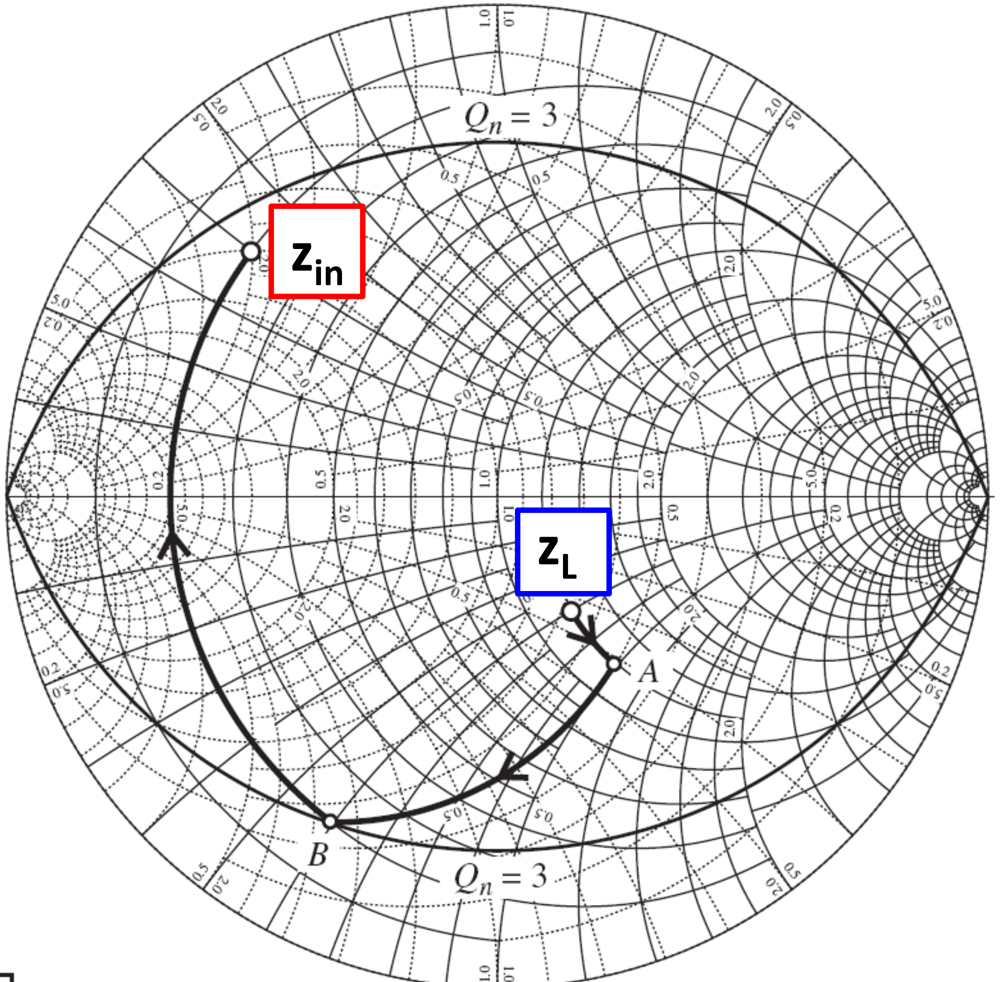
Gives the name **T**



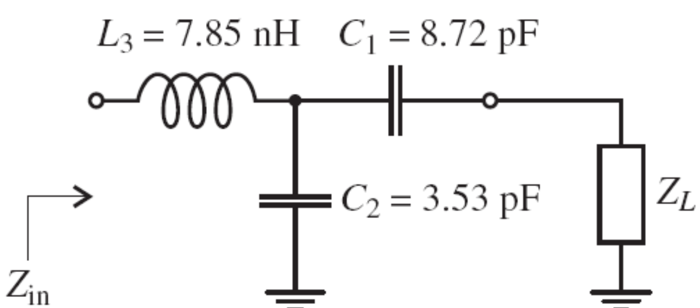
- First element in series (Z_1) is purely reactive, therefore the **combined impedance of (Z_L and Z_1)** will reside on the constant resistance circle of **$r = r_L$**
- Similarly, **Z_3** (that is purely reactive!) **is connected in series** with the input, therefore **the combined impedance Z_B** (consisting of Z_L, Z_1, and Z_2) lies on the constant resistance circle **$r = r_{in}$**
- Network needs to have a Q_n of 3 → we should choose impedance in such a way that **Z_B** is located on the **intersection** of constant resistance circle **$r = r_{in}$** and **Q_n = 3** circle → helps in the determination of **Z_3**

Example – 1 (contd.)

- The constant resistance circle of **z**$_{in}$ intersects the $Q_n = 3$ circle at point **B**. This gives value of Z_3.
- The constant resistance circle $r = r_L$ and a constant conductance circle that passes through B helps in the determination of Z_2 and Z_1.



Final solution at 1 GHz

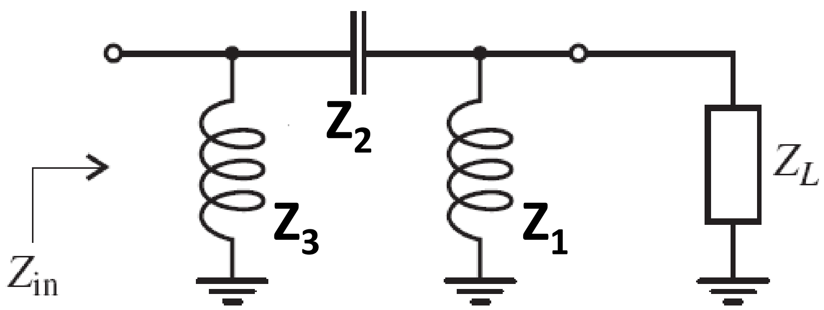


Example – 2

- For a broadband amplifier, it is required to develop a Pi-type matching network that transforms a load impedance $Z_L = (10 - j10)\Omega$ into an input impedance of $Z_{in} = (20 + j40)\Omega$. The design should involve the lowest possible $\mathbf{Q_n}$. Compute the values for the matching network components, assuming that matching is required at f = 2.4GHz.

Solution

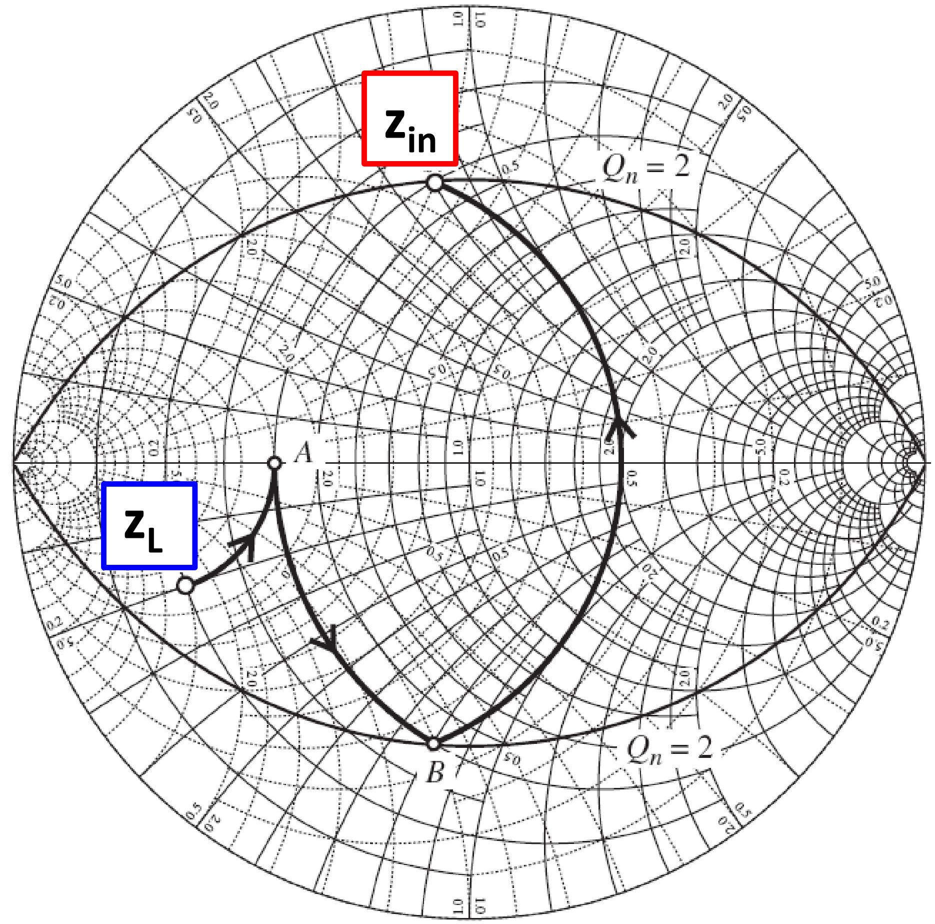
- Several Configurations possible (including the forbidden!). One such is below:



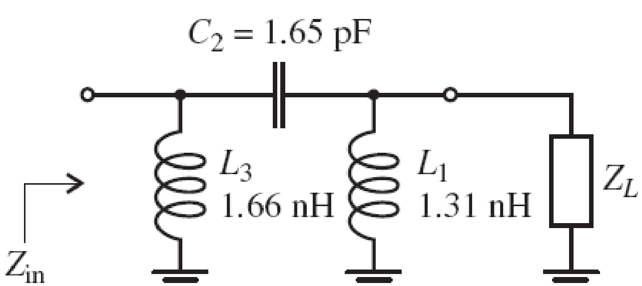
- Since the load and source impedances are fixed, we can't develop a matching network that has Q_n lower than the values at locations Z_L and Z_{in}
- Therefore in this example, the minimum value of Q_n is determined at the input impedance location as $Q_n = |X_{in}|/R_{in} = 40/20 = 2$

Example – 2 (contd.)

- In the design, we first plot constant conductance circle $g = g_{in}$ and find its intersection with Q_n=2 circle (point B) → determines the value of Z_3
- Next find the intersection point (labeled as A) of the $g=g_L$ circle and constant-resistance circle that passes through B → determines value of Z_2 and Z_1



Final solution at 2.4 GHz



Example – 2 (contd.)

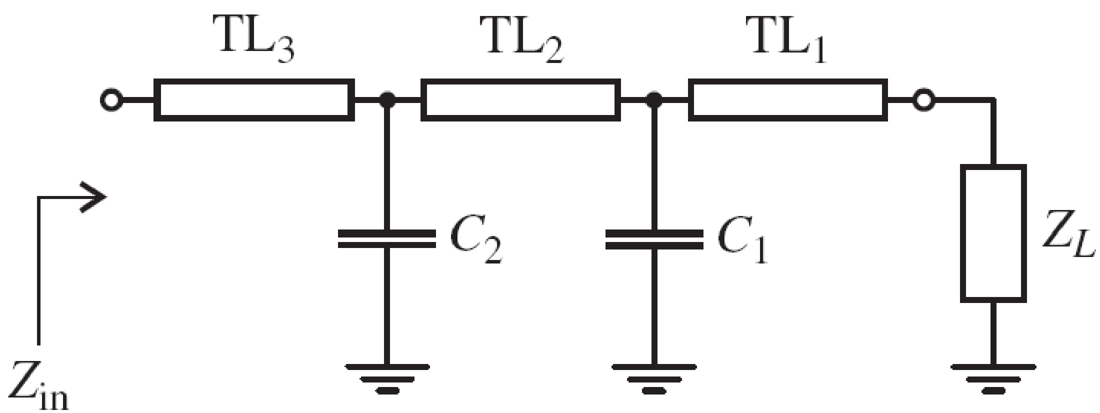
- It is important to note that the relative positions of Z_{in} and Z_L **allows only one** optimal Pi-type network for a given specification.
- All other realizations will result in higher Q_n → **essentially smaller BW!**
- Furthermore, **for smaller Z_L** the Pi-matching isn't possible!

It is thus apparent that BW can't be enhanced arbitrarily by reducing the Q_n. The limits are set by the desired complex Z_{in} and Z_L.

With increasing frequency and correspondingly reduced wavelength the influence of parasitics in the discrete elements are noticeable → distributed matching networks overcome most of the limitations (of discrete components) at high frequency

Microstrip Line Matching Networks

- In the lower RF region, its often a standard practice to use a hybrid approach that combines lumped and distributed elements.
- These types of matching circuits usually contain TL segments in series and capacitors in shunt.



- Inductors are avoided in these designs as they tend to have higher resistive losses as compared to capacitors.
- In principle, only one shunt capacitor with two TL segments connected in series on both sides is sufficient to transform any given load impedance to any input impedance.

Microstrip Line Matching Networks (contd.)

- Similar to the L-type matching network, these configurations may also involve the additional requirement of a fixed Q_n, necessitating additional components to control the bandwidth of the circuit.
- In practice, these configurations are extremely useful as they permit tuning of the circuits even after manufacturing → changing the values of capacitors as well as placing them at different locations along the TL offers a wide range of flexibility → In general, all the TL segments have the same width to simplify the actual tuning →the tuning ability makes these circuits very appropriate for prototyping.

Example – 3

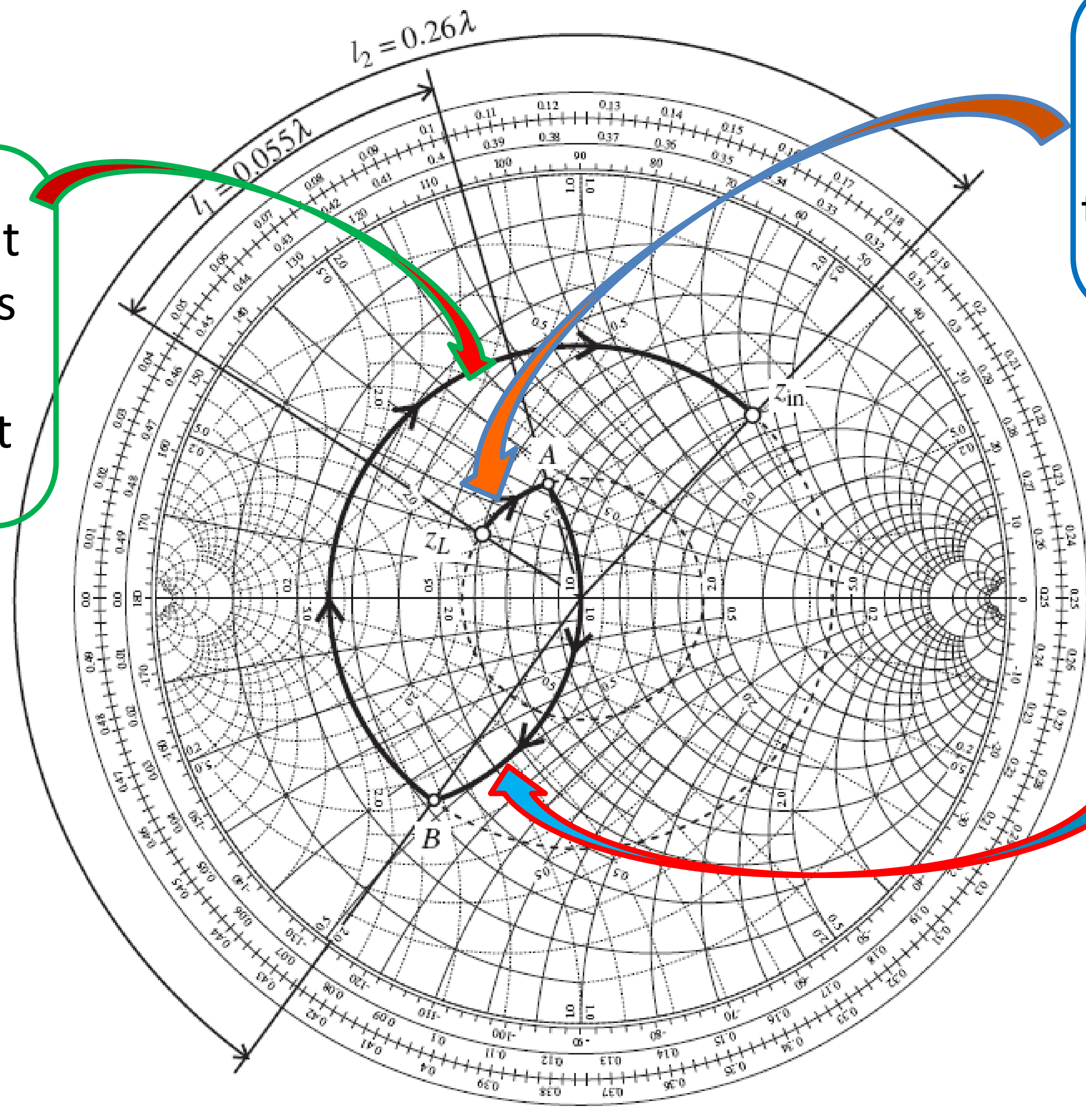
Design a hybrid matching network that transforms the load Z_L = (30 + j10) Ω to an input impedance Z_{in} = (60 + j80) Ω. The matching network should contain only two series TL segments and one shunt capacitor. Both TLs have a 50Ω characteristic impedance, and the frequency at which the matching is required is f = 1.5 GHz

Solution

- Mark the normalized load impedance (0.6 + j0.2) on the Smith chart.
- Draw the corresponding SWR circle.
- Mark the normalized input impedance (1.2 + j1.6) on the Smith chart.
- Draw the corresponding SWR circle.
- The choice of the point from which we transition from the load SWR circle to the input SWR circle can be made arbitrarily.

Example – 3 (contd.)

point B to normalized input impedance gives length of the second segment of TL

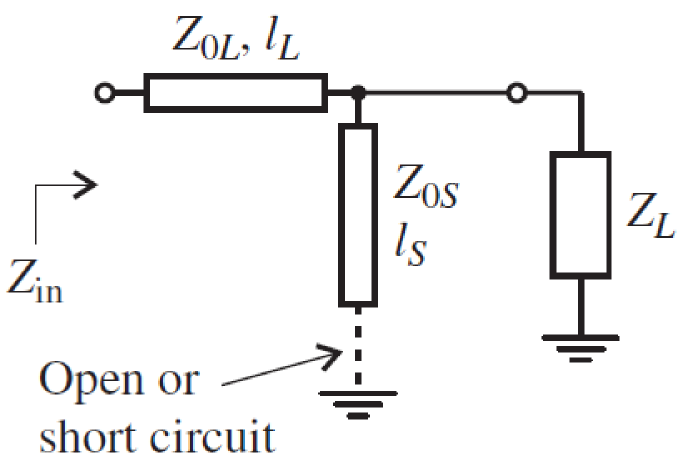


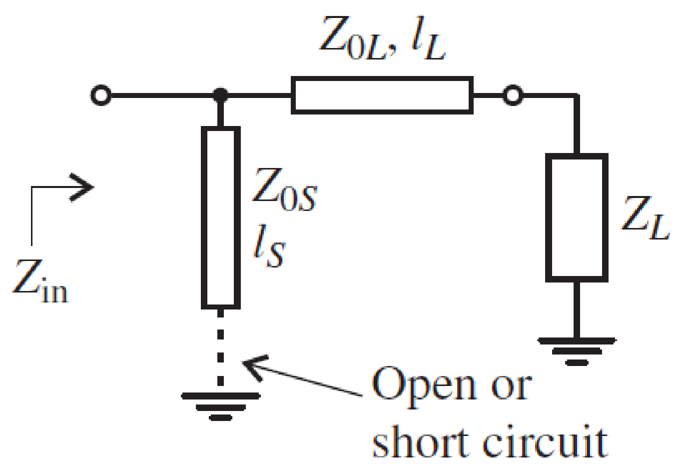
Normalized load to the point A gives length of the first segment of TL

A to B provides the necessary susceptance value for the shunt capacitor

Stub Matching Networks

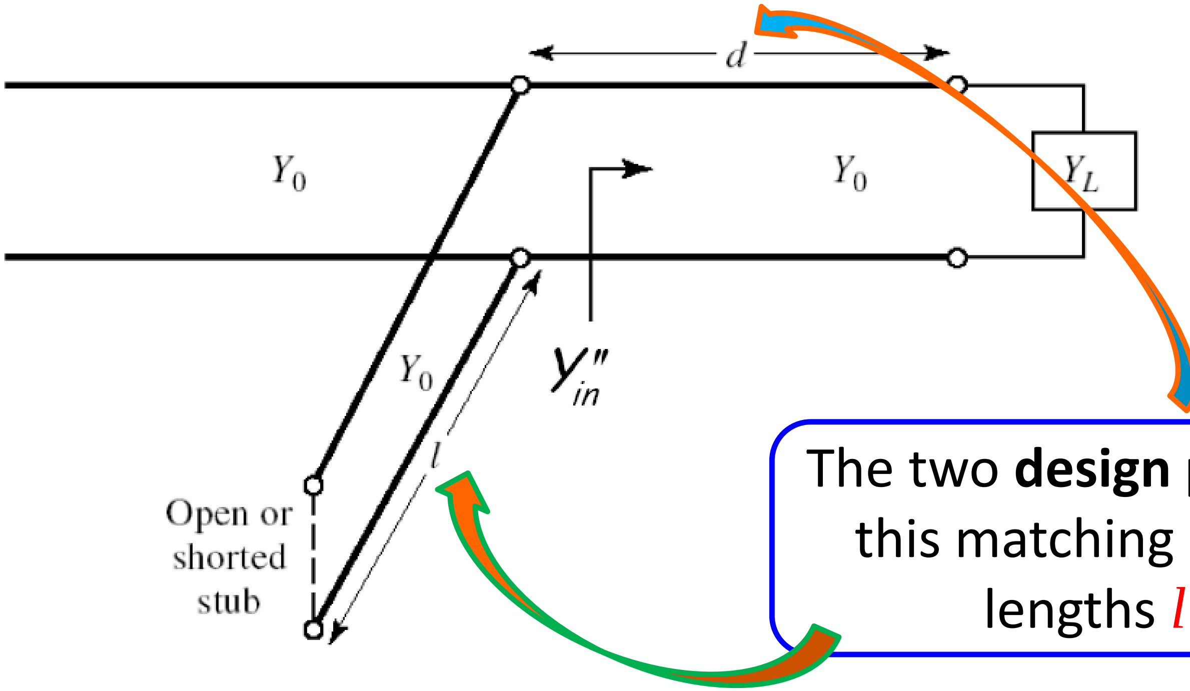
- The next logical step in the transition from lumped to distributed element networks is the complete elimination of all lumped components → this can be achieved by employing open – and/or short – circuited stub lines





Shunt-stub Matching Networks

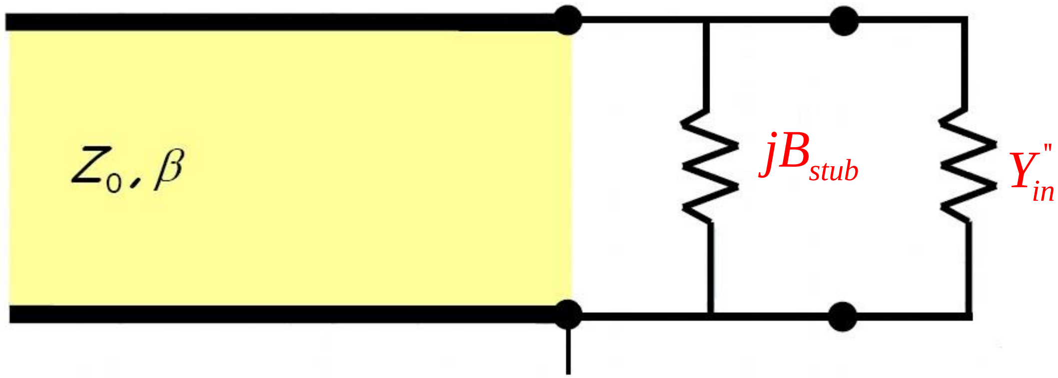
- Let us consider the following TL configuration with shunt stub.



The two **design parameters** of this matching network are lengths l and d.

Shunt-stub Matching Networks (contd.)

- An equivalent circuit for the shunt-tub TL can be:



$\mathbf{z = 0}$

Where:

$$Y''_{in} = Y_0 \left(\frac{Y_L + jY_0 \tan(\beta d)}{Y_0 + jY_L \tan(\beta d)} \right)$$

$$jB_{stub} = \begin{cases} jY_0 \tan(\beta l) & \text{For open-stub} \\ -jY_0 \cot(\beta l) & \text{For short-stub} \end{cases}$$

Shunt-stub Matching Networks (contd.)

- Therefore, for a matched circuit, we require:

$$\boxed{jB_{stub} + Y_{in}'' = Y_0}$$

- Note this complex equation is actually **two real equations**!

$$\boxed{\mathrm{Re}\{Y_{in}''\} = Y_0}$$

$$\boxed{\mathrm{Im}\{jB_{stub} + Y_{in}''\} = 0 \quad \Rightarrow B_{stub} = -B_{in}''}$$

Where: $\boxed{-B_{in}'' = \mathrm{Im}\{Y_{in}''\}}$

- Since Y_{in}'' is dependent on ***d*** only, our **design procedure** is:

1) Set d such that $\mathrm{Re}\{Y_{in}''\} = Y_0$.

2) Then set ℓ such that $B_{stub} = -B_{in}''$.

We have two choice, either Analytical or Smith chart for finding out the lengths d and l

Shunt-stub Matching Networks (contd.)

Use of the **Smith Chart** to determine the lengths!

- Rotate **clockwise** around the Smith Chart from y_l until you intersect the $\boldsymbol{g_s}$**=1 circle**. The "length" of this rotation determines the value $\boldsymbol{d}$. Recall there are **two** possible solutions!
- Rotate **clockwise** from the short/open circuit point around the $\boldsymbol{g=0}$ **circle**, until b_{stub} equals $-b_{in}''$. The "length" of this rotation determines the stub length l.

Example – 4

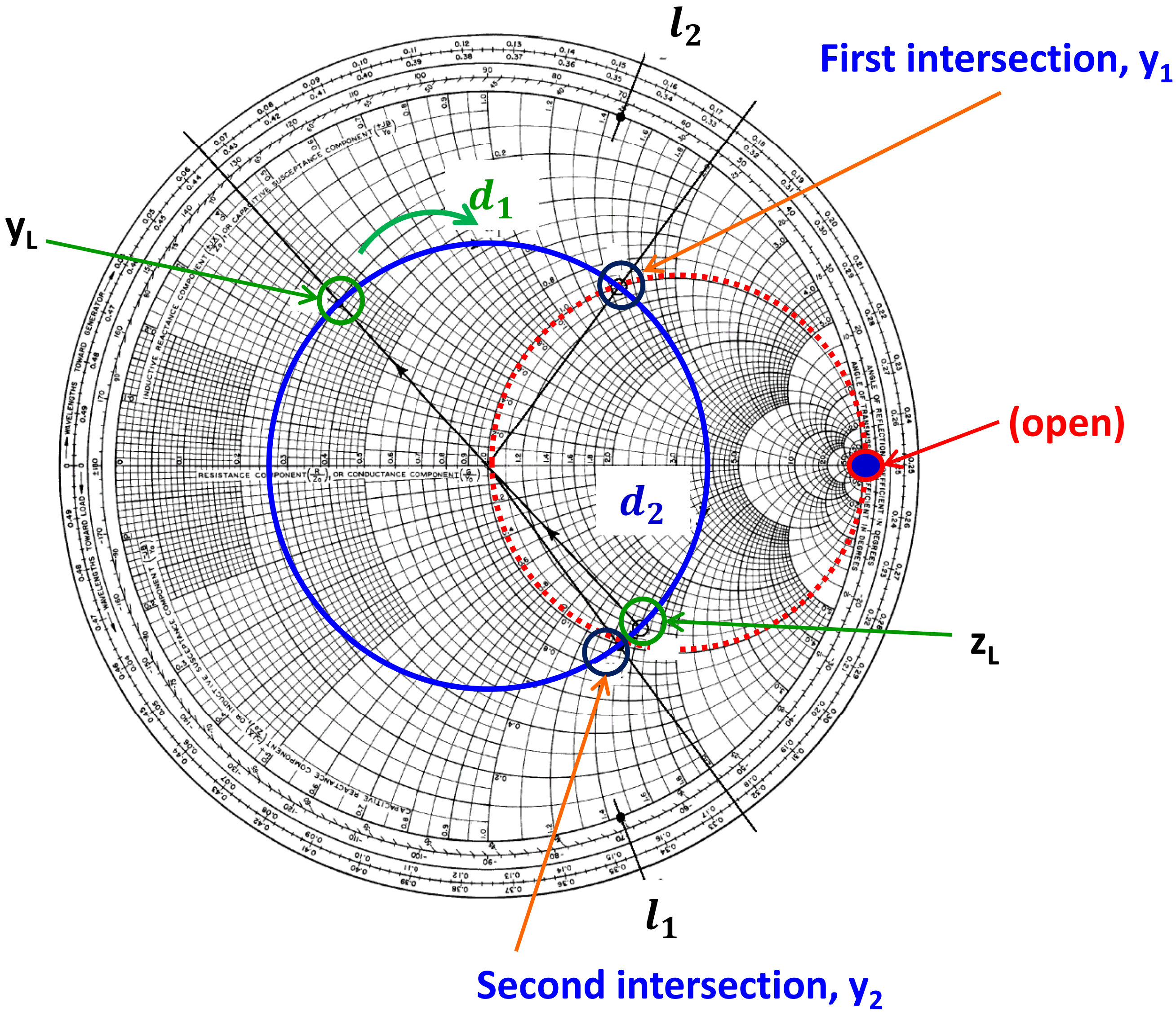
Let us take the case where we want to match a load of Z_L= (60−j80)Ω (at 2 GHz) to a transmission line of Z_0 =50Ω.

Example – 4 (contd.)

Solution

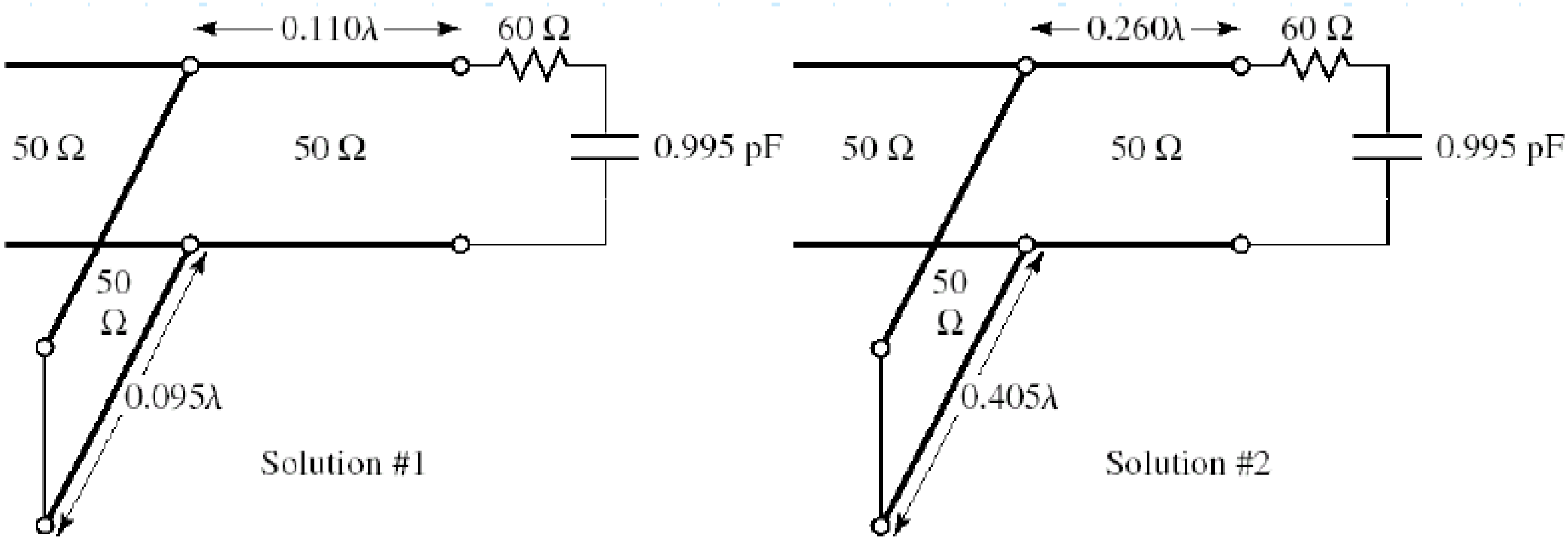
y_L to y_1 towards generator (clockwise) gives length d_1 (first solution)

y_L to y_2 towards generator (clockwise) gives length d_2 (second solution)



Example – 4 (contd.)

- Determine the respective admittances at the two intersection points
- These are of the form 1 + jx and 1 – jx
- Cancel these imaginary part of the admittances by introducing shunt-stubs of length l_1 and l_2 respectively
- l_1 and l_2 are the lengths from open circuit point in the Smith chart (if open stub is used) along the g = 0 circle until the achieved admittances are of opposite signs to those at the intersection points in the earlier step



Example – 4 (contd.)

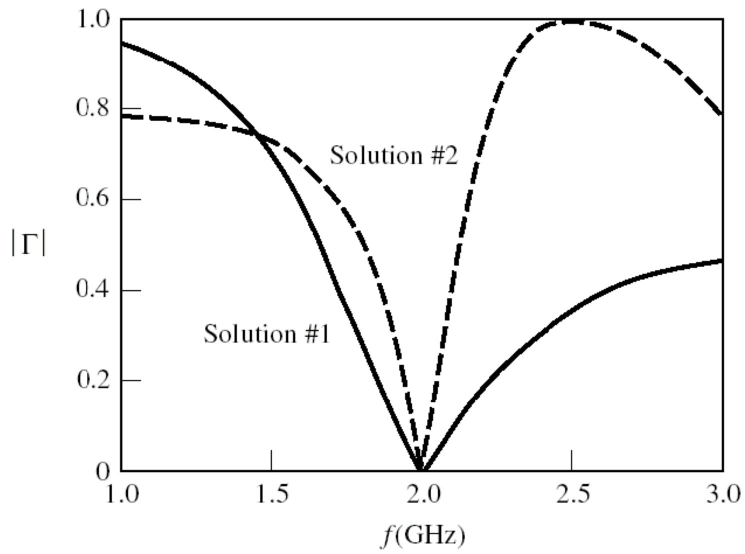
Q: **Two** solutions! Which one do we use?

A: The one with the **shortest** lengths of transmission line!

Q: Oh, I see! Shorter transmission lines provide **smaller** and (slightly) cheaper matching networks.

A: True! But there is a more **fundamental** reason why we select the solution with the **shortest** lines—the matching **bandwidth** is **larger**!

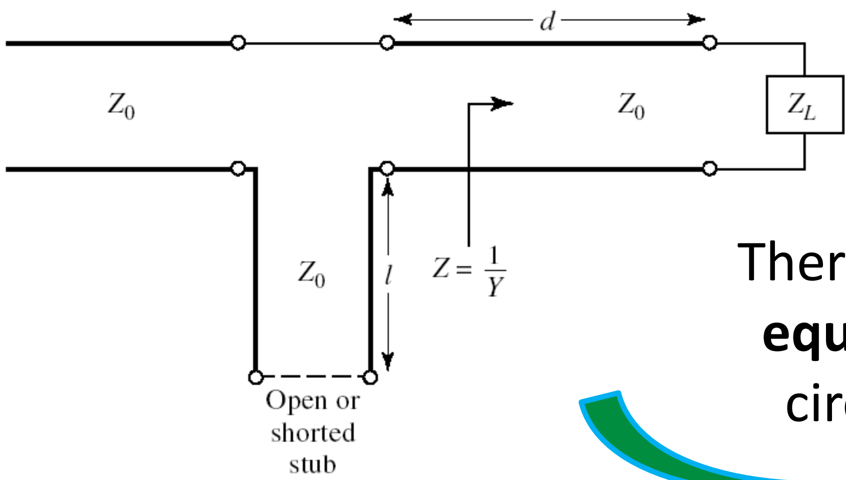
- For example, consider the **frequency response** of the two solutions:



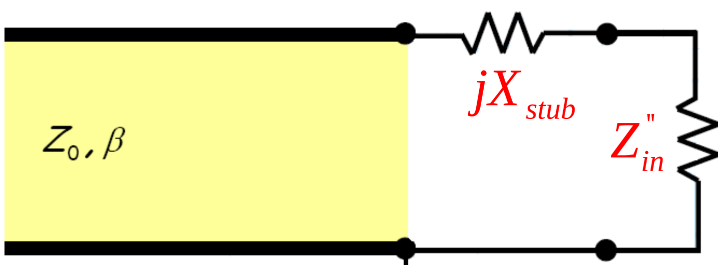
Clearly, solution 1 provides a **wider** bandwidth!

Series-stub Matching Networks

- Consider the following transmission line structure, with a **series** stub:



Therefore an **equivalent** circuit is:



where of course:

$$Z_{in}'' = Z_0\left(\frac{Z_L + jZ_0\tan(\beta d)}{Z_0 + jZ_L\tan(\beta d)}\right)$$

$$jX_{stub} = \begin{cases} -jZ_0\cot(\beta l) & \text{For open-stub} \\ jZ_0\tan(\beta l) & \text{For short-stub} \end{cases}$$

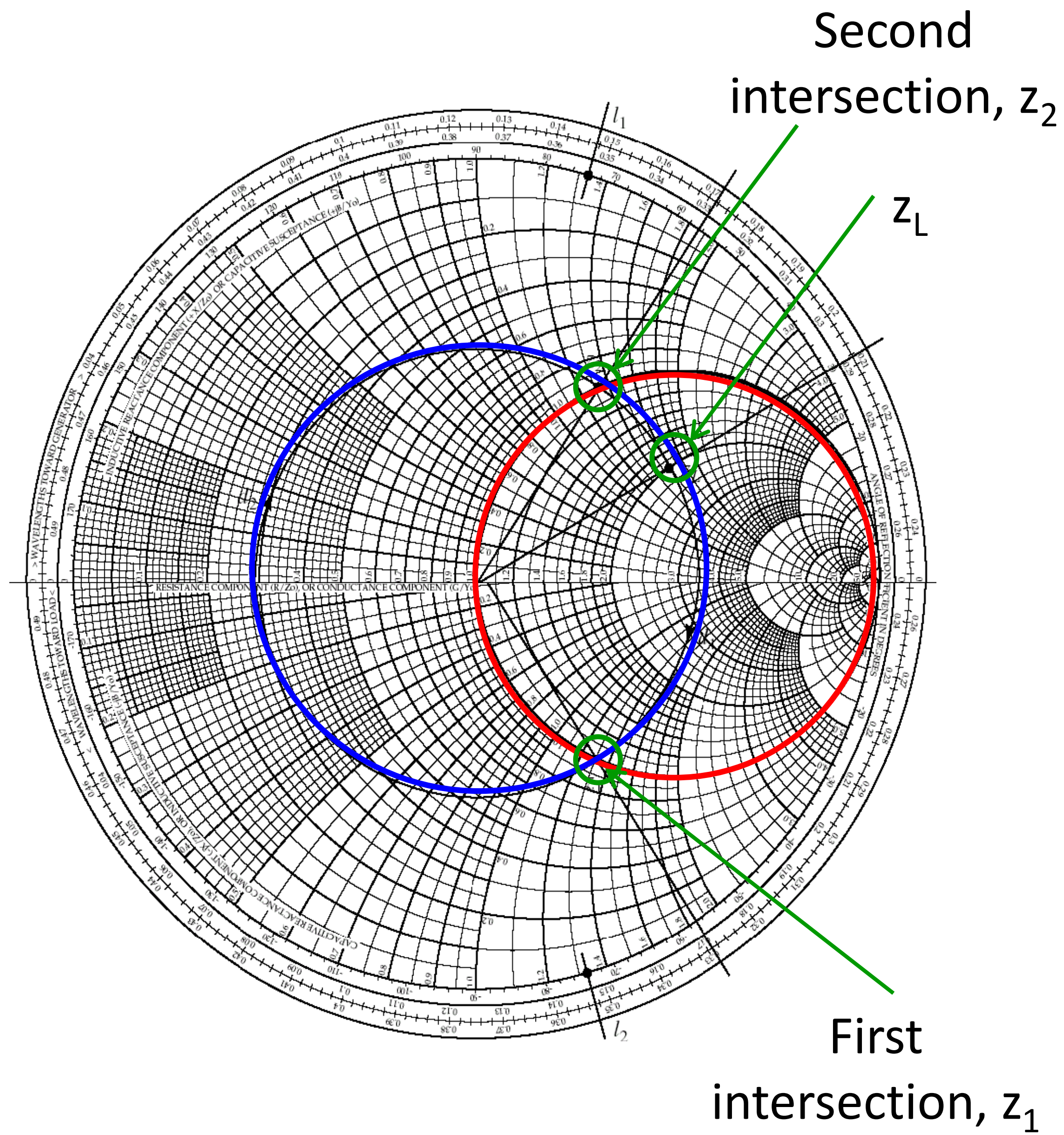
Example – 5

Let us take the case where we want to match a load of Z_L= (100 + j80)Ω (at 2 GHz) to a transmission line of Z_0 =50Ω.

Example – 2 (contd.)

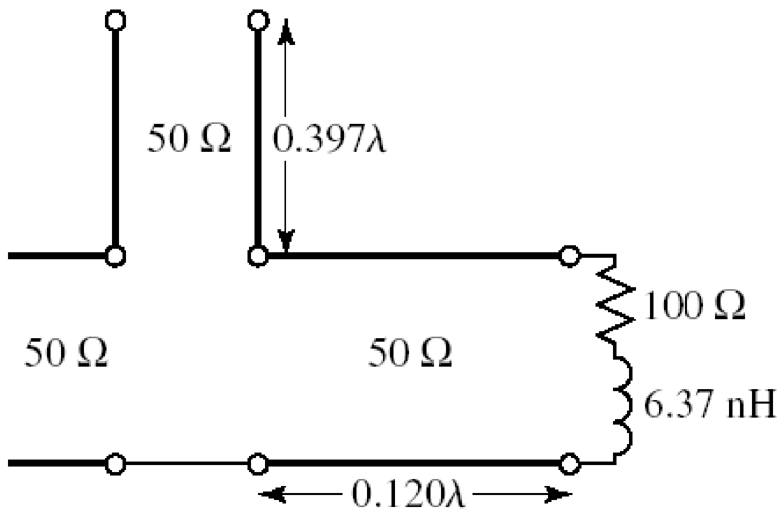
z_l to z_1 towards generator (clockwise) gives length d_1 (first solution)

z_l to z_2 towards generator (clockwise) gives length d_2 (second solution)

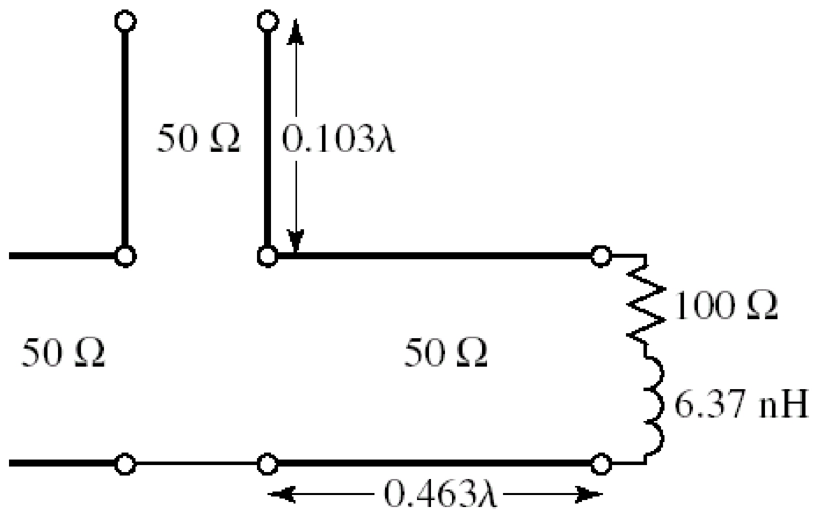


Example – 5 (contd.)

- Determine the respective impedances at the two intersection points and these are of the form $1 + jx$ and $1 - jx$
- Cancel these imaginary part of the impedances by introducing series-stubs of length l_1 and l_2 respectively
- l_1 and l_2 are the lengths from open circuit point in the Smith chart (if open stub is used) along the r = 0 circle until the achieved impedances are of opposite signs to those at the intersection points in the earlier step



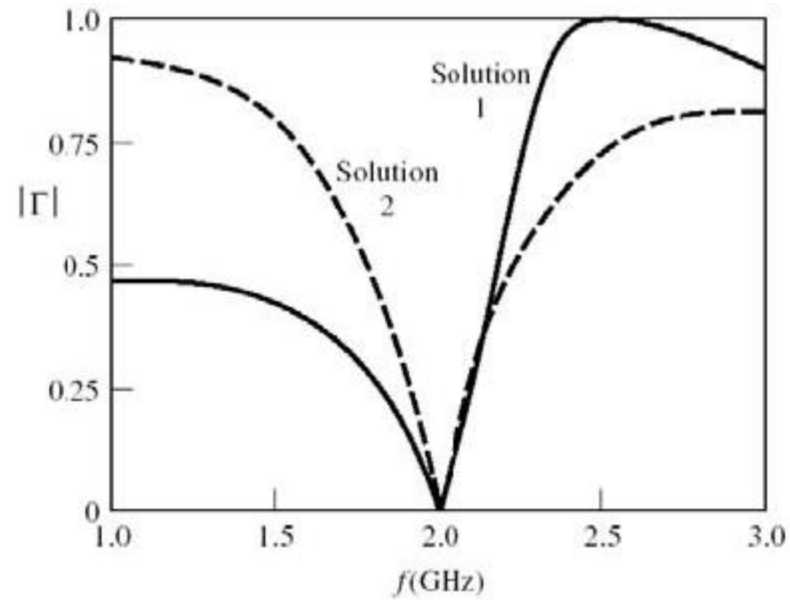
Solution 1



Solution 2

Example – 5 (contd.)

Again, we should use the solution with the **shortest** transmission lines, although in **this** case that distinction is a bit **ambiguous**. As a result, the bandwidth of each design is about the same (depending on how **you** define **bandwidth**!).



Example – 6

For a load impedance of $Z_L = (60 - j45)\Omega$, design single-stub (shunt) matching networks that transform the load to a $Z_{in} = (75 + j90)\Omega$ input impedance. Assume both the stub and transmission line have a characteristic impedance of $Z_0 = 75\Omega$

Solution

- Normalize the Z_L and Z_{in} with 75Ω
- Mark these normalized impedances on the Z-Smith chart
- Move to Y-Smith chart or better use ZY-Smith chart
- Plot constant conductance (g_L) circle
- Plot SWR circle for normalized input impedance (z_{in})
- Two intersection points between constant conductance circle and SWR circle can be observed
- Rotation from intersection points to z_{in} give the lengths d_1 and d_2 and corresponding changes in admittance
- Look for cancelling the additional admittances using shunt stub by equating corresponding stub lengths from **'open'** in Smith chart

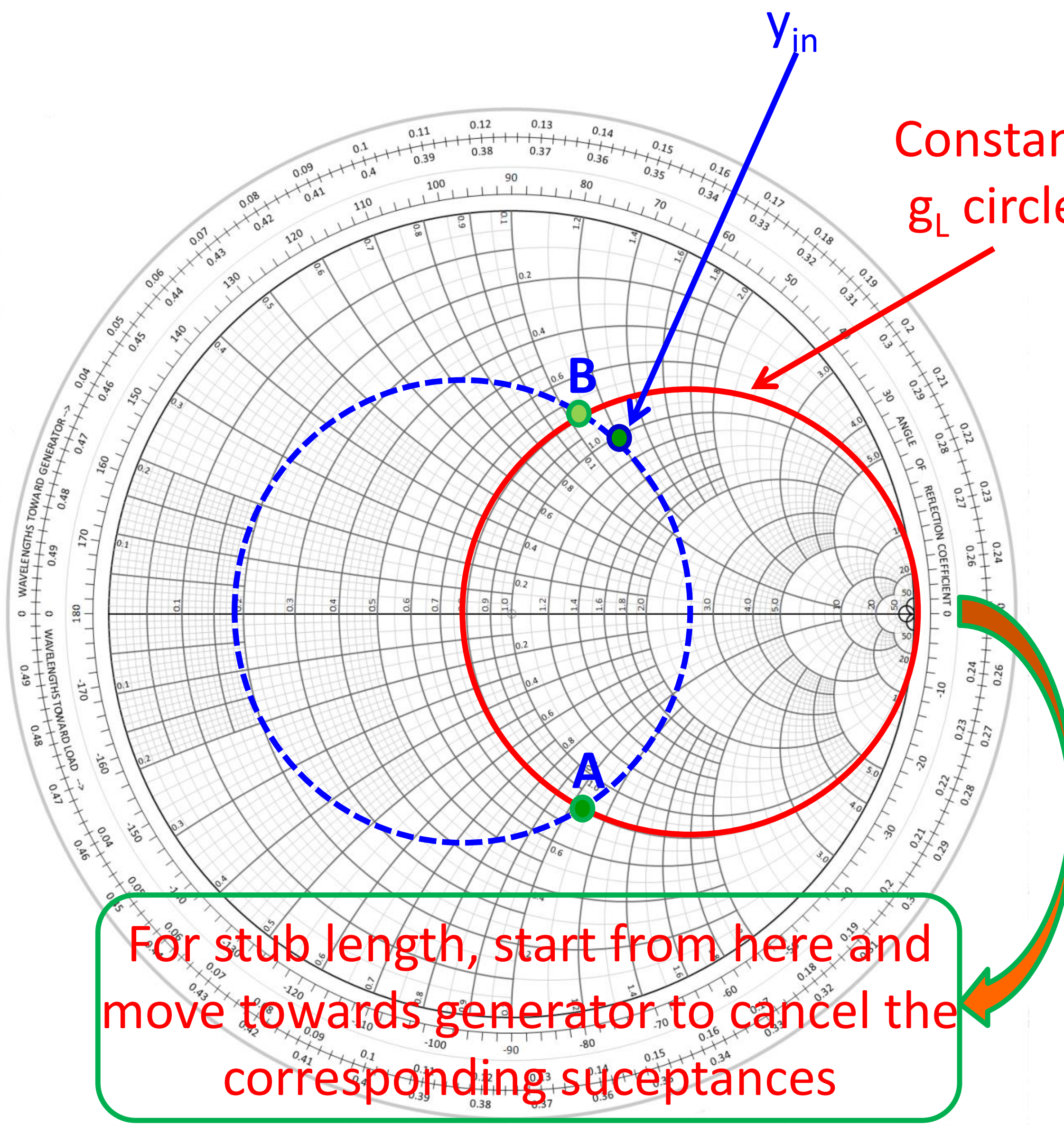
Example – 6 (contd.)

Here:

$z_L = 0.8 - j0.6$

y_{in} to A towards generator (clockwise) gives length d_1 (first solution)

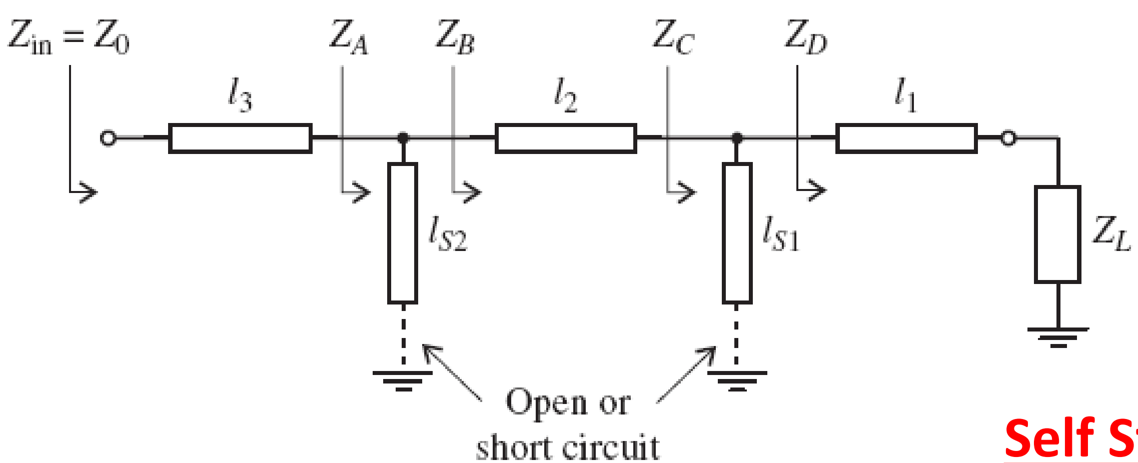
y_{in} to B towards generator (clockwise) gives length d_2 (second solution)



For stub length, start from here and move towards generator to cancel the corresponding suceptances

Double-stub Matching Networks

- The single-stub matching networks are quite versatile → allows matching between any input and load impedances, so long as they have a non-zero real part.
- Main drawback is the requirement of variable length TL between the stub and the input port or the stub and the stub and the load impedance → many a times problematic when variable impedance tuner is needed.
- In a double-stub matching networks, **two short- or open-circuited stubs are connected in shunt** with a fixed-length TL separating them → the usual separation is $\lambda/8$, $3\lambda/8$ or $5\lambda/8$.



Self Study

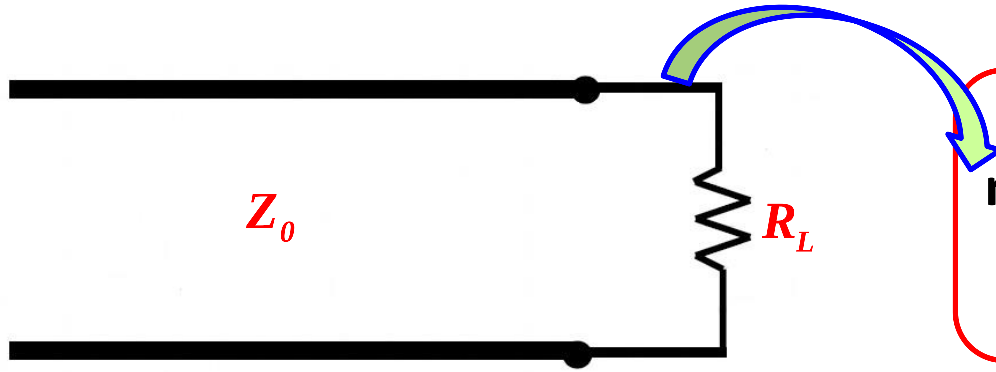
The Quarter Wave Transformer

- By now you must have noticed that a **quarter-wave length** of transmission line (l = $\lambda/4$, $2\beta l = \pi$) appears **often** in RF/microwave engineering problems.
- Another application of the l = $\lambda/4$ transmission line is as an **impedance matching network.**

Q: Why does the quarter-wave matching network work — after all, the quarter-wave line is **mismatched** at both ends?

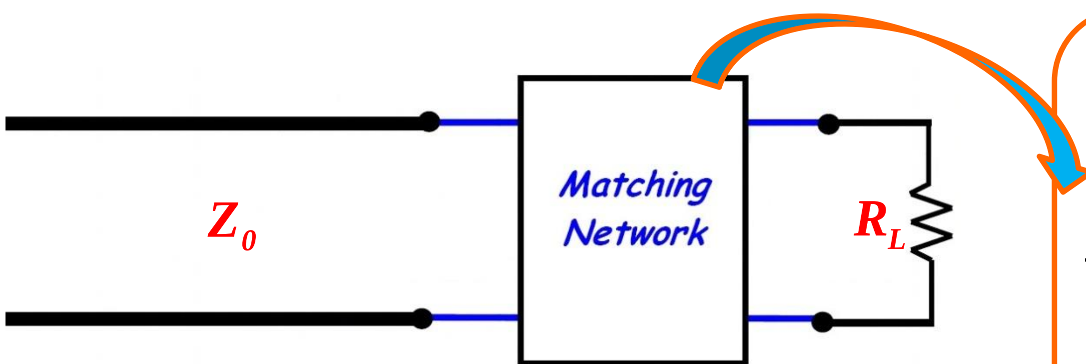
The Quarter Wave Transformer (contd.)

- Let us consider a TL (with characteristic impedance Z_0) where the end is terminated with a **resistive** (i.e., real) load:



Unless $R_L = Z_0$, the resistor is **mismatched** to the line, and thus some of the incident power will be **reflected**.

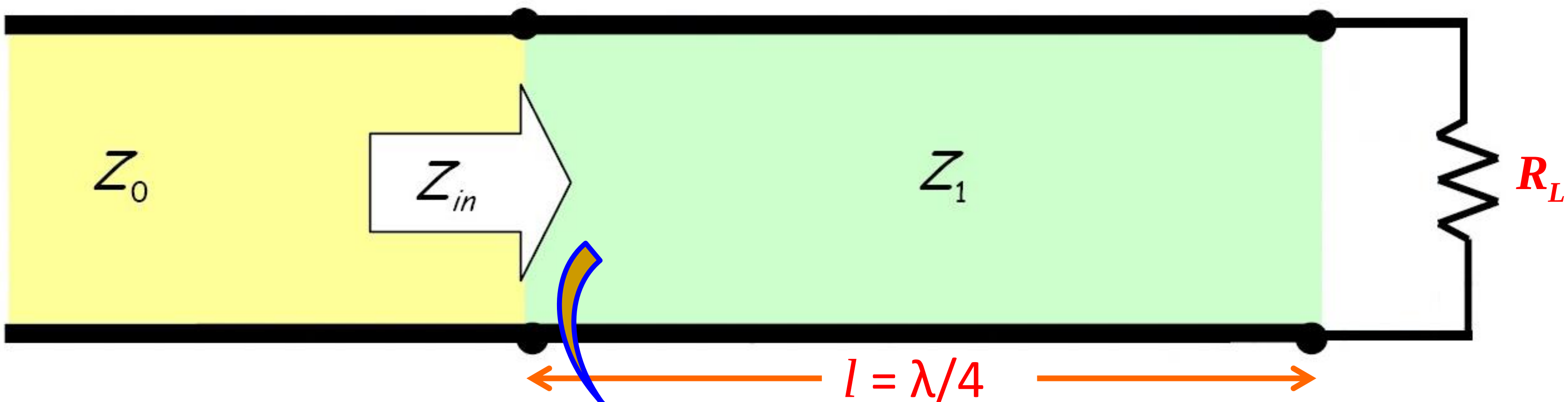
- We can of course correct this situation by placing a matching network between the line and the load:



In addition to the designs we have just studied (e.g., L-networks, stub tuners), one of the simplest matching network designs is the **quarter-wave transformer**.

The Quarter Wave Transformer (contd.)

- The quarter-wave transformer is simply a transmission line with characteristic impedance Z_1 and length $l = \lambda/4$ (i.e., a quarter-wave line).



This λ/4 line **is** the **matching network**!

Q: But what about the characteristic impedance Z_1; what **should** its value be??

The Quarter Wave Transformer (contd.)

A: Remember, the quarter wavelength case is one of the **special** cases that we studied. We know that the **input** impedance of the quarter wavelength line is:

$$\boxed{\boldsymbol{Z}_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}}$$

- Thus, if we wish for Z_{in} to be numerically equal to Z_0, we find:

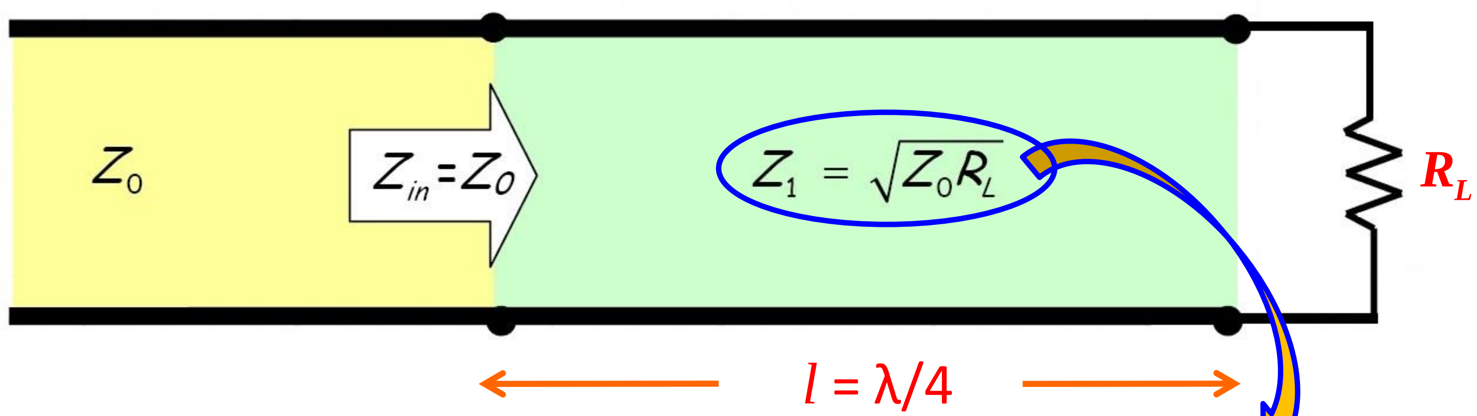
$$\boxed{\boldsymbol{Z}_{in} = \frac{(Z_1)^2}{R_L} = Z_0}$$

- Solving for Z_1, we find its **required** value to be: $\boxed{Z_1 = \sqrt{Z_0 R_L}}$

In other words, the characteristic impedance of the quarter wave line is the **geometric average** of Z_0 and R_L!

The Quarter Wave Transformer (contd.)

Therefore, a λ/4 line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will **match** a transmission line with characteristic impedance Z_0 **to** a resistive load R_L



This ensures that **all power** is delivered to load R_L!

Alas, the quarter-wave transformer (like all our designs) have a few problems!

The Quarter Wave Transformer (contd.)

Problem #1

- The matching **bandwidth** is **narrow** !
- In other words, we obtain a **perfect** match at precisely the frequency where the length of the matching transmission line is a **quarter-wavelength.**

remember, this length can be a quarter-wavelength at just **one** frequency!

- **Wavelength** is related to **frequency** as:

$$\boxed{\lambda = \frac{v_p}{f} = \frac{1}{f\sqrt{LC}}} \Rightarrow$$

v_p is propagation velocity of wave

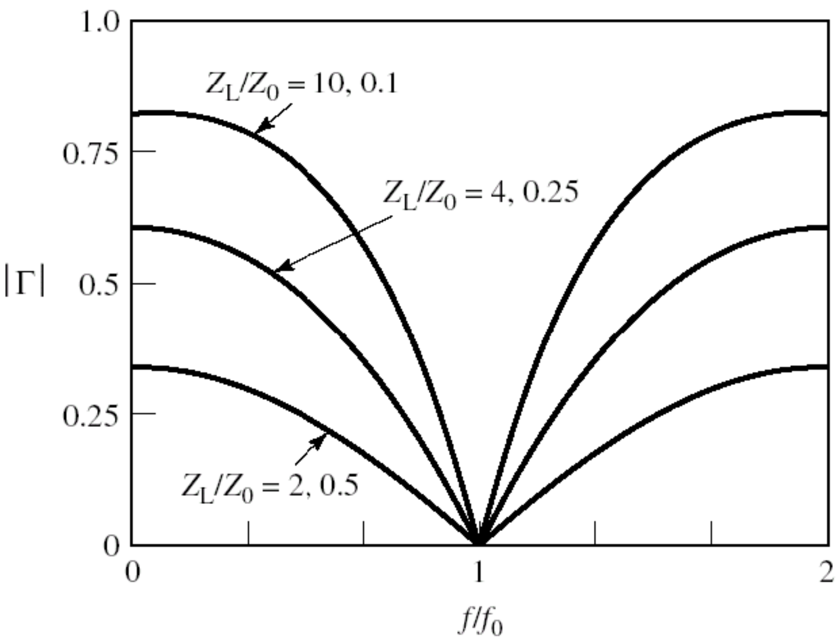
- For **example**, assuming that v_p = c (c = the speed of light in a vacuum), one wavelength at 1 GHz is 30 cm (λ = 0.3m), while one wavelength at 3 GHz is 10 cm (λ = 0.1m). As a result, a TL length l = 7.5cm is a quarter wavelength for a signal at 1GHz **only**.

Thus, a quarter-wave transformer provides a perfect match (Γ_{in} = 0) at one and only one signal frequency!

The Quarter Wave Transformer (contd.)

In other words, as the signal frequency (i.e., wavelength) changes, the **electrical** length of the matching TL segment changes. It will **no longer** be a **quarter** wavelength, and thus we **no longer** will have a **perfect** match

It can be observed that the **closer** R_L (or R_{in}) is to characteristic impedance Z_0, the **wider** the bandwidth of the quarter wavelength transformer



In principle, the bandwidth can be **increased** by adding **multiple** $\lambda/4$ sections!

The Quarter Wave Transformer (contd.)

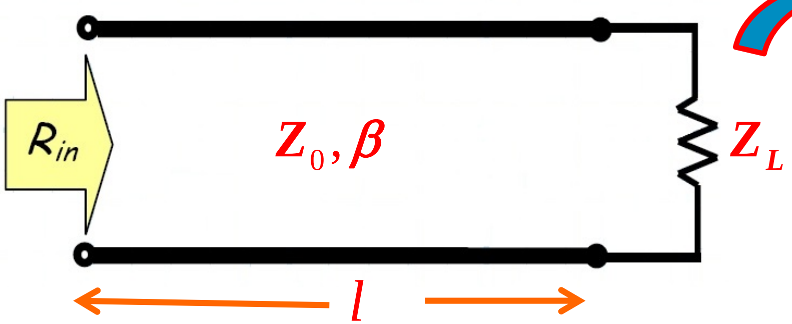
Problem #2

Recall the matching solution was limited to loads that were **purely real**! i.e.:

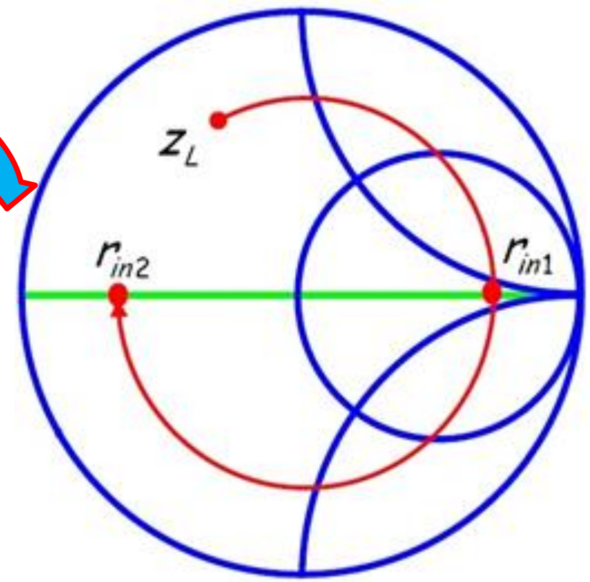
$$Z_L = R_L + j0$$

Obviously, this is a BIG problem, as most loads will have a **reactive** component!

- Fortunately, we have a relatively easy **solution** to this problem, as we can always add some **length** l of TL to the load to make the impedance completely **real**:



Transforms $R_L + jX_L$ into R_{in}

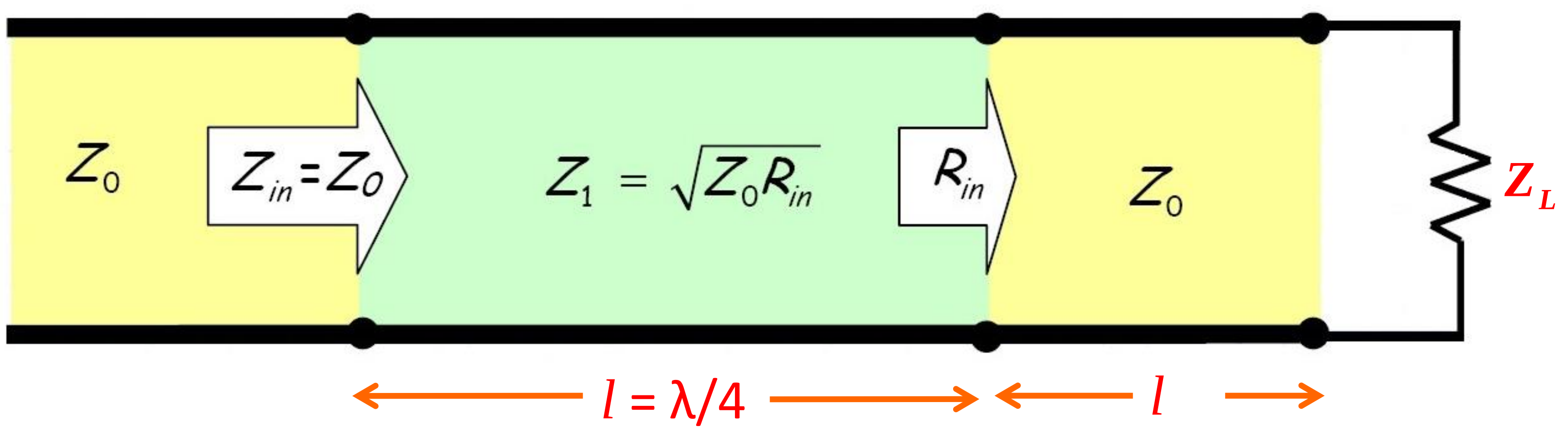


Clearly two possible solutions

The Quarter Wave Transformer (contd.)

However, it should be understood that the input impedance will be purely real at only **one** frequency!

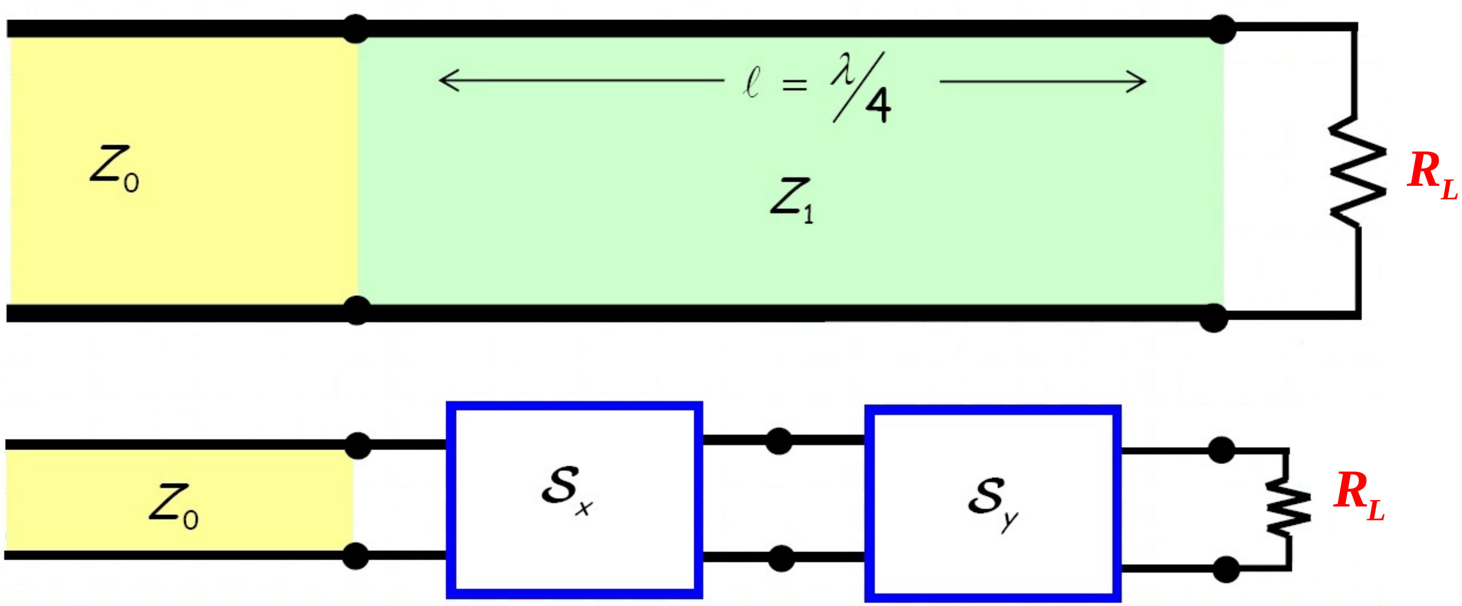
Once the output impedance has been converted to purely real, one can then build a quarter-wave transformer to **match** the line Z_0 to resistance R_{in}



Again, since the transmission lines are lossless, **all** of the incident power is delivered to the **load** Z_L .

The Quarter Wave Transformer (contd.)

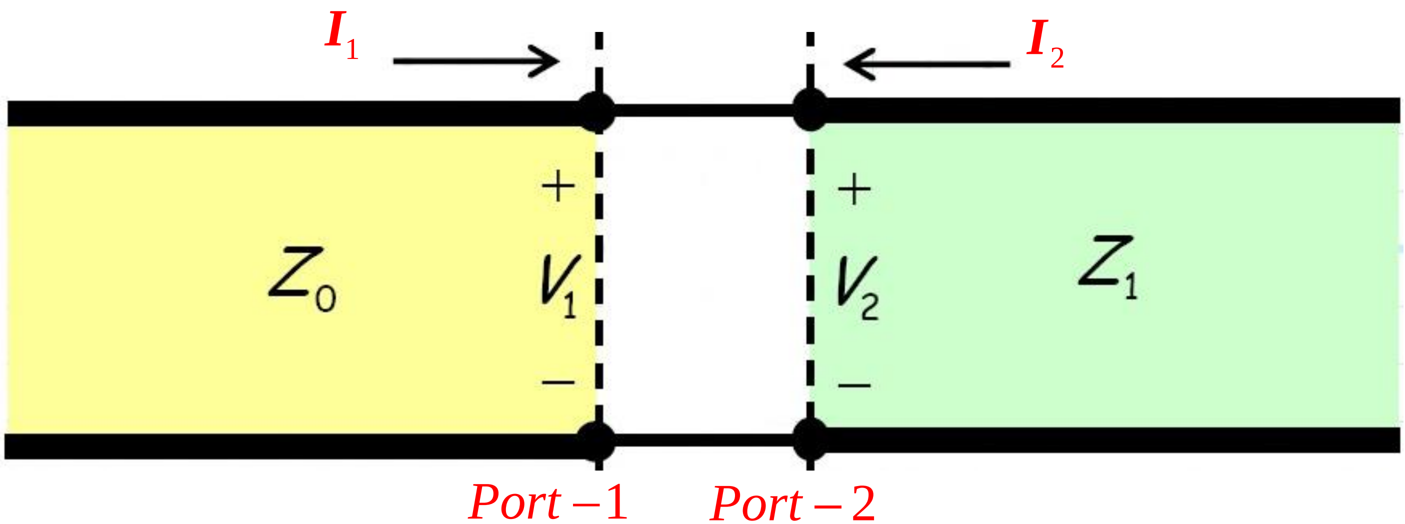
- A quarter wave transformer can be thought of as a cascaded series of **two** two-port devices, terminated with a load R_L:



Q: Two two-port devices? It appears to me that a quarter-wave transformer is **not** that complex. What **are** the **two** two–port devices?

A: The **first** is a "**connector**". Note a connector is the interface between one transmission line (characteristic impedance Z_0) to a second transmission line (characteristic impedance Z_1).

The Quarter Wave Transformer (contd.)

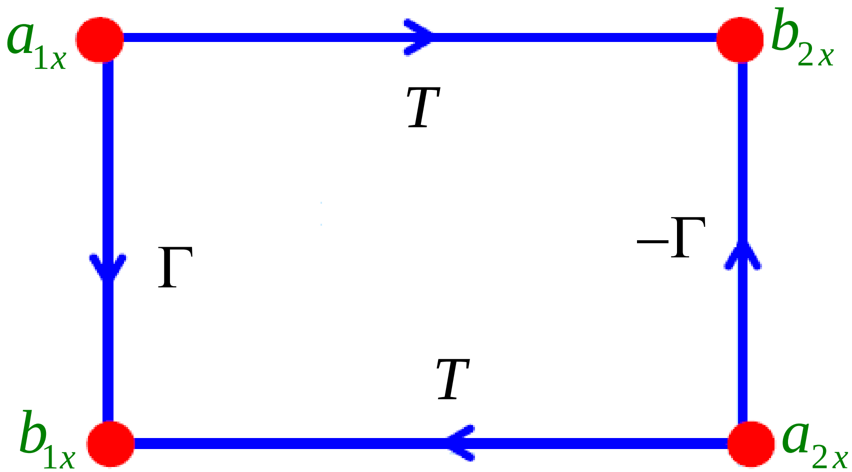


- we **earlier** determined the scattering matrix of this two-port device as:

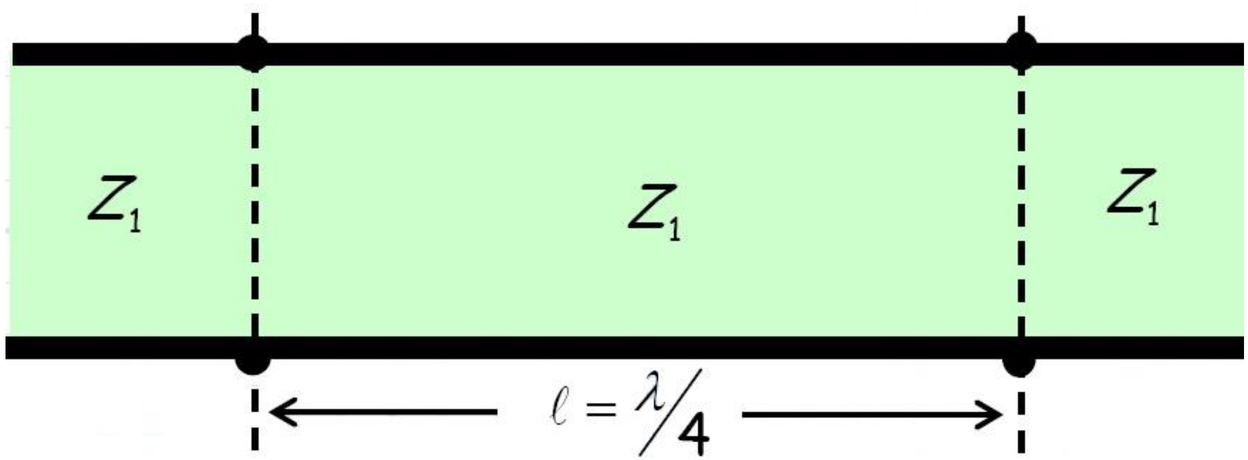
$$\mathbf{S}_x = \begin{bmatrix} \dfrac{Z_1 - Z_0}{Z_1 + Z_0} & \dfrac{2\sqrt{Z_0 Z_1}}{Z_1 + Z_0} \\ \dfrac{2\sqrt{Z_0 Z_1}}{Z_1 + Z_0} & \dfrac{Z_0 - Z_1}{Z_1 + Z_0} \end{bmatrix} \xrightarrow{\text{Compact Form}} \mathbf{S}_x = \begin{bmatrix} \Gamma & T \\ T & -\Gamma \end{bmatrix}$$

The Quarter Wave Transformer (contd.)

- Therefore signal flow graph of the connector can be given as:



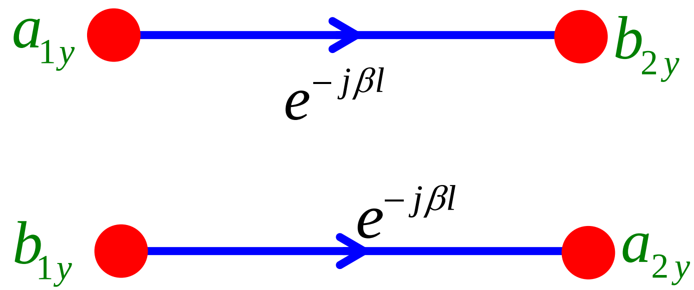
- Now, the **second** two-port device is a quarter wavelength of **TL:**



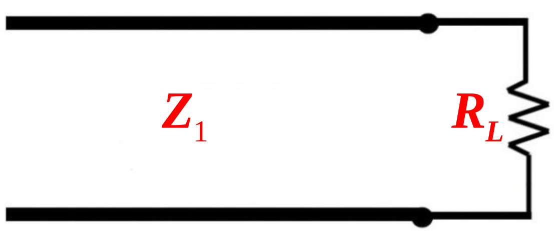
The Quarter Wave Transformer (contd.)

- The second device has the scattering matrix and SFG as:

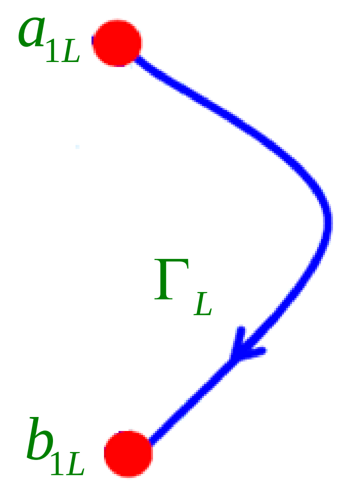
$$\boldsymbol{S}_y = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix}$$



- Finally, a **load** has a “scattering matrix” and SFG as:

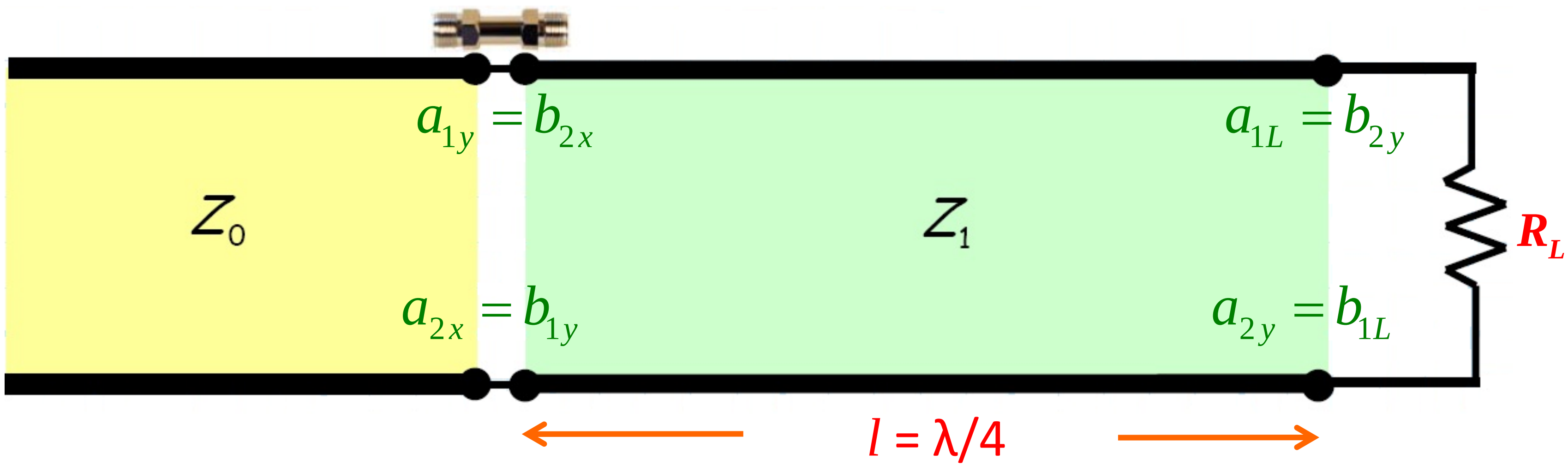


$$\boldsymbol{S} = \left[\frac{R_L - Z_1}{R_L + Z_1}\right] = \Gamma_L$$



The Quarter Wave Transformer (contd.)

- Of course, if we connect the ideal connector to a quarter wavelength of transmission line, and terminate the whole thing with load R_L, we have formed a **quarter wave matching network**!

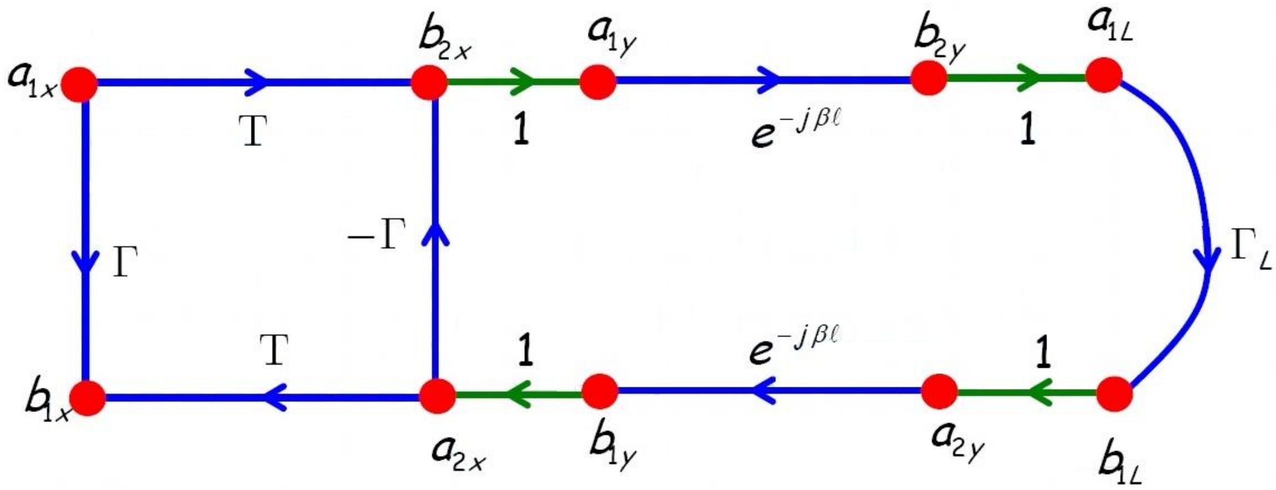


- The boundary conditions associated with these connections are likewise:

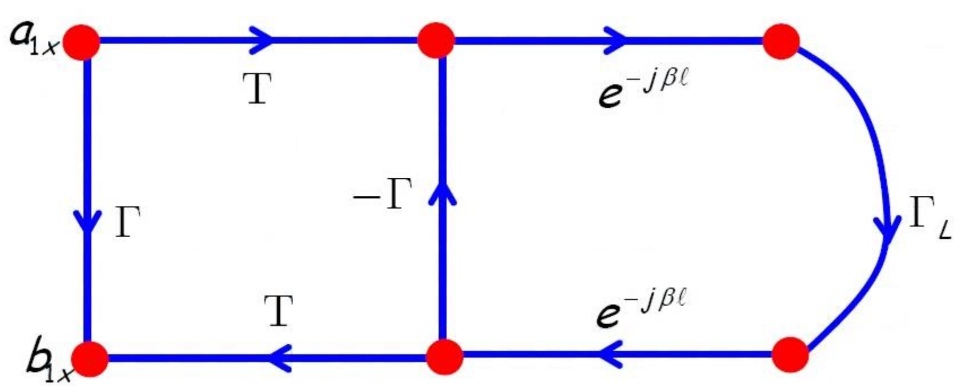
$$a_{1y} = b_{2x} \qquad a_{2x} = b_{1y} \qquad a_{1L} = b_{2y} \qquad a_{2y} = b_{1L}$$

The Quarter Wave Transformer (contd.)

- Therefore, we can put the signal-flow graph pieces together to form the **signal-flow graph** of the quarter wave network:



- **Simplification gives:**



The Quarter Wave Transformer (contd.)

Simplification:

Therefore:

$$\boxed{\Gamma_{in} \doteq \frac{b_{1x}}{a_{1x}} = \Gamma + \frac{T^2\Gamma_L e^{-j2\beta l}}{1-\Gamma\Gamma_L}}$$

Q: Hey wait! If the quarter-wave transformer is a **matching network**, shouldn't Γ_{in} = 0??

A: Who says it isn't! Consider now **three important facts**.

The Quarter Wave Transformer (contd.)

- For a **quarter wave transformer**, we set Z_1 such that:

$$Z_1^2 = Z_0 R_L \qquad \Rightarrow \qquad Z_0 = \frac{Z_1^2}{R_L}$$

- **Inserting** this into the scattering parameter S_{11} of the connector, we find:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{Z_1 - Z_1^2 / R_L}{Z_1 + Z_1^2 / R_L} = \frac{R_L - Z_1}{R_L + Z_1}$$

- For the quarter-wave transformer, the **connector** S_{11} value (i.e., Γ) is the **same** as the **load** reflection coefficient Γ_L :

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L$$

Fact 1

- Since the connector is **lossless** (unitary scattering matrix!), we can conclude (and likewise show) that:

$$1 = |S_{11}|^2 + |S_{21}|^2 = |\Gamma|^2 + |T|^2$$

The Quarter Wave Transformer (contd.)

- Since Z_0 , Z_1 , and R_L are all real, the values Γ and T are also **real valued**. As a result, $|\Gamma|^2 = \Gamma^2$ and $|T|^2 = T^2$, and we can likewise conclude:

$$\boxed{|\Gamma|^2 + |T|^2 = \Gamma^2 + T^2 = 1} \Leftarrow \textbf{Fact 2}$$

- Likewise, the Z_1 transmission line has $l = \lambda/4$, so that:

$$\boxed{2\beta l = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi} \Rightarrow \boxed{e^{-j\beta l} = e^{-j\pi} = -1} \Leftarrow \textbf{Fact 3}$$

- As a result:

$$\Gamma_{in} = \Gamma + \frac{T^2\Gamma_L e^{-j2\beta l}}{1-\Gamma\Gamma_L} = \Gamma - \frac{T^2\Gamma_L}{1-\Gamma\Gamma_L}$$

- And using the **newly discovered** fact that (for a correctly designed transformer) $\Gamma_L = \Gamma$:

$$\Gamma_{in} = \Gamma - \frac{T^2\Gamma}{1-\Gamma^2}$$

The Quarter Wave Transformer (contd.)

- We also have a **recent** discovery that says $T^2 = 1 - \Gamma^2$, therefore:

$$\Gamma_{in} = \Gamma - \frac{T^2\Gamma}{1-\Gamma^2} = \Gamma - \frac{T^2\Gamma}{T^2} = 0$$

A **perfect match**! The quarter-wave transformer does indeed work!