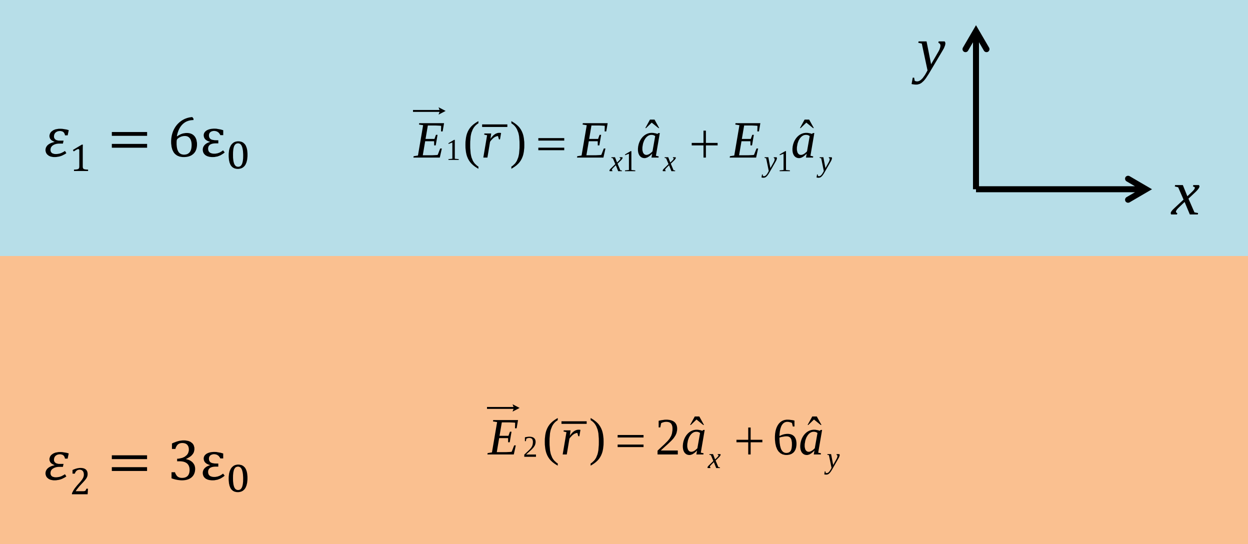
Lecture – 9

Date: 05.02.2015

- Boundary Conditions (contd.)
- Electrostatic Boundary Value Problems

Example – 1: Boundary Conditions

- Two slabs of dissimilar **dielectric** material share a common **boundary**, as shown below. The respective electric field is also shown.



In **each** dielectric region, let's determine (in terms of ε_0):

(1) the **electric field,** **(2)** the **electric flux density,** **(3)** the **bound volume charge density** (i.e., the equivalent polarization charge density) within the dielectric, and **(4)** the **bound surface charge density** (i.e., the equivalent polarization charge density) at the dielectric interface

Example – 1 (contd.)

- Since we already know the electric field in the second region, let's evaluate **region 2** first.
- We can easily determine the **electric flux density** within the region:

$$\vec{D}_2(\overline{r}) = \varepsilon_2 \vec{E}_2(\overline{r}) \quad \Longrightarrow \quad \vec{D}_2(\overline{r}) = 3\varepsilon_0\left(2\hat{a}_x + 6\hat{a}_y\right)$$

$$\boxed{\therefore \vec{D}_2(\overline{r}) = 6\varepsilon_0\hat{a}_x + 18\varepsilon_0\hat{a}_y}$$

- **Likewise, the polarization vector within the region is:**

$$\vec{P}_2(\overline{r}) = \varepsilon_0 \chi_{e2} \vec{E}_2(\overline{r}) \quad \Longrightarrow \quad \vec{P}_2(\overline{r}) = \varepsilon_0\left(\varepsilon_{r2} - 1\right)\left(2\hat{a}_x + 6\hat{a}_y\right)$$

$$\Rightarrow \vec{P}_2(\overline{r}) = \varepsilon_0\left(3 - 1\right)\left(2\hat{a}_x + 6\hat{a}_y\right) \quad \Longrightarrow \quad \boxed{\therefore \vec{P}_2(\overline{r}) = 4\varepsilon_0\hat{a}_x + 12\varepsilon_0\hat{a}_y}$$

Example – 1 (contd.)

Q: Why did we determine the **polarization** vector? It is **not** one of the quantities this problem asked for!

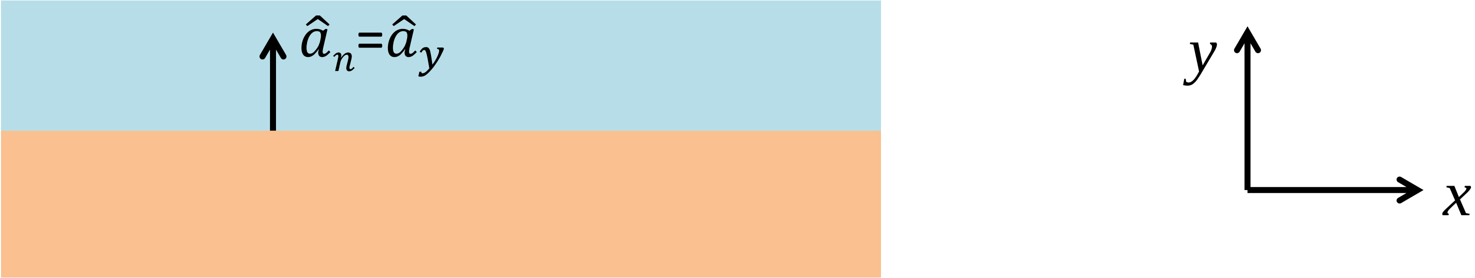
A: True! But the problem **did** ask for the equivalent **bound charge densities** (both volume and surface) within the dielectric. We need to know polarization vector $\vec{P}(\bar{r})$ to find this **bound** charge!

- Recall the bound **volume** charge density is: $\boxed{\rho_{vp}(\bar{r}) = -\nabla.\vec{P}(\bar{r})}$
- and the bound **surface** charge density is: $\boxed{\rho_{sp}(\bar{r}) = \vec{P}(\bar{r}).\hat{a}_n}$
- Since the polarization vector $\vec{P}(\bar{r})$ is a **constant** (i.e., it has precisely the same magnitude and direction at every point within region 2), we find that the divergence of $\vec{P}(\bar{r})$ is **zero,** and thus the volume bound charge density is zero within the region:

$$\rho_{vp2}(\bar{r}) = -\nabla.\vec{P}_2(\bar{r}) \Longrightarrow \rho_{vp2}(\bar{r}) = -\nabla.\left(4\varepsilon_0\hat{a}_x + 12\varepsilon_0\hat{a}_y\right) \Longrightarrow \boxed{\therefore \rho_{vp2}(\bar{r}) = 0}$$

Example – 1 (contd.)

- However, we find that the **surface** bound charge density is **not** zero!
- Note that the unit vector normal to the **surface** of the bottom dielectric slab is $\hat{a}_{n2}=\hat{a}_y$:



- Since the polarization vector is constant, we know that its value at the **dielectric interface** is likewise equal to $4\varepsilon_0\hat{a}_x+12\varepsilon_0\hat{a}_y$. Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the top of region 2 (at the dielectric interface) is:

$$\rho_{sp2}(\bar{r}_b)=\vec{P}_2(\bar{r}_b).\hat{a}_{n2} \Rightarrow \rho_{sp2}(\bar{r}_b)=\left(4\varepsilon_0\hat{a}_x+12\varepsilon_0\hat{a}_y\right).\hat{a}_y \Rightarrow \boxed{\therefore\ \rho_{sp2}(\bar{r}_b)=12\varepsilon_0}$$

Example – 1 (contd.)

- Now, let’s determine these same quantities for **region 1** (i.e., the **top** dielectric slab).

Q1: How the heck can we do this? We don’t know **anything** about the fields in region 1 !

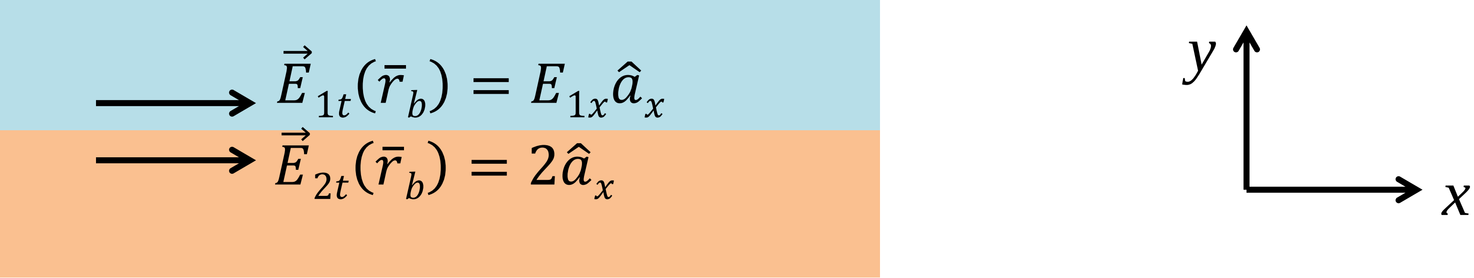
A1: True! We don’t know $\vec{E}_1(\bar{r})$ or $\vec{D}_1(\bar{r})$ or even $\vec{P}_1(\bar{r})$. However, we know the **next** best thing—we know $\vec{E}_2(\bar{r})$ and $\vec{D}_2(\bar{r})$ and even $\vec{P}_2(\bar{r})$!

Q2: Huh!?!

A2: We can use **boundary conditions** to transfer our solutions from region 2 into region 1!

Example – 1 (contd.)

- First, we note that **at the dielectric interface**, the vector components of the electric fields **tangential** to the interface are $\vec{E}_{1t}(\bar{r}_b) = E_{1x}\hat{a}_x$ and $\vec{E}_{2t}(\bar{r}_b) = 2\hat{a}_x$:

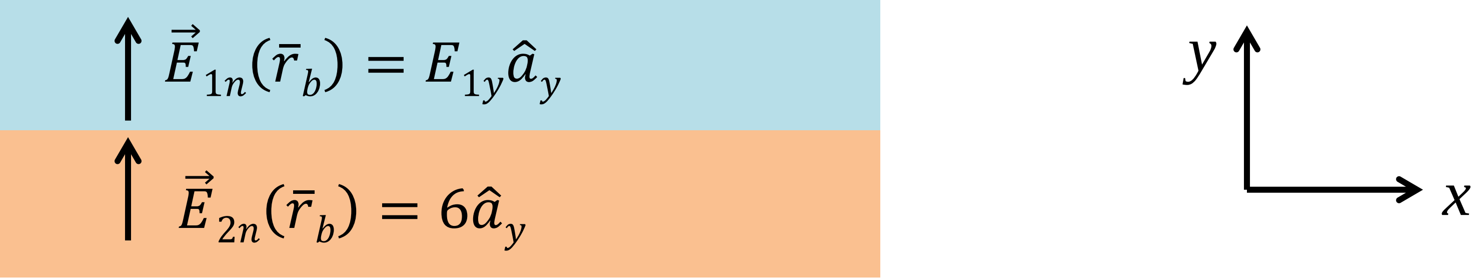


- Thus, applying the **boundary condition** $\vec{E}_{1t}(\bar{r}_b) = \vec{E}_{2t}(\bar{r}_b)$, we find:

$$E_{1x}\hat{a}_x = 2\hat{a}_x \quad \Longrightarrow \quad \boxed{\therefore E_{1x} = 2}$$

Example – 1 (contd.)

- Likewise, we note that **at the dielectric interface**, the vector components of the electric fields **normal** to the interface are $\vec{E}_{1n}(\bar{r}_b) = E_{1y}\hat{a}_y$ and $\vec{E}_{2n}(\bar{r}_b) = 6\hat{a}_y$:



- Here, we can apply a **second boundary condition**, $\varepsilon_1\vec{E}_{1n}(\bar{r}_b) = \varepsilon_2\vec{E}_{2n}(\bar{r}_b)$:

$$6\varepsilon_0 * E_{1y}\hat{a}_y = 3\varepsilon_0 * 6\hat{a}_y \Rightarrow E_{1y}\hat{a}_y = 3\hat{a}_y \Rightarrow \boxed{\therefore E_{1y} = 3}$$

- Thus, the electric field in the top region is:

$$\vec{E}_1(\bar{r}) = E_{1x}\hat{a}_x + E_{1y}\hat{a}_y \Rightarrow \boxed{\therefore \vec{E}_1(\bar{r}) = 2\hat{a}_x + 3\hat{a}_y}$$

Example – 1 (contd.)

- We can then find the **electric flux density** by multiplying by the permittivity of **region 1** ($\varepsilon_1 = 6\varepsilon_0$).

$$\vec{D}_1(\bar{r}) = \varepsilon_1 \vec{E}_1(\bar{r}) \quad \Longrightarrow \quad \boxed{\therefore \vec{D}_1(\bar{r}) = 12\varepsilon_0 \hat{a}_x + 18\varepsilon_0 \hat{a}_y}$$

- Note we could have solved this problem **another** way!
- **Instead** of applying boundary conditions to $\vec{E}_2(\bar{r})$, we **could** have applied them to **electric flux density** $\vec{D}_2(\bar{r})$:

$$\vec{D}_2(\bar{r}) = 6\varepsilon_0 \hat{a}_x + 18\varepsilon_0 \hat{a}_y$$

- We know that the **electric flux density** within region 1 must be constant, i.e.:

$$\boxed{\vec{D}_1(\bar{r}) = D_{1x}\hat{a}_x + D_{1y}\hat{a}_y}$$

Example – 1 (contd.)

- The vector fields $\vec{D}_1(\bar{r})$ and $\vec{D}_2(\bar{r})$ **at the interface** are related by the **boundary conditions**:

$$\frac{\vec{D}_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{\vec{D}_{2t}(\bar{r}_b)}{\varepsilon_2} \qquad \vec{D}_{1n}(\bar{r}_b) = \vec{D}_{2n}(\bar{r}_b)$$

- After simplification, we find that the **electric flux density** in **region 1** is:

$$\boxed{\vec{D}_1(\bar{r}) = 12\varepsilon_0\hat{a}_x + 18\varepsilon_0\hat{a}_y}$$

Precisely the **same** result as before!

- We can then find the **electric field** in **region 1** by **dividing** the **obtained electric flux density** by the dielectric permittivity:

$$\boxed{\vec{E}_1(\bar{r}) = \frac{\vec{D}_1(\bar{r})}{\varepsilon_1} = 2\hat{a}_x + 3\hat{a}_y}$$

the **same** result as before!

Example – 1 (contd.)

- Now, finishing this problem, we need to find the **polarization** vector $\vec{P}_1(\bar{r})$:

$$\vec{P}_1(\bar{r}) = \varepsilon_0\left(\varepsilon_{r1} - 1\right)\vec{E}_1(\bar{r}) \Longrightarrow \vec{P}_1(\bar{r}) = \varepsilon_0\left(6-1\right)\left(2\hat{a}_x + 3\hat{a}_y\right)$$

$$\boxed{\therefore \vec{P}_1(\bar{r}) = 10\varepsilon_0\hat{a}_x + 15\varepsilon_0\hat{a}_y}$$

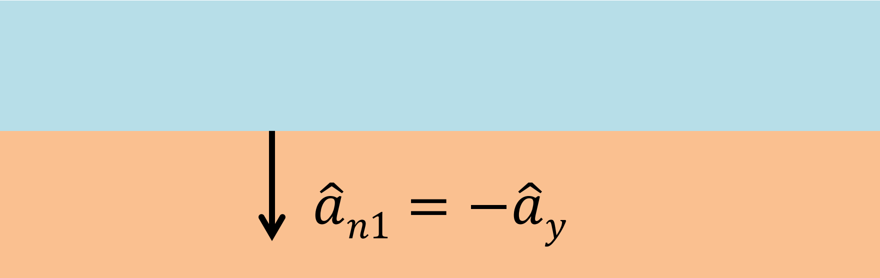
- Thus, the **volume** charge density of **bound** charge is again **zero**:

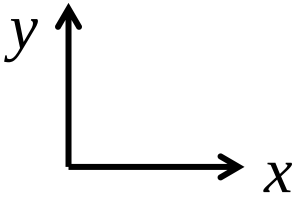
$$\rho_{vp1}(\bar{r}) = -\nabla.\vec{P}_1(\bar{r}) \Longrightarrow \rho_{vp1}(\bar{r}) = -\nabla.\left(10\varepsilon_0\hat{a}_x + 15\varepsilon_0\hat{a}_y\right) \Longrightarrow \boxed{\therefore \rho_{vp1}(\bar{r}) = 0}$$

However, we again find that the **surface** bound charge density is **not** zero!

Example – 1 (contd.)

- Note that the unit vector **normal** to the **bottom** surface of the **top** dielectric slab points **downward**, i.e., $\hat{a}_{n1} = -\hat{a}_y$:





- Since the polarization vector is **constant**, we know that its value **at the dielectric interface** is likewise equal to: $\vec{P}_1(\bar{r}) = 10\varepsilon_0\hat{a}_x + 15\varepsilon_0\hat{a}_y$

- Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the bottom of region 1 (at the dielectric interface) is:

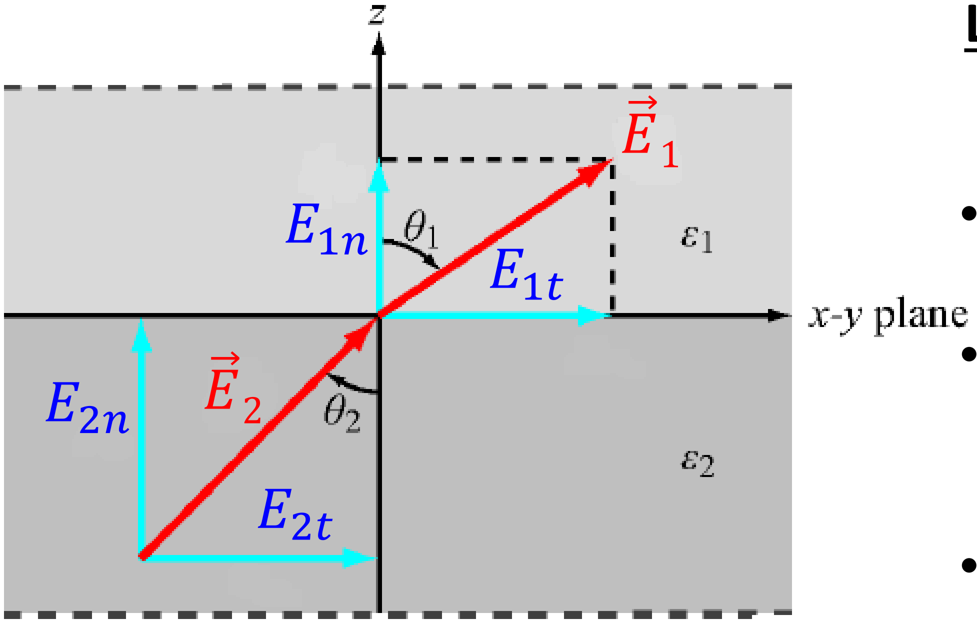
$$\rho_{sp1}(\bar{r}_b) = \vec{P}_1(\bar{r}_b).\hat{a}_{n1} \Rightarrow \rho_{sp1}(\bar{r}_b) = \left(10\varepsilon_0\hat{a}_x + 15\varepsilon_0\hat{a}_y\right).\left(-\hat{a}_y\right) \Rightarrow \boxed{\therefore \rho_{sp1}(\bar{r}_b) = -15\varepsilon_0}$$

- Now, we can determine the **net** surface charge density of **bound** charge that is lying **on the dielectric interface**:

$$\rho_{sp}(\bar{r}_b) = \rho_{sp1}(\bar{r}_b) + \rho_{sp2}(\bar{r}_b) \Rightarrow \boxed{\rho_{sp}(\bar{r}_b) = -15\varepsilon_0 + 12\varepsilon_0 = -3\varepsilon_0}$$

Example – 2

- In the following figure, the x-y plane is a charge free boundary separating two dielectric media with permittivities ε_{r1} and ε_{r2}. If the electric field in medium 1 is $\vec{E}_1 = E_{1x}\hat{a}_x + E_{1y}\hat{a}_y + E_{1z}\hat{a}_z$, find (a) the electric field $\vec{E}_2$ in medium 2, and (b) the angles θ_1 and θ_2.



Let,

$$\vec{E}_2 = E_{2x}\hat{a}_x + E_{2y}\hat{a}_y + E_{2z}\hat{a}_z$$

- Here, the normal to the boundary is $\hat{a}_z$
- Therefore, the x and y components are tangential and z components are normal to the boundary
- **At the charge free interface,** the tangential components of $\vec{E}$ **and** normal component of $\vec{D}$ **are continuous.**

Example – 2 (contd.)

- Therefore

$$\underbrace{E_{2x} = E_{1x} \qquad E_{2y} = E_{1y}}_{\text{Tangential Components}} \qquad \underbrace{D_{2z} = D_{1z}}_{\text{Normal Component}} \rightarrow \boxed{\varepsilon_2 E_{2z} = \varepsilon_1 E_{1z}}$$

- **Thus,**

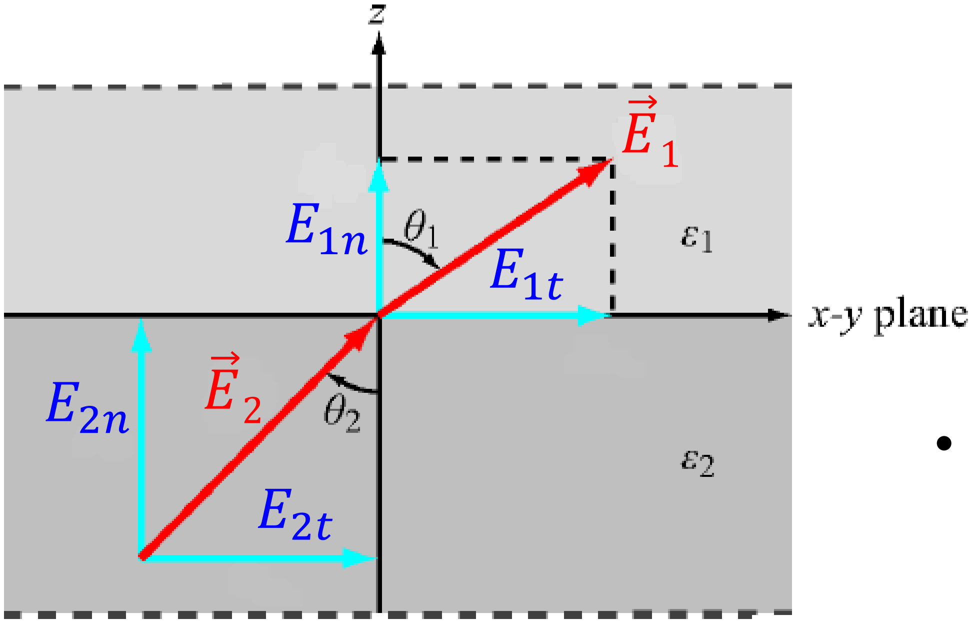
$$\boxed{\vec{E}_2 = E_{2x}\hat{a}_x + E_{2y}\hat{a}_y + E_{2z}\hat{a}_z = E_{1x}\hat{a}_x + E_{1y}\hat{a}_y + \frac{\varepsilon_1}{\varepsilon_2} E_{1z}\hat{a}_z}$$

Example – 2 (contd.)

- The tangential components of $\vec{E}_1$ and $\vec{E}_2$ are:

$$\boxed{E_{1t} = \sqrt{E_{1x}^2 + E_{1y}^2}} \qquad \boxed{E_{2t} = \sqrt{E_{2x}^2 + E_{2y}^2}} \Rightarrow \boxed{E_{2t} = \sqrt{E_{1x}^2 + E_{1y}^2}}$$

- Therefore the angles θ_1 and θ_2 can be written as:



$$\tan\theta_1 = \frac{E_{1t}}{E_{1n}} \Rightarrow \tan\theta_1 = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}}$$

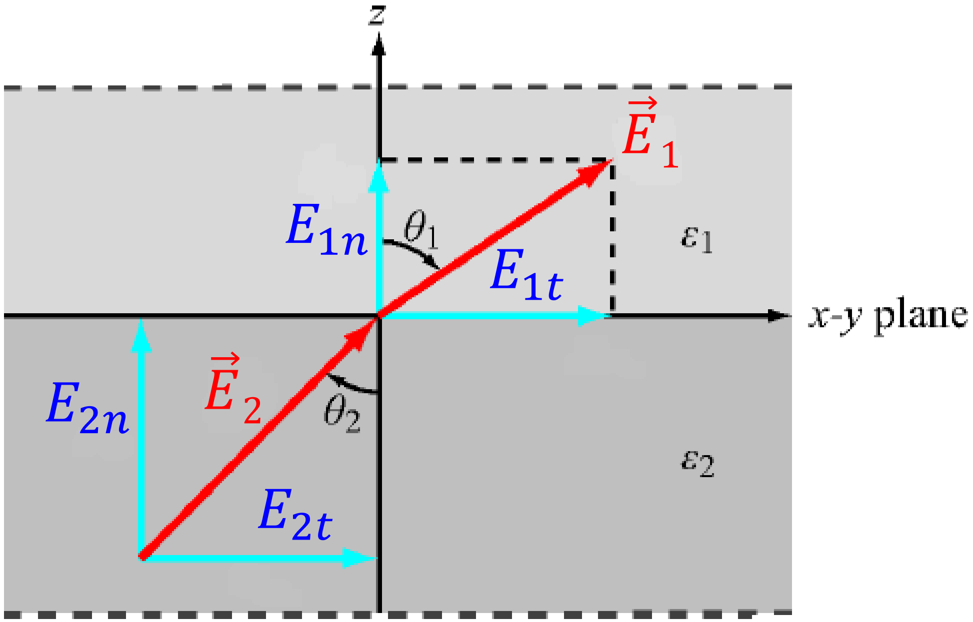
$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} \Rightarrow \tan\theta_2 = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{\left(\frac{\varepsilon_1}{\varepsilon_2}\right)E_{1z}}$$

- The two angles are related as:

$$\boxed{\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1}}$$

Example – 3

- Find $\vec{E}_1$ in the following figure, if $\vec{E}_2 = 2\hat{a}_x - 3\hat{a}_y + 3\hat{a}_z$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 8\varepsilon_0$ and the boundary is charge free.



- Given that the x–y plane is the boundary between the two media, the x- and y-components of $\vec{E}_2$ are parallel to the boundary, and therefore are the same across the two sides of the boundary. Thus,

$$E_{1x} = E_{2x} = 2 \qquad E_{1y} = E_{2y} = -3$$

For the z-component

$$\varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \Longrightarrow E_{1z} = \frac{8\varepsilon_0}{2\varepsilon_0} E_{2z} = 12$$

- Therefore: $\vec{E}_1 = E_{1x}\hat{a}_x + E_{1y}\hat{a}_y + E_{1z}\hat{a}_z \Longrightarrow \vec{E}_1 = 2\hat{a}_x - 3\hat{a}_y + 12\hat{a}_z$ V/m

Example – 4

- **Repeat example – 3** for a boundary with surface charge density $\rho_s = 3.54 \times 10^{-11}\ C/m^2$.

From example-2: $E_{1x} = 2$ $E_{1y} = -3$

For z-component: $\boxed{\varepsilon_1 E_{1z} - \varepsilon_2 E_{2z} = \rho_s}$

$$\boxed{\Rightarrow E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1}} \quad \boxed{\Rightarrow E_{1z} = \frac{3.54\times 10^{-11} + 8\varepsilon_0 \times 3}{2\varepsilon_0}} \quad \boxed{\therefore E_{1z} = 14}$$

- Therefore: $\vec{E}_1 = E_{1x}\hat{a}_x + E_{1y}\hat{a}_y + E_{1z}\hat{a}_z$

$$\boxed{\vec{E}_1 = 2\hat{a}_x - 3\hat{a}_y + 14\hat{a}_z \text{ V/m}}$$

Example – 5

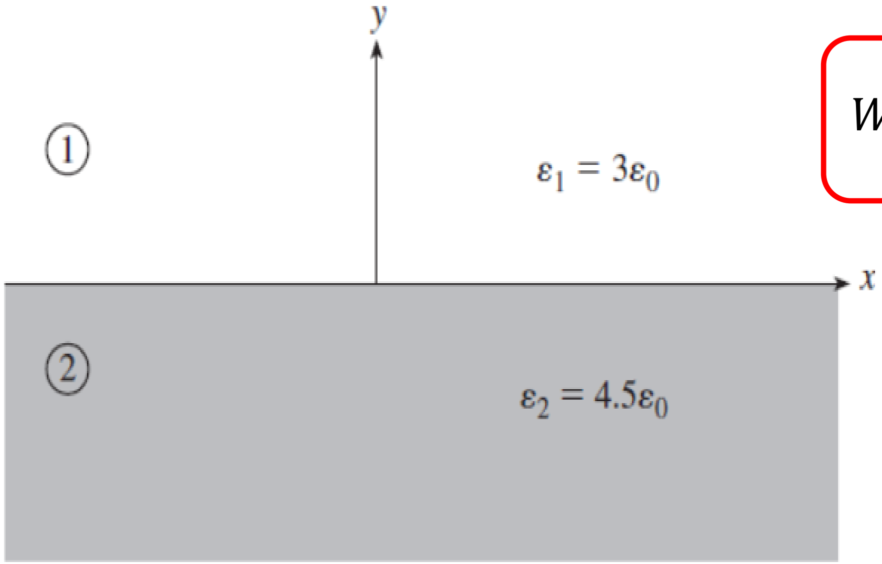
- Given that $\vec{E}_1 = 10\hat{a}_x - 6\hat{a}_y + 12\hat{a}_z$ V/m in following figure. Find (a) $\vec{P}_1$, (b) $\vec{E}_2$ and the angle $\vec{E}_2$ makes with the y-axis, (c) the energy density in each region.

$$\vec{P}_1(\vec{r}) = 0.1768\hat{a}_x - 0.1061\hat{a}_y + 0.2122\hat{a}_z \ {}^{nC}/_{m^2}$$

$$\vec{E}_2 = 10\hat{a}_x - 4\hat{a}_y + 12\hat{a}_z \quad \text{V/m}$$

$$\theta_2 = 75.64°$$

$$W_{E1} = 3.7136 \ {}^{nJ}/_{m^3} \qquad W_{E2} = 5.1725 \ {}^{nJ}/_{m^3}$$



Example – 6

- A silver-coated sphere of radius 5cm carries a total charge of 12nC uniformly distributed on its surface in free space. Calculate (a) $|\vec{D}|$ on the surface of the sphere, (b) $\vec{D}$ external to the sphere, (c) the total energy stored in the field.

$$\boxed{\left|\vec{D}\right| = 381.97 \quad {}^{nC}/_{m^2}}$$

$$\boxed{\vec{D} = \frac{0.955}{r^2}\hat{a}_r \quad {}^{nC}/_{m^2}}$$

$$\boxed{W = 12.96\ \mu J}$$

Example – 7

- A dielectric interface is defined by $4x + 3y = 10m$. The region including the origin is free space, where $\vec{D}_1 = 2\hat{a}_x - 4\hat{a}_y + 6.5\hat{a}_z$ nC/m2. In the other region, $\varepsilon_{r2} = 2.5$. Find $\vec{D}_2$ and the angle θ_2 that $\vec{D}_2$ makes with the normal.

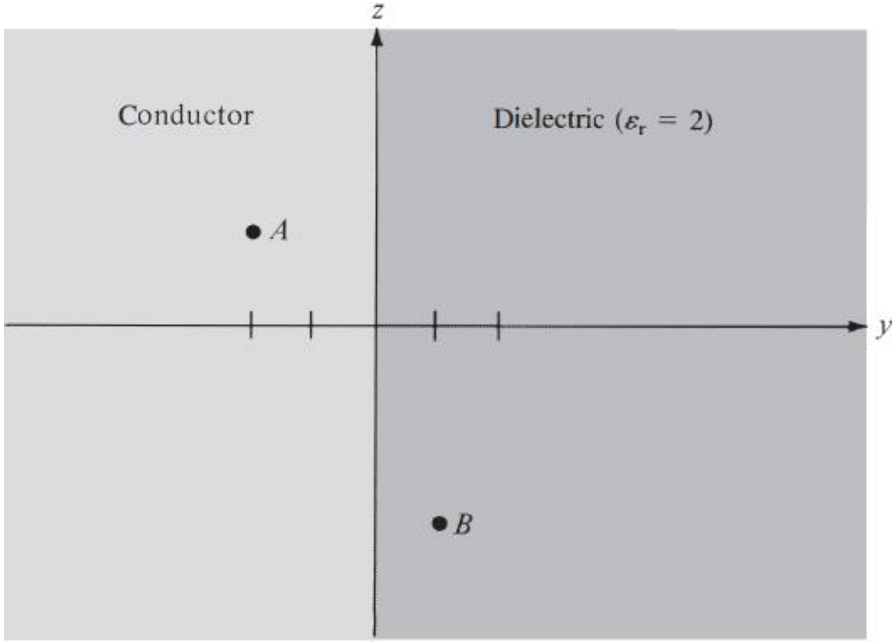
$$\boxed{\vec{D}_2 = 5.96\hat{a}_x - 9.28\hat{a}_y + 16.25\hat{a}_z}$$

$$\boxed{\theta_2 = -86.74°}$$

Example – 8

- Region $y < 0$ consists of a perfect conductor while region $y > 0$ is a dielectric medium $(\varepsilon_{1r} = 2)$ as shown below. If there is a surface charge of $2\ nC/m^2$ on the conductor , determine $\vec{E}$ and $\vec{D}$ at:

(a) $A(3, -2, 2)$ (b) $B(-4, 1, 5)$



Example – 8 (contd.)

(a) Point $A(3, -2, 2)$ is in the conductor since $y = -2 < 0$ at A. Hence:

$$\vec{E} = 0 = \vec{D}$$

(b) Point $B(-4, 1, 5)$ is in the dielectric medium since $y = 1 > 0$ at B. Hence:

$$D_n = \rho_s = 2\, nC/m^2$$

<u>Therefore:</u>

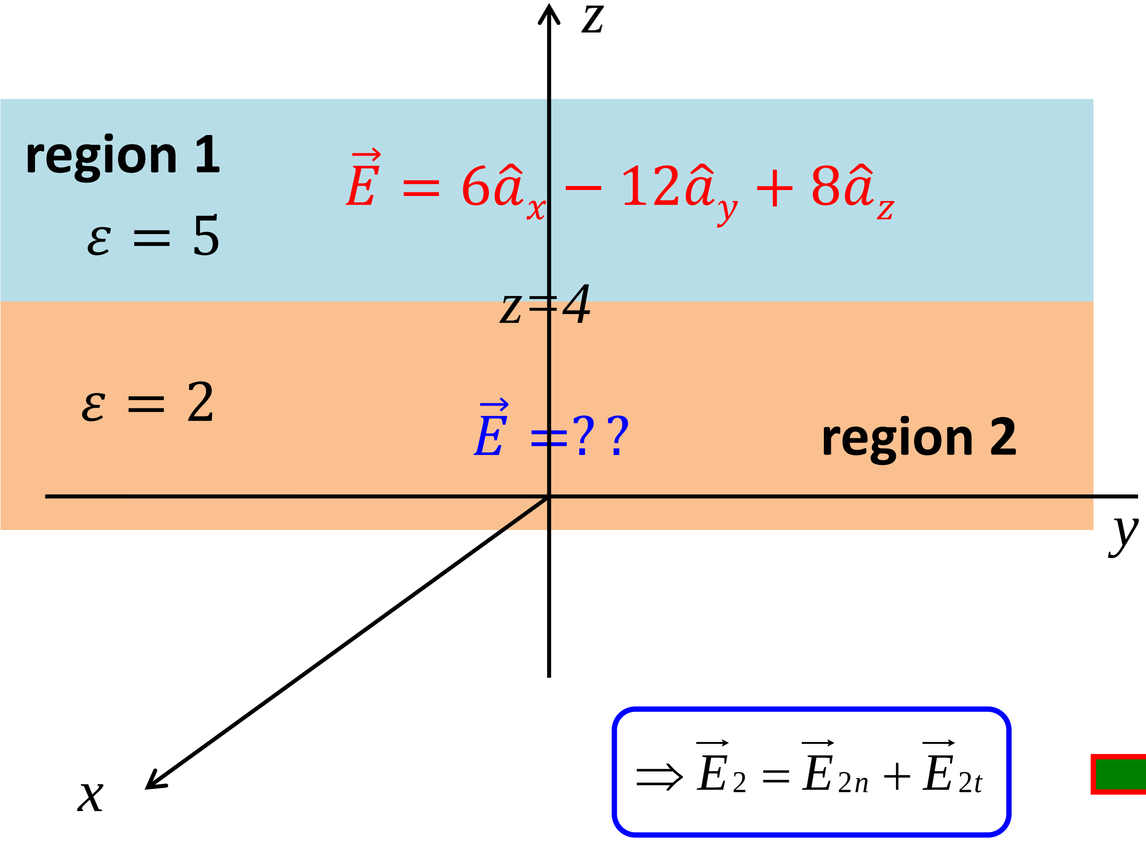
$$\vec{D} = 2\hat{a}_y \quad nC/m^2$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0 \varepsilon_{1r}} \quad V/m$$

$$\vec{E} = 113.1\hat{a}_y \quad V/m$$

Example – 9

- The plane $z = 4$ is the interface between two dielectrics. The dielectric region $z > 4$ has dielectric constant of 5 and $\vec{E} = 6\hat{a}_x - 12\hat{a}_y + 8\hat{a}_z$ (V/m). If the dielectric constant is 2 in region $z < 4$, find the electric field intensity in that region.



$$\vec{E}_{1n} = \vec{E}_{1z} = 8\hat{a}_z$$

$$\vec{E}_{1t} = 6\hat{a}_x - 12\hat{a}_y = \vec{E}_{2t}$$

$$\boxed{\vec{E}_{2n} = \frac{\varepsilon_1}{\varepsilon_2}\vec{E}_{1n}}$$

$$\boxed{\vec{E}_{2n} = \frac{5}{2} \times 8\hat{a}_z = 20\hat{a}_z}$$

$$\boxed{\Rightarrow \vec{E}_2 = \vec{E}_{2n} + \vec{E}_{2t}}$$

$$\boxed{\therefore \vec{E}_2 = 6\hat{a}_x - 12\hat{a}_y + 20\hat{a}_z}$$

Until now: we used Coulomb's law and Gauss's law to determine $\vec{E}$ when the charge distribution is known or $\vec{E} = -\nabla \mathrm{V}$ when the potential is known throughout the region.

Now: we will consider practical electrostatics problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find $\vec{E}$ and V throughout the region. Such problems are usually solved using Poisson's or Laplace's equation or "Method of Images"

Poisson's and Laplace's Equation

- From Gauss's Law: $\nabla.\vec{E} = \frac{\rho_v}{\varepsilon}$
- We have: $\vec{E} = -\nabla V$

$$\nabla.(\nabla V) = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

(Poisson's Equation)

- If the medium under consideration contains no charge then: $\nabla^2 V = 0$

Laplace's Equations

These formulations are extremely useful for determining the electrostatic potential V in regions with boundaries on which V is known, such as the regions between the plates of a capacitor with specified voltage difference across it.

Poisson's and Laplace's Equation (contd.)

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho 2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}}$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_v}{\varepsilon}}$$

- The corresponding Laplace's equations are:

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho 2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0}$$

Uniqueness Theorem

- One can use any of the available methods (analytical, graphical, numerical, experimental etc.) to solve Laplace‘s or Poisson’s equations.
- **If the solution exits** then **that solution is unique** irrespective of the method used to determine them.
- This is known as **Uniqueness Theorem**.
- **Proof of this theorem** – through contradiction [follow your text book]
- **Before we begin to solve Boundary-Value-Problems**, we should bear in mind the **three things** that uniquely describe a problem:

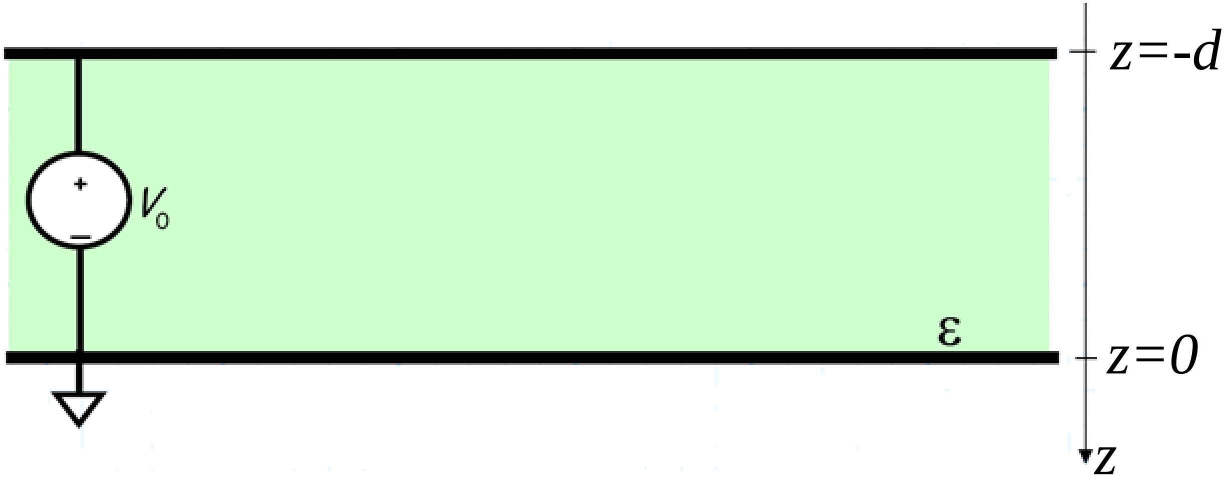
1. **The appropriate differential equation** (Laplace’s or Poisson’s equation)
2. **The solution region**
3. **The prescribed boundary conditions**

Procedure for Solving Poisson’s or Laplace’s Equations

1. Solve Laplace’s (if $\rho_v = 0$) or Poisson’s (if $\rho_v \neq 0$) equation using either **(a) direct integration when V is a function of one variable** **or** **(b) separation of variables if V is a function of more than one variable**. The solution at this point is not unique but is expressed in terms of unknown integration constants to be determined.
2. Apply the **boundary conditions** to determine the unique solution for V. Imposing the given boundary conditions makes the solution unique.
3. Having obtained V, find $\vec{E}$ using $\vec{E} = -\nabla V$, $\vec{D}$ from $\vec{D} = \varepsilon \vec{E}$, and $\vec{J}$ from $\vec{J} = \sigma \vec{E}$.
4. If required, find the charge Q induced on a conductor using $Q = \int \rho_s \, dS$, **where** $\rho_s = D_n$ **and** D_n is the component of $\vec{D}$ normal to the conductor. **If necessary,** the capacitance of two conductors can be found using $C = Q/V$ or the resistance of an object can be found using $R = \frac{V}{I}$, where $I = \int \vec{J}.\overline{dS}$.

Example – 10: Dielectric Filled Parallel Plates

- Consider two infinite, parallel **conducting** plates, spaced a distance d apart. The region between the plates is filled with a dielectric ε. Say a voltage V_0 is placed across these plates.



Q: What electric potential field $V(\bar{r})$, electric field $\vec{E}(\bar{r})$ and charge density $\rho_s(\bar{r})$ is produced by this situation?

Example – 10 (contd.)

A: We must solve a **boundary value problem**! We must find solutions that:

a) Satisfy the **differential equations** of electrostatics (e.g., Poisson's, Laplace's, Gauss's).

b) Satisfy the electrostatic **boundary conditions**.

Q: Yikes! Where do we even start ?

A: We might start with the electric potential field $V(\bar{r})$, since it is a **scalar** field.

a) The electric potential function must satisfy **Poisson's equation**:

$$\boxed{\nabla^2 V(\bar{r}) = -\frac{\rho_v(\bar{r})}{\varepsilon}}$$

b) It must also satisfy the **boundary conditions**:

$$\boxed{V(z=-d)=V_0 \qquad\qquad V(z=0)=0}$$

Example – 10 (contd.)

- Consider first the dielectric region $(-d < z < 0)$. Since the region is a dielectric, there is **no** free charge, and:

$$\boxed{\rho_v(\bar{r}) = 0}$$

- Therefore, Poisson’s equation reduces to **Laplace’s equation**:

$$\boxed{\nabla^2 V(\bar{r}) = 0}$$

- This problem is greatly simplified, as it is evident that the solution $V(\bar{r})$ is independent of coordinates x and y. In other words, the electric potential field will be a function of coordinate z **only**:

$$\boxed{V(\bar{r}) = V(z)}$$

- This make the problem **much** easier! Laplace’s equation becomes:

$$\boxed{\nabla^2 V(\bar{r}) = 0} \Rightarrow \boxed{\nabla^2 V(z) = 0} \Rightarrow \boxed{\frac{\partial^2 V(z)}{\partial z^2} = 0}$$

Example – 10 (contd.)

- Integrating **both** sides of Laplace’s equation, we get:

$$\int\left(\frac{\partial^2 V(z)}{\partial z^2}\right)dz = \int(0)\,dz \quad \Longrightarrow \quad \frac{\partial V(z)}{\partial z} = C_1$$

- And integrating **again** we find:

$$\int\left(\frac{\partial V(z)}{\partial z}\right)dz = \int(C_1)dz \quad \Longrightarrow \quad V(z) = C_1 z + C_2$$

- We find that the equation $V(z) = C_1 z + C_2$ **will** satisfy Laplace’s equation (try it!). We must now apply the **boundary conditions** to determine the value of **constants** C_1 and C_2.
- We know that the value of the electrostatic potential at every point **on the top** plate $(z = -d)$ is $V(-d) = V_0$, while the electric potential **on the bottom** plate $(z = 0)$ is zero $V(0) = 0$. Therefore:

$$\boxed{V(z=-d) = -C_1 d + C_2 = V_0} \qquad \boxed{V(z=0) = C_1(0) + C_2 = 0}$$

Example – 10 (contd.)

- **Two** equations and **two** unknowns (C_1 and C_2)!
- **Solving** for C_1 and C_2 we get:

$$C_2 = 0 \qquad\qquad C_1 = -\frac{V_0}{d}$$

- and therefore, the **electric potential** field within the dielectric is found to be:

$$V(z) = C_1 z + C_2 \Longrightarrow \boxed{V(\bar{r}) = -\frac{V_0}{d} z}$$

- Before we proceed, let's do a **sanity check**!
- In other words, let's evaluate our answer at $z = 0$ and $z = -d$, to make **sure** our result is correct:

$$V(z=-d) = -\frac{V_0}{d}(-d) = V_0 \qquad ✓$$

$$V(z=0) = -\frac{V_0}{d}(0) = 0 \qquad ✓$$

Example – 10 (contd.)

- Now, we can find the **electric field** within the dielectric by taking the **gradient** of our result:

$$\vec{E}(\bar{r}) = -\nabla V(\bar{r}) \qquad \Longrightarrow \qquad \vec{E}(\bar{r}) = \frac{V_0}{d}\hat{a}_z \qquad -d \leq z \leq 0$$

- And thus we can easily determine the **electric flux density** by multiplying by the dielectric constant of the material:

$$\boxed{\vec{D}(\bar{r}) = \varepsilon \vec{E}(\bar{r}) = \varepsilon \frac{V_0}{d}\hat{a}_z \qquad -d \leq z \leq 0}$$

- Finally, we need to determine the **charge density** that actually created these fields!

Q: Charge density !?! I thought that we already determined that the charge density $\rho_v(\bar{r})$ is equal to **zero**?

A: We know that the free charge density **within the dielectric** is zero—but there must be charge **somewhere**, otherwise there would be no fields!

Example – 10 (contd.)

- Recall that we found that **at** a conductor/dielectric **interface**, the **surface charge density** on the conductor is related to the **electric flux density** in the dielectric as:

$$\boxed{D_n = \vec{D}(\bar{r}).\hat{a}_n = \rho_s(\bar{r})}$$

- First, we find that the electric flux density on the **bottom** surface of the **top** conductor (i.e., at $z = -d$) is:

$$\boxed{\vec{D}(\bar{r})|_{z=-d} = \left(\varepsilon \frac{V_0}{d}\hat{a}_z\right)|_{z=-d} = \varepsilon \frac{V_0}{d}\hat{a}_z}$$

- For **every** point on **bottom** surface of the **top** conductor, we find that the unit vector **normal** to the conductor is:

$$\boxed{\hat{a}_n = \hat{a}_z}$$

Example – 10 (contd.)

- Therefore, we find that the **surface charge density** on the bottom surface of the top conductor is:

$$\rho_{s+}(\overline{r}) = \vec{D}(\overline{r}).\hat{a}_n \,|_{z=-d} = \varepsilon \frac{V_0}{d}\hat{a}_z.\hat{a}_z \quad \Rightarrow \quad \boxed{\therefore \rho_{s+}(\overline{r}) = \varepsilon \frac{V_0}{d} \qquad (z = -d)}$$

- Likewise, we find the unit vector **normal** to the **top** surface of the **bottom** conductor is (do you see why):

$$\boxed{\hat{a}_n = -\hat{a}_z}$$

- Therefore, evaluating the **electric flux density** on the top surface of the bottom conductor (i.e., $z = 0$), we find:

$$\rho_{s-}(\overline{r}) = \vec{D}(\overline{r}).\hat{a}_n \,|_{z=0} = \varepsilon \frac{V_0}{d}\hat{a}_z.(-\hat{a}_z) \quad \Rightarrow \quad \boxed{\therefore \rho_{s-}(\overline{r}) = \frac{-\varepsilon V_0}{d} \qquad (z = 0)}$$

Example – 10 (contd.)

- We should **note** several things about these solutions:

1) $\nabla \times \vec{E}(\bar{r}) = 0$

2) $\nabla . \vec{D}(\bar{r}) = 0$ and $\nabla^2 V(\bar{r}) = 0$

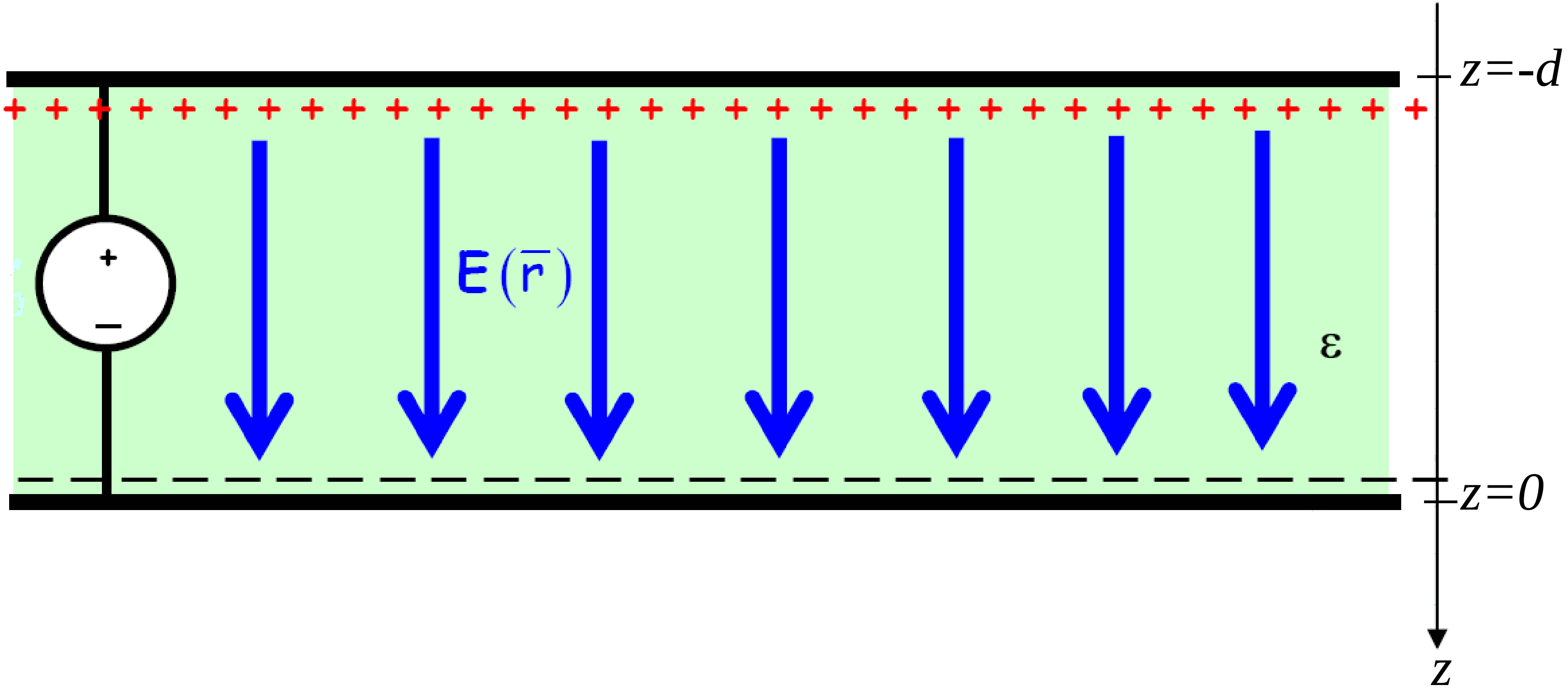
3) $\vec{D}(\bar{r})$ and $\vec{E}(\bar{r})$ are normal to the surface of the conductor (i.e., their tangential components equal zero!

4) The electric field is precisely the same as calculated earlier. i.e.,

$$\boxed{\vec{E}(\bar{r}) = \frac{\rho_{s+}(\bar{r})}{2\varepsilon}\hat{a}_z - \frac{\rho_{s-}(\bar{r})}{2\varepsilon}\hat{a}_z = \frac{V_0}{d}\hat{a}_z \qquad (-d < z < 0)}$$

Example – 10 (contd.)

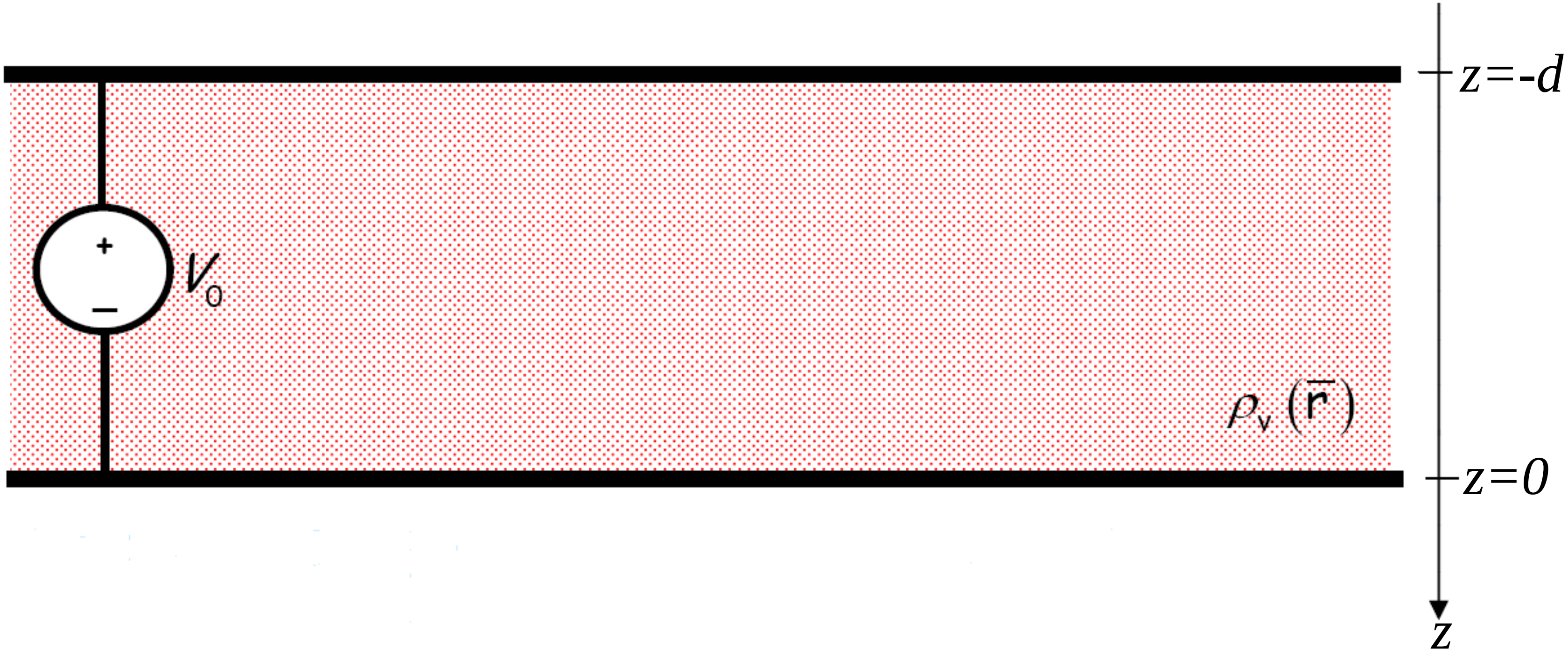
- In other words, the **fields** $\vec{E}(\bar{r})$, $\vec{D}(\bar{r})$, and $V(\bar{r})$ are attributable to **charge densities** $\rho_{s+}(\bar{r})$ and $\rho_{s-}(\bar{r})$.



Example – 11

- Consider now a problem similar to the previous example (i.e., dielectric filled parallel plates), with the exception that the space between the infinite, conducting parallel plates is filled with **free charge**, with a density:

$$\boxed{\rho_v(\bar{r}) = -z\varepsilon_0 \qquad\qquad (-d < z < 0)}$$



Example – 11 (contd.)

Q: How do we determine the fields within the parallel plates for **this** problem?

A: Same as before! However, since the charge density between the plates is **not** equal to zero, we recognize that the electric potential field must satisfy **Poisson's equation**:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0}$$

- For the specific charge density $\rho_v(\bar{r}) = -z\varepsilon_0$:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0} = z$$

- Since **both** the charge density and the plate geometry are **independent** of coordinates x and y, we know the electric potential field will be a function of coordinate z **only** (i.e., $V(\bar{r}) = V(z)$).
- Therefore, Poisson's equation becomes:

$$\nabla^2 V(\bar{r}) = \frac{\partial^2 V(z)}{\partial z^2} = z$$

Example – 11 (contd.)

- We can solve this differential equation by first **integrating** both sides:

$$\int\left(\frac{\partial^2 V(z)}{\partial z^2}\right)dz=\int(z)\,dz \quad\Longrightarrow\quad \frac{\partial V(z)}{\partial z}=\frac{z^2}{2}+C_1$$

- And integrating **again** we find:

$$\int\left(\frac{\partial V(\bar{r})}{\partial z}\right)dz=\int\left(\frac{z^2}{2}+C_1\right)dz \quad\Longrightarrow\quad V(\bar{r})=\frac{z^3}{6}+C_1 z+C_2$$

- Note that this expression for $V(\bar{r})$ **satisfies** Poisson’s equation for this case. The question remains, however: what are the values of **constants** C_1 and C_2?
- We find them in the same manner as before—**boundary conditions**!

Example – 10 (contd.)

- Note the boundary conditions for **this** problem are:

$$V(z=-d)=V_0 \qquad\qquad V(z=0)=0$$

- Therefore, we can construct **two** equations with **two** unknowns:

$$\boxed{V(z=-d)=V_0=\frac{(-d)^3}{6}+C_1(-d)+C_2}$$

$$\boxed{V(z=0)=0=\frac{(0)^3}{6}+C_1(0)+C_2}$$

- It is evident that C_2 = 0, therefore constant C_1 is: $\boxed{C_1=-\left(\frac{V_0}{d}+\frac{d^2}{6}\right)}$

- The **electric potential field** between the two plates is therefore:

$$\boxed{V(\bar{r})=\frac{z^3}{6}-\left(\frac{V_0}{d}+\frac{d}{6}\right)z \qquad (-d<z<0)}$$

Example – 11 (contd.)

- Performing our **sanity check**, we find:

$$V(z=-d)=\frac{(-d)^3}{6}-\left(\frac{V_0}{d}+\frac{d}{6}\right)(-d) \Rightarrow V(z=-d)=\frac{-d^3}{6}+V_0+\frac{d^3}{6}=V_0 \quad ✓$$

and $$V(z=0)=\frac{(0)^3}{6}-\left(\frac{V_0}{d}+\frac{d}{6}\right)(0) \Rightarrow V(z=0)=0 \quad ✓$$

From this result, we can determine the **electric field** $\vec{E}(\bar{r})$, the **electric flux density** $\vec{D}(\bar{r})$, and the **surface charge density** $\rho_s(\bar{r})$, as before.

Note, however, that the permittivity of the material between the plates is ε_0, as the “dielectric” between the plates is **free space**.