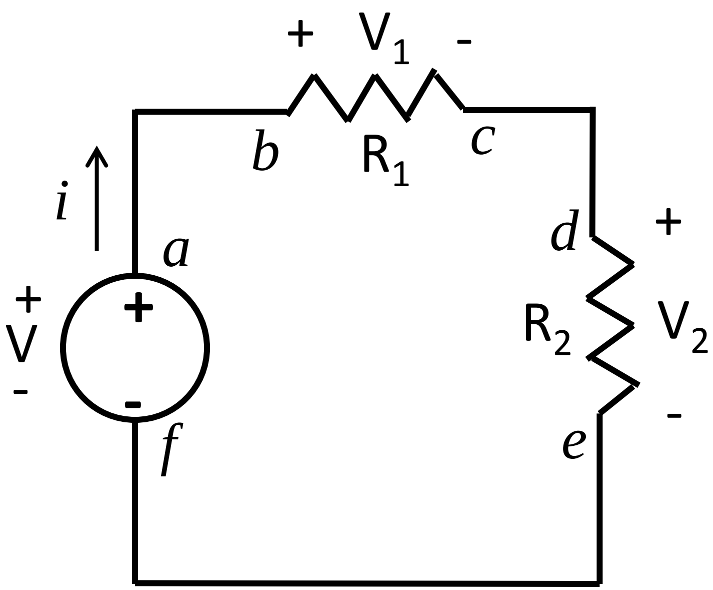
Lecture – 8

Date: 02.02.2015

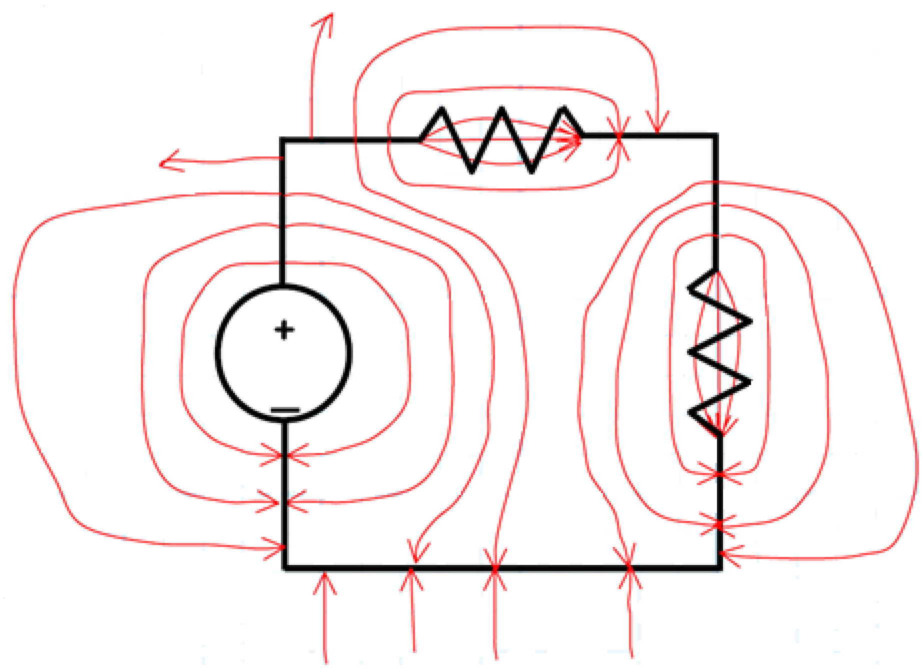
- Kirchoff's Voltage Law
- Joule's Law
- Polarization Vector
- Continuity Equation and Relaxation Time
- Electrostatic Boundary Conditions

Kirchoff's Voltage Law

- Consider a simple electrical **circuit**:



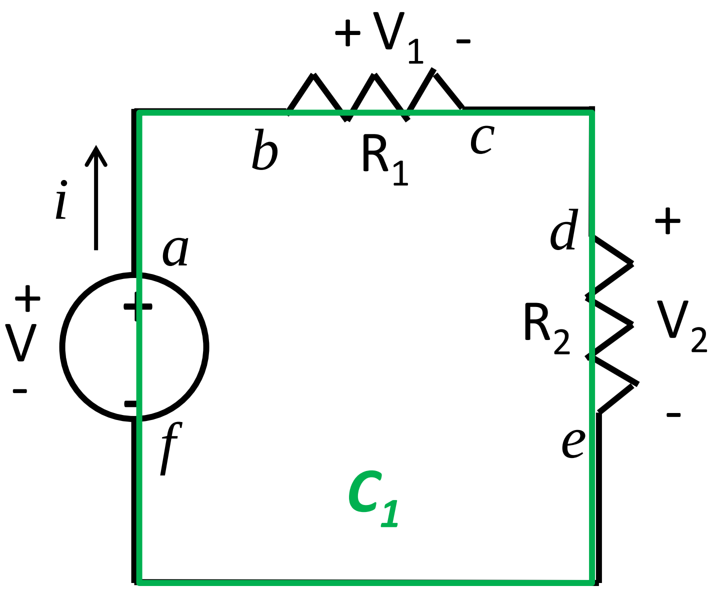
The electric field in this circuit will "look" **something** like this:



We find that if the **voltage source** is on (i.e., $V \neq 0$), there will be electric potential differences (i.e., voltage) between different points of the circuit. This can **only** be true if **electric fields** are present!

Kirchoff's Voltage Law (contd.)

- So, instead of using circuit theory, let's use our new **electrostatics** knowledge to **analyze** this circuit.
  - First, consider a **contour** C_1 that follows the circuit path.



- Using this path, let's **evaluate** the contour integral:

$$\int_{C_1} \vec{E}(\overline{r}).\overline{dl}$$

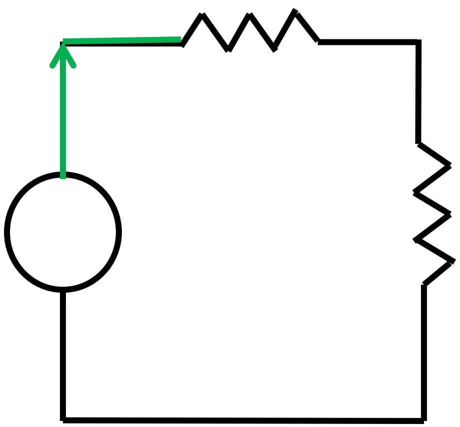
- This is most easily done by breaking the contour C_1 into **six sections**: section 1 extends from point a to point b, section 2 extends from point b to point c, etc. Thus, the integral becomes:

$$\int_{C_1} \vec{E}(\overline{r}).\overline{dl} = \int_a^b \vec{E}(\overline{r}).\overline{dl} + \int_b^c \vec{E}(\overline{r}).\overline{dl} + \int_c^d \vec{E}(\overline{r}).\overline{dl} + \int_d^e \vec{E}(\overline{r}).\overline{dl} + \int_e^f \vec{E}(\overline{r}).\overline{dl} + \int_f^a \vec{E}(\overline{r}).\overline{dl}$$

Kirchoff's Voltage Law (contd.)

- Let's evaluate each term **individually**:

Section 1 (a to b)



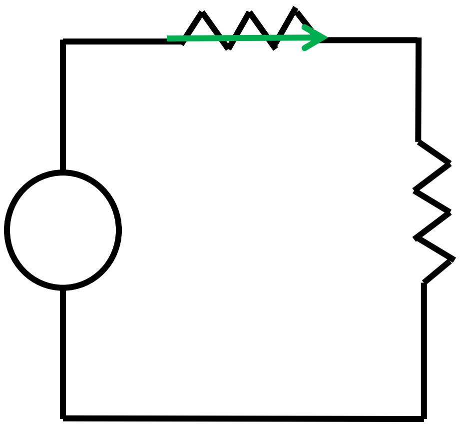
In this section, the contour follows the **wire** from the voltage source to the first resistor. We know that the electric field in a perfect conductor is **zero**, and likewise in a good conductor it is **very small**. Assuming the wire is in fact made of a **good conductor** (e.g. copper), we can approximate the electric field **within** the wire (and thus at **every** point along section 1) as **zero** (i.e., $\vec{E}(\bar{r}) = 0$). Therefore, this first integral equals zero!

$$\int_a^b \vec{E}(\bar{r}).\overline{dl} = 0$$

This of course makes sense! We know that the electric potential difference across a **wire** is **zero volts**.

Kirchoff’s Voltage Law (contd.)

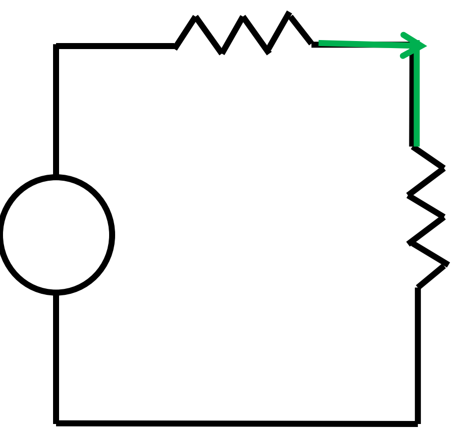
Section 2 (b to c)



In this section, the contour moves through the first **resistor**. The contour integral along this section therefore allows us to determine the electric **potential difference** across this resistor. Let’s denote this potential difference as V_1:

$$\int_b^c \vec{E}(\overline{r}).\overline{dl} = V_b - V_c = V_1$$

Section 3 (c to d)

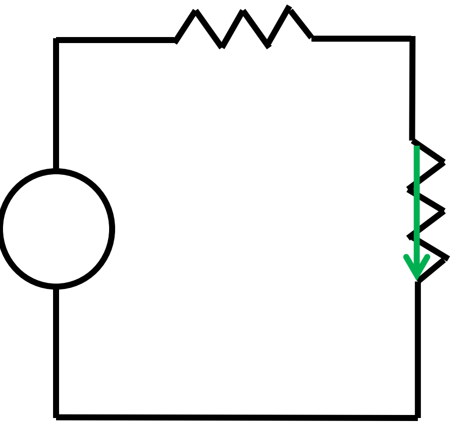


Just like section 1, the contour follows a **wire**, and thus the electric field long this section of the contour is **zero**, as is the potential difference between point c and point d.

$$\int_c^d \vec{E}(\overline{r}).\overline{dl} = 0$$

Kirchoff's Voltage Law (contd.)

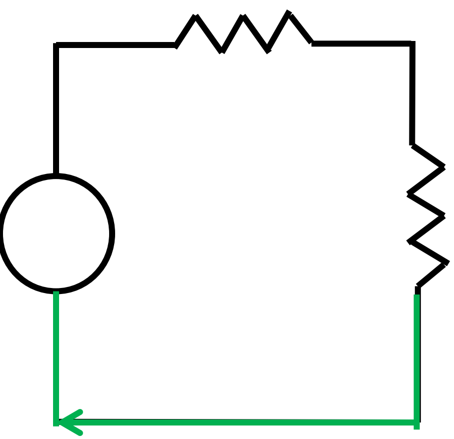
Section 4 (d to e)



Just like section 2, the contour moves through a **resistor**. The contour integral for this section is thus equal to the potential difference across this **second** resistor, which we denote as V_2:

$$\int_d^e \vec{E}(\overline{r}).\overline{dl} = V_d - V_e = V_2$$

Section 5 (e to f)

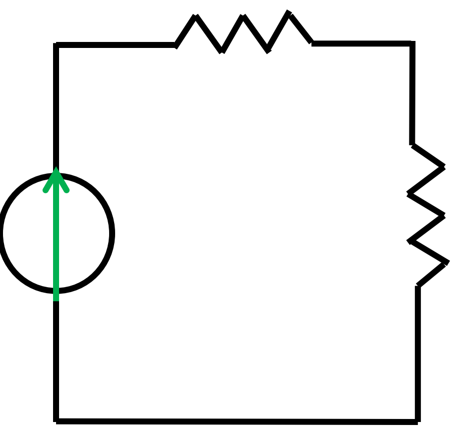


Again, the contour follows a conducting **wire**—and again, the electric field along the contour and the potential difference across it are both **zero**:

$$\int_e^f \vec{E}(\overline{r}).\overline{dl} = 0$$

Kirchoff's Voltage Law (contd.)

Section 6 (f to a)



This **final** section of contour C_1 extends through the **voltage source**, thus the contour integral of this section provides the electric potential difference between the two terminals of the this voltage source (i.e., $V_f - V_a$). By **definition**, the potential difference between points a and f is a value of V volts (i.e., $V_a - V_f = V$). Therefore, we find that the contour integral of section 6 is:

$$\int_f^a \vec{E}(\bar{r}).\overline{dl} = V_f - V_a = -\left(V_a - V_f\right) = -V$$

- **Whew**! Now let's **combine** these results to determine the contour integral for the **entire** contour C_1.

$$\oint_{C_1} \vec{E}(\bar{r}).\overline{dl} = 0 + V_1 + 0 + V_2 + 0 - V$$

$$\Rightarrow \oint_{C_1} \vec{E}(\bar{r}).\overline{dl} = V_1 + V_2 - V$$

Kirchoff's Voltage Law (contd.)

Q: Wait; I've forgotten, Why are we evaluating these contour integrals?

A: Remember, since the electric field is **static**, we also know that integral around any closed contour is **zero**. Thus, we can conclude that:

$$0 = \oint_{C_1} \vec{E}(\overline{r}).\overline{dl} = V_1 + V_2 - V$$

- In other words, we find by performing an **electromagnetic** analysis of the circuit, the voltages across each circuit element are related as:

$$V_1 + V_2 - V = 0$$

Kirchoff's Voltage Law (contd.)

Q: You **have** wasted my time! Using **only** Kirchoff's Voltage Law (KVL), **I** arrived at **precisely** the same result ($V_1 + V_2 - V = 0$). **I** think the above equation is true because of **KVL**, not because of your **fancy** electromagnetic theory!

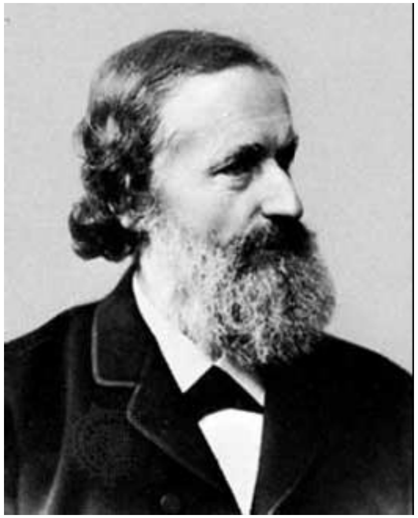
A: It **is** true that the result we obtained by integrating the electric field around the circuit contour is **likewise** apparent from **KVL**. However, this result is **still** attributable to electrostatic physics, because KVL is a **direct** result of electrostatics!

Kirchoff's Voltage Law (contd.)

- The electrostatic equation: $$\oint_{C_1} \vec{E}(\bar{r}).\overline{dl} = 0$$

  when applied to the closed contour of any **circuit**, results in **Kirchoff's Voltage Law**, i.e.: $$\sum_n V_n = 0$$

  where V_n are the electric potential differences across each element of a circuit "loop" (i.e., closed contour).



Gustav Robert Kirchhoff (1824-1887), German physicist, announced the laws that allow calculation of the currents, voltages, and resistances of electrical networks in 1845, when he was only **twenty-one**! His other work established the technique of spectrum analysis that he applied to determine the composition of the Sun.

Joule's Law

- Recall that the **work** done **on** charge Q **by** an electric field in moving the charge along some **contour** *C* is:

$$W = Q\int_C \vec{E}(\overline{r}).\overline{dl}$$

Q: Say instead of one charge Q, we have a steady **stream** of charges (i.e., electric current) flowing along contour *C*?

A: We would need to determine the **rate** of work **per unit time**, i.e., the **power** applied by the field to the current.

- Recall also that the **time derivative** of work is power!

$$P = \frac{dW}{dt} = \frac{d}{dt}\left(Q\int_C \vec{E}(\overline{r}).\overline{dl}\right)$$

$$P = \frac{dQ}{dt}\int_C \vec{E}(\overline{r}).\overline{dl} = I\int_C \vec{E}(\overline{r}).\overline{dl}$$

- But look! The **contour integral** we know is equal to the **potential difference** V between either end of the contour. Therefore:

$$P = I\int_C \vec{E}(\overline{r}).\overline{dl} = IV$$

Look familiar!?

The **power** delivered to charges by the field is equal to the **current** **"I"** flowing along the contour, **times** the **potential difference** (i.e., voltage V) across the contour.

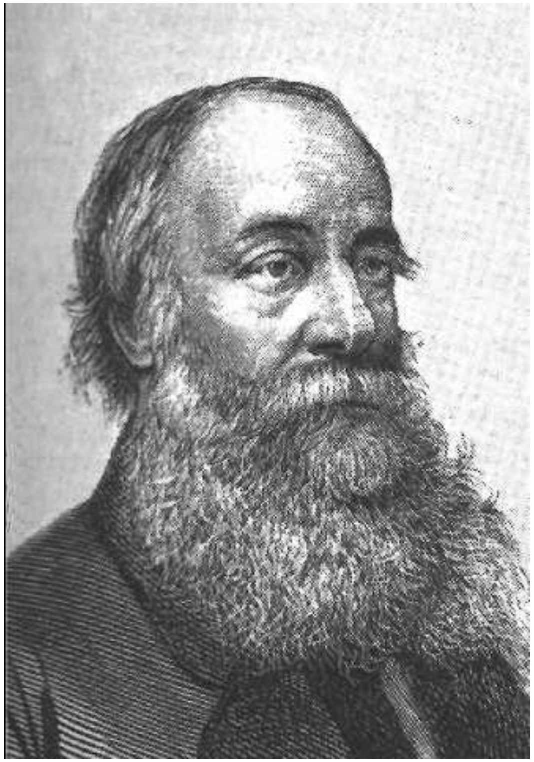
Joule's Law (contd.)

- Consider now the power delivered in some **volume** v, say the volume of a resistor. Recall the electric field has units of **volts/m**, and the current density has units of **amps/m²**.
- We find that the **dot product** of the electric field and the current density is a **scalar** value with units of Watts/m³. We call this scalar value the **power density**:

$$\text{Power Density} = \vec{E}(\bar{r}).\vec{J}(\bar{r}) \left[\left(\frac{V}{m}\right)\left(\frac{A}{m^2}\right) = \left(\frac{Watt}{m^3}\right) \right]$$

- Integrating power density over some volume v gives the **total power** delivered by the field **within that volume**:

$$P = \iiint_v \vec{E}(\bar{r}).\vec{J}(\bar{r})dv = \iiint_v \sigma(\bar{r})\left|\vec{E}(\bar{r})\right|^2 dv = \iiint_v \frac{1}{\sigma(\bar{r})}\left|\vec{J}(\bar{r})\right|^2 dv \qquad \text{[Watt]}$$

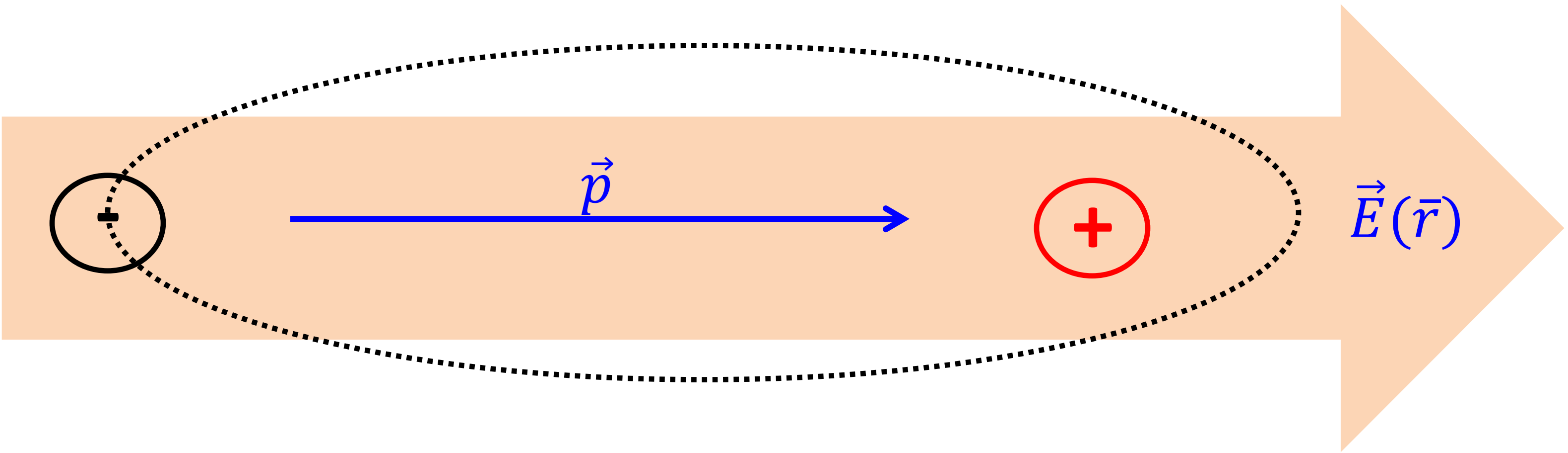


James Prescott Joule (1818-1889), born into a well-to-do family prominent in the brewery industry, studied at Manchester under Dalton. At age twenty-one he published the "I-squared-R" law which bears his name. Two years later, he published the first determination of the mechanical equivalent of heat. He became a collaborator with Thomson and they discovered that the temperature of an expanding gas falls. The "Joule-Thomson effect" was the basis for the large refrigeration plants constructed in the 19th century (but not used by the British brewery industry). Joule was a patient, methodical and devoted scientist; it became known that he had taken a thermometer with him on his honeymoon and spent time attempting to measure water temperature differences at the tops and bottoms of waterfalls.

Recall that if a **dielectric** material is immersed in an **electric field**, each atom/molecule in the material will form an **electric dipole!**

The Polarization Vector

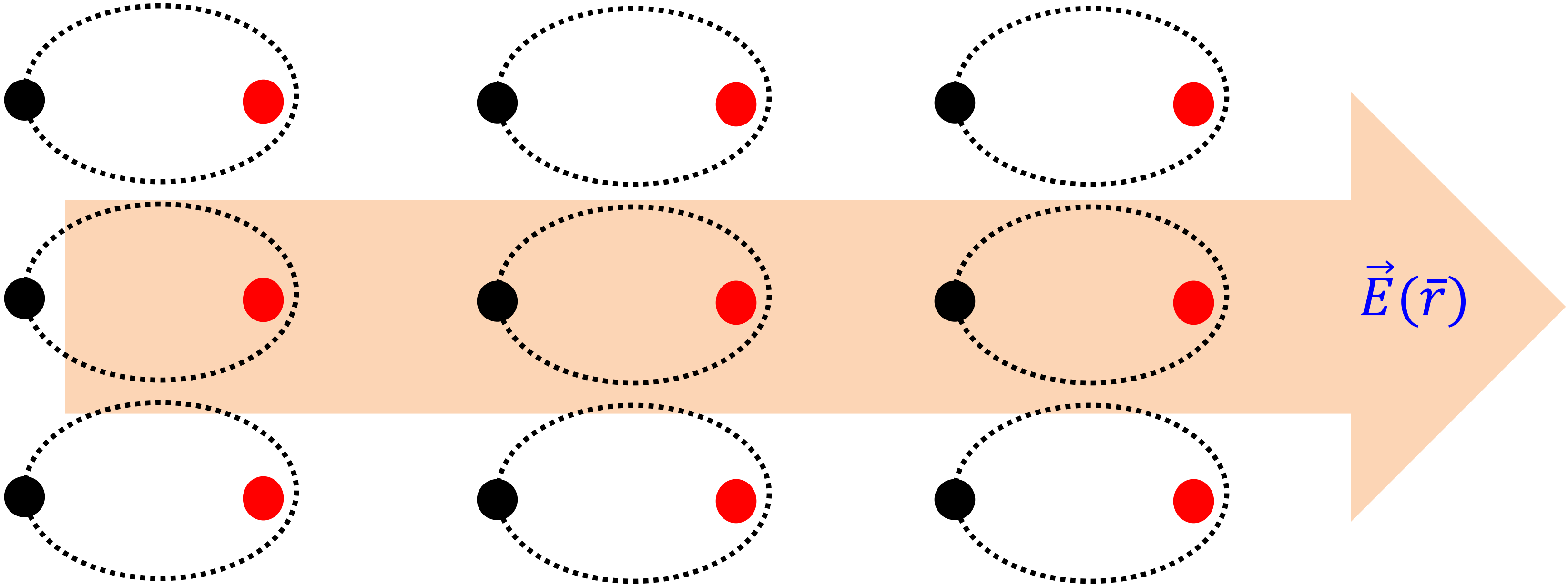
- Recall that in **dielectric materials** (i.e., insulators), the charges are **bound**.



As a result, atoms/molecules form **electric dipoles** when an electric field is present!

The Polarization Vector (contd.)

- Note that even for some **small** volume Δv, there are **many** atoms/molecules present; therefore there will be **many** electric dipoles.



- We therefore define an **average** dipole moment, per unit volume, called the **Polarization Vector** $\vec{P}(\bar{r})$.

$$\vec{P}(\bar{r}) \doteq \frac{\sum \vec{p}_n}{\Delta v}\left[\frac{dipole\_moment}{unit\_volume} = \frac{C}{m^2}\right]$$

The Polarization Vector (contd.)

$$\vec{P}(\bar{r}) \doteq \frac{\sum \vec{p}_n}{\Delta v}\left[\frac{dipole\_moment}{unit\_volume} = \frac{C}{m^2}\right]$$

$\overrightarrow{p_n}$ is one of N dipole moments in volume Δv, centered at position $\bar{r}$. Note the polarization vector is a **vector field**. As a result, the direction and magnitude of the Polarization vector can change as function of position (i.e., a function of $\bar{r}$).

Q: How are vector fields $\vec{P}(\bar{r})$ and $\vec{E}(\bar{r})$ related??

A: Recall that the direction of each dipole moment is the same as the polarizing electric field. Thus $\vec{P}(\bar{r})$ and $\vec{E}(\bar{r})$ have the same direction. Their magnitudes are related by a unitless scalar value $\chi_e(\bar{r})$, called **electric susceptibility:**

$$\vec{P}(\bar{r}) = \varepsilon_0 \chi_e(\bar{r}) \vec{E}(\bar{r})$$

Electric susceptibility is a **material parameter** indicating the "stretchability" of the dipoles.

The Polarization Vector (contd.)

Q: Can we determine the **fields** created by a polarized material?

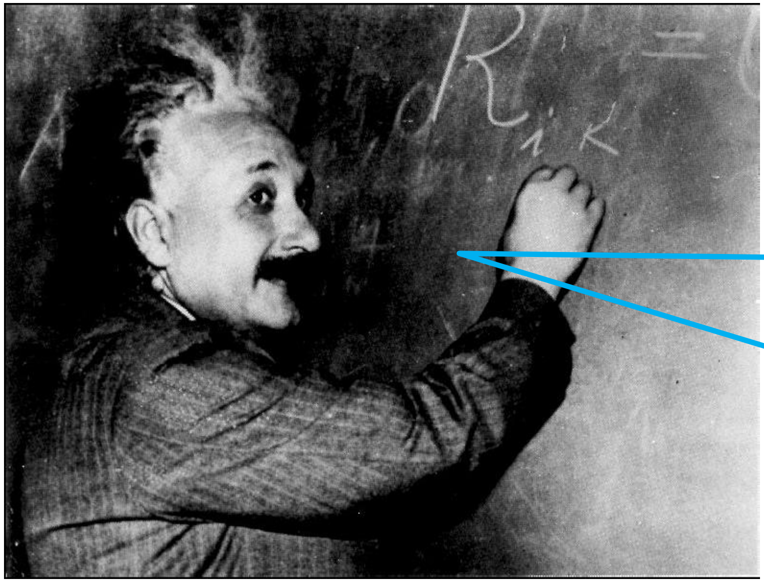
A: Recall the **electric potential field** created by **one** dipole is:

$$V(\bar{r}) = \frac{\vec{p}.\left(\bar{r} - \bar{r}'\right)}{4\pi\varepsilon_0 \left|\bar{r} - \bar{r}'\right|^3}$$

Therefore, the electric potential field created by a **distribution of** dipoles (i.e., $\vec{P}(\bar{r})$) across some volume v is:

$$V(\bar{r}) = \iiint_v \frac{\vec{P}(\bar{r}').\left(\bar{r} - \bar{r}'\right)}{4\pi\varepsilon_0 \left|\bar{r} - \bar{r}'\right|^3} dv'$$

The Polarization Vector (contd.)



Q: But I thought **scalar** charge distributions $\rho_v(\bar{r})$ and $\rho_s(\bar{r})$created the electric potential field $V(\bar{r})$. Now you are saying that potential fields are created by the **vector** field $\vec{P}(\bar{r})$!?!

A: As we will soon see, the polarization **vector** $\vec{P}(\bar{r})$ creates equivalent charge **distributions**—we will get the correct answer for $V(\bar{r})$ from **either** source!

Polarization Charge Distributions

- Consider a chunk of **dielectric** material with volume v.

Say this dielectric material is immersed in an **electric field** $\vec{E}(\bar{r})$, therefore creating atomic **dipoles** with density $\vec{P}(\bar{r})$.

Q: What **electric potential field** $V(\bar{r})$ is created by these diploes?

A: We know that:

$$V(\bar{r}) = \iiint_v \frac{\vec{P}(\bar{r}').(\bar{r}-\bar{r}')}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3} dv'$$

Using Divergence Theorem

It can be shown that:

$$V(\bar{r}) = \iiint_v \frac{\vec{P}(\bar{r}').(\bar{r}-\bar{r}')}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3} dv' = \frac{1}{4\pi\varepsilon_0}\iiint_v \frac{-\nabla.\vec{P}(\bar{r}')}{|\bar{r}-\bar{r}'|} dv' + \frac{1}{4\pi\varepsilon_0}\oiint_S \frac{\vec{P}(\bar{r}').\hat{a}_n(\bar{r})}{|\bar{r}-\bar{r}'|} dS'$$

where S is the **closed** surface that surrounds volume v, and $\hat{a}_n(\bar{r})$ is the unit vector **normal** to surface S (pointing **outward**).

Polarization Charge Distributions (contd.)

$$V(\bar{r}) = \iiint_v \frac{\vec{P}(\bar{r}').(\bar{r}-\bar{r}')}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3} dv' = \frac{1}{4\pi\varepsilon_0} \iiint_v \frac{-\nabla.\vec{P}(\bar{r}')}{|\bar{r}-\bar{r}'|} dv' + \frac{1}{4\pi\varepsilon_0} \oiint_S \frac{\vec{P}(\bar{r}').\hat{a}_n(\bar{r})}{|\bar{r}-\bar{r}'|} dS'$$

This complicated result is only important when we compare it to the electric potential created by **volume** charge density $\rho_v(\bar{r})$ and **surface** charge density $\rho_s(\bar{r})$.

- If both volume and surface charge are present, the **total** electric potential field is:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_v \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|} dv' + \frac{1}{4\pi\varepsilon_0} \oiint_S \frac{\rho_S(\bar{r}')}{|\bar{r}-\bar{r}'|} dS'$$

- The comparison gives:

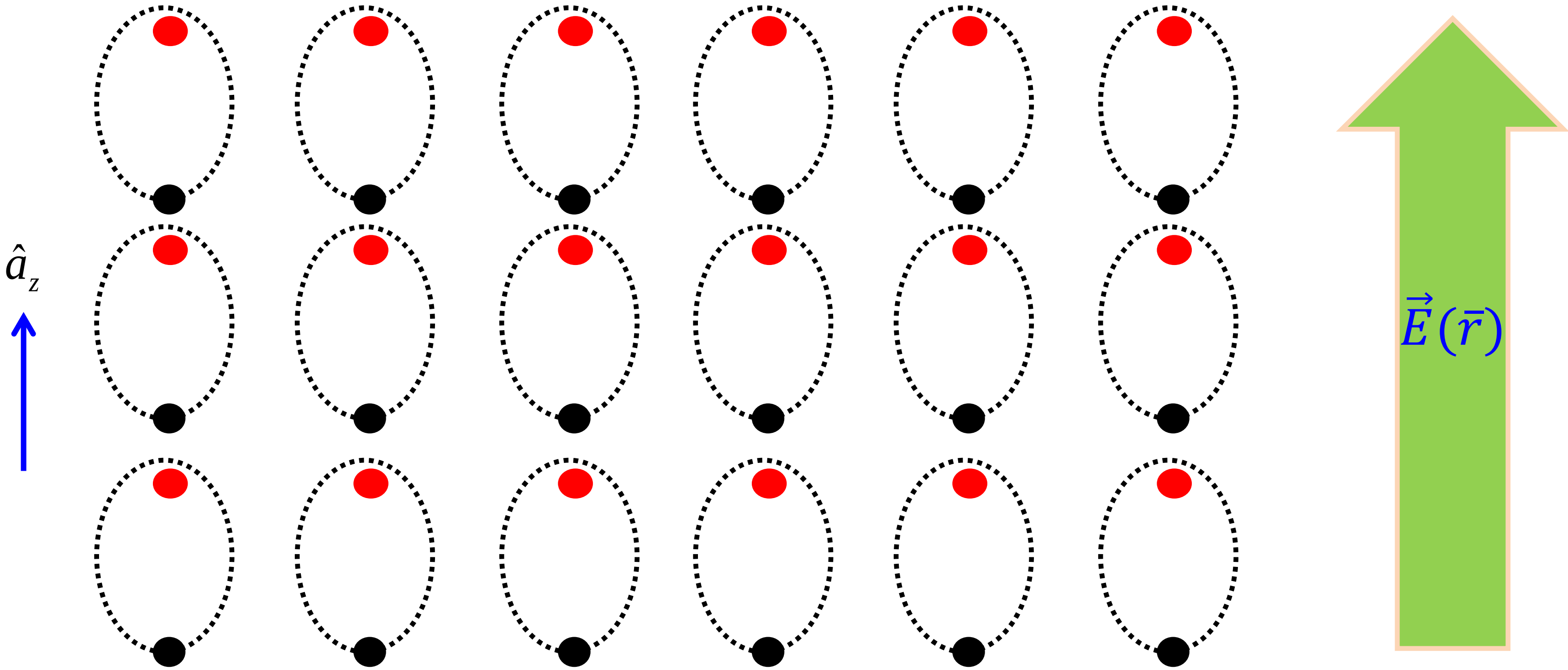
$$\rho_{vp}(\bar{r}) = -\nabla.\vec{P}(\bar{r})$$

$$\rho_{sp}(\bar{r}) = \vec{P}(\bar{r}).\hat{a}_n$$

The subscript *p* (e.g., ρ_{vp}, ρ_{sp}) indicates that these functions represent **equivalent charge densities** due to the **dipoles** created in the dielectric.

Polarization Charge Distributions (contd.)

- In other words, the electric potential field $V(\bar{r})$ (and thus electric field $\vec{E}(\bar{r})$) created by the dipoles in the dielectric (i.e., $\vec{P}(\bar{r})$) is **indistinguishable** from the electric potential field created by the equivalent charge densities $\rho_{vp}(\bar{r})$ and $\rho_{sp}(\bar{r})$!
- For example, consider a dielectric material immersed in an electric field, such that its polarization vector $\vec{P}(\bar{r})$ is: $\vec{P}(\bar{r}) = 3\hat{a}_z$ C/m²



Polarization Charge Distributions (contd.)

- Note since the polarization vector is a **constant**, the equivalent volume charge density is **zero**:

$$\rho_{vp}(\bar{r}) = -\nabla.\vec{P}(\bar{r}) = -\nabla.3\hat{a}_z = 0$$

- On the **top** surface of the dielectric ($\hat{a}_n = \hat{a}_z$), the equivalent **surface** charge is:

$$\rho_{sp}(\bar{r}) = \vec{P}(\bar{r}).\hat{a}_n = 3\hat{a}_z.\hat{a}_z = 3\ \text{C/m}^2$$

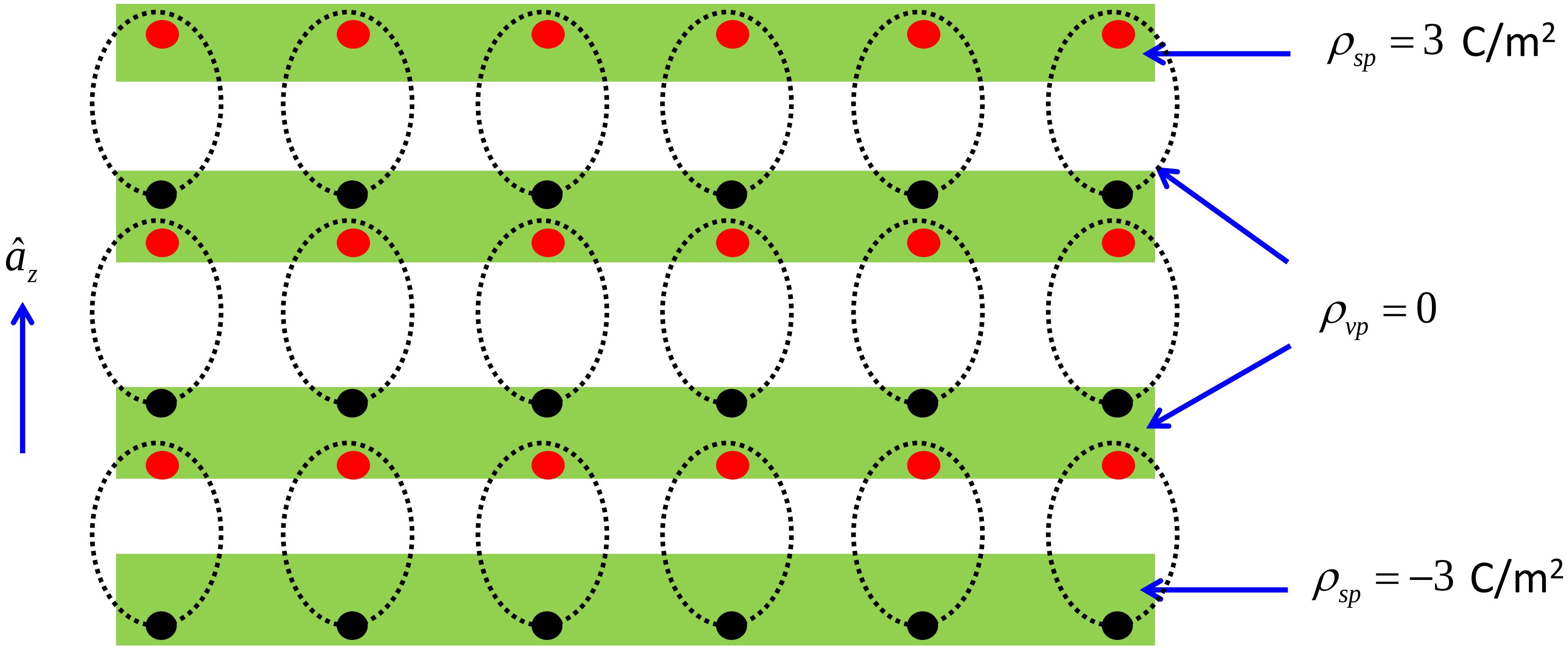
- On the **bottom** surface of the dielectric ($\hat{a}_n = -\hat{a}_z$), the equivalent **surface** charge is:

$$\rho_{sp}(\bar{r}) = \vec{P}(\bar{r}).\hat{a}_n = -3\hat{a}_z.\hat{a}_z = -3\ \text{C/m}^2$$

- On the **sides** of the dielectric material, the **surface** charge is **zero**, since ($\hat{a}_n.\hat{a}_z = 0$).

Polarization Charge Distributions (contd.)

- The result actually makes **physical** sense! Note at the **top** of dielectric, there is a layer of **positive** charge, and at the **bottom**, there is a layer of **negative** charge.



- In the **middle** of the dielectric, there are **positive** charge layers on top of **negative** charge layers. The two add together and **cancel** each other, so that equivalent **volume** charge density is **zero.**

Polarization Charge Distributions (contd.)

- Finally, recall that there is no perfect dielectric, all materials will have some non-zero conductivity $\sigma(\bar{r})$.
- As a result, we find that the total charge density within some material is the sum of the polarization charge density and the free charge (i.e., conducting charge) density:

$$\rho_{vT}(\bar{r}) = \rho_v(\bar{r}) + \rho_{vp}(\bar{r})$$

Where:

$\rho_{vT}(\bar{r}) \doteq$ total volume charge density

$\rho_v(\bar{r}) \doteq$ free charge density

$\rho_{vp}(\bar{r}) \doteq$ polarization charge density

- This is likewise (as well as **more frequently!**) true for **surface** charge density:

$$\rho_{sT}(\bar{r}) = \rho_s(\bar{r}) + \rho_{sp}(\bar{r})$$

Electric Flux Density

- Yikes! Things have gotten **complicated**!
- In free space, we found that charge $\rho_v(\bar{r})$ creates an electric field $\vec{E}(\bar{r})$.

Pretty **simple**! $\rho_v(\bar{r}) \Rightarrow \vec{E}(\bar{r})$

- But, if dielectric material is present, we find that charge $\rho_v(\bar{r})$ creates an **initial** electric field $\vec{E_i}(\bar{r})$. This electric field in turn **polarizes** the material, forming bound charge $\rho_{vp}(\bar{r})$. This bound charge, however, then creates its **own** electric field $\vec{E}_s(\bar{r})$ (sometimes called a **secondary** field), which modifies the initial electric field!

Not so simple! $\rho_v(\bar{r}) \Rightarrow \vec{E_i}(\bar{r}) \Rightarrow \rho_{vp}(\bar{r}) \Rightarrow \vec{E_s}(\bar{r})$

The **total** electric field created by free charge when dielectric material is present is thus $\vec{E}(\bar{r}) = \vec{E_i}(\bar{r}) + \vec{E_s}(\bar{r})$.

Electric Flux Density (contd.)

Q: Isn't there some **easier** way to account for the effect of dielectric material??

A: Yes there is! We use the concept of dielectric **permittivity**, and a new vector field called the **electric flux density** $\vec{D}(\bar{r})$.

- To see how this works, first consider the point form of **Gauss's Law**:

$$\nabla.\vec{E}(\bar{r}) = \frac{\rho_{vT}(\bar{r})}{\varepsilon_0}$$

where $\rho_{vT}(\bar{r})$ is the **total** charge density, consisting of both the **free** charge density $\rho_v(\bar{r})$ and **bound** charge density $\rho_{vp}(\bar{r})$:

$$\rho_{vT}(\bar{r}) = \rho_v(\bar{r}) + \rho_{vp}(\bar{r})$$

- Therefore, we can write Gauss's Law as:

$$\varepsilon_0 \nabla.\vec{E}(\bar{r}) = \rho_v(\bar{r}) + \rho_{vp}(\bar{r})$$

Electric Flux Density (contd.)

- Recall the **bound** charge density is equal to: $\rho_{vp}(\bar{r}) = -\nabla.\vec{P}(\bar{r})$
- Therefore, expression for Gauss's Law becomes: $\varepsilon_0 \nabla.\vec{E}(\bar{r}) = \rho_v(\bar{r}) - \nabla.\vec{P}(\bar{r})$

$$\nabla.\left(\varepsilon_0 \vec{E}(\bar{r}) + \vec{P}(\bar{r})\right) = \rho_v(\bar{r})$$

Note this final result says that the divergence of vector field $\varepsilon_0\vec{E}(\bar{r}) + \vec{P}(\bar{r})$ is equal to the **free** charge density $\rho_v(\bar{r})$. Let's define this vector field the **electric flux density** $\vec{D}(\bar{r})$:

$$\vec{D}(\bar{r}) \doteq \varepsilon_0 \vec{E}(\bar{r}) + \vec{P}(\bar{r})$$

- Therefore, we can write a **new** form of Gauss's Law: $\vec{D}(\bar{r}) \doteq \varepsilon_0 \vec{E}(\bar{r}) + \vec{P}(\bar{r})$

This equation says that the electric flux density $\vec{D}(\bar{r})$ **diverges** from **free** charge $\rho_v(\bar{r})$. In other words, the source of electric flux density is free charge $\rho_v(\bar{r})$ --and free charge **only**!

Electric Flux Density (contd.)

- The electric field $\vec{E}(\bar{r})$ is created by **both** free charge and bound charge within the dielectric material.
- However, the electric flux density $\vec{D}(\bar{r})$ is created by **free** charge **only**—the bound charge within the dielectric material makes no difference with regard to $\vec{D}(\bar{r})$!
- We can further simplify the expression. Recall that the polarization vector is related to electric field by susceptibility $\chi_e(\bar{r})$:

$$\vec{P}(\bar{r}) = \varepsilon_0 \chi_e(\bar{r}) \vec{E}(\bar{r})$$

- Therefore the electric flux density is:

$$\vec{P}(\bar{r}) = \varepsilon_0 \chi_e(\bar{r}) \vec{E}(\bar{r})$$

- We can further simplify this by defining the **permittivity** of the medium (the dielectric material):

$$\varepsilon(\bar{r}) = \varepsilon_0 \left(1 + \chi_e(\bar{r})\right)$$

- This enables us to define **relative** permittivity:

$$\varepsilon_r(\bar{r}) \doteq \frac{\varepsilon(\bar{r})}{\varepsilon_0} = 1 + \chi_e(\bar{r})$$

Electric Flux Density (contd.)

- We can thus write a **simple** relationship between electric flux density and electric field:

$$\vec{D}(\bar{r}) = \varepsilon(\bar{r})\vec{E}(\bar{r}) = \varepsilon_0\varepsilon_r(\bar{r})\vec{E}(\bar{r})$$

Like conductivity $\sigma(\bar{r})$, permittivity $\varepsilon(\bar{r})$ is a fundamental **material** parameter. Also like conductivity, it relates the electric field to another vector field.

Thus, we have an **alternative** way to view electrostatics:

1. Free charge $\rho_v(\bar{r})$ creates electric flux density $\vec{D}(\bar{r})$.
2. The electric field can be then determined by simply dividing $\vec{D}(\bar{r})$ by the material permittivity $\varepsilon(\bar{r})$ (i.e., $\vec{E}(\bar{r}) = \vec{D}(\bar{r})/\varepsilon(\bar{r})$.

$$\rho_v(\bar{r}) \Rightarrow \vec{D}(\bar{r}) \Rightarrow \vec{E}(\bar{r})$$

Electrostatic Field Equations in Dielectrics

- The electrostatic equations for fields in **dielectric materials** are:

$$\nabla \times \vec{E}(\bar{r}) = 0 \qquad \nabla . \vec{D}(\bar{r}) = \rho_v(\bar{r}) \qquad \vec{D}(\bar{r}) = \varepsilon(\bar{r})\vec{E}(\bar{r})$$

- In **integral** form, these equations are:

$$\oint_C \vec{E}(\bar{r}).\overline{dl} = 0 \qquad \oiint_S \vec{D}(\bar{r}).\overline{dS} = Q_{enc} \qquad \vec{D}(\bar{r}) = \varepsilon(\bar{r})\vec{E}(\bar{r})$$

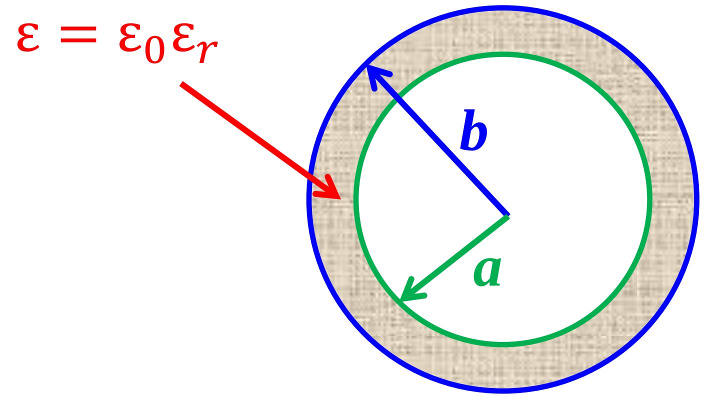
- Likewise, for free charge located in some **homogeneous** (i.e., constant) material with permittivity *ε*, we get the following relations:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon}\iiint_v \frac{\rho_v(\bar{r}')}{|\bar{r} - \bar{r}'|} dv' \qquad \left|\vec{E}(\bar{r})\right| = \frac{1}{4\pi\varepsilon}\iiint_v \frac{\rho_v(\bar{r}')}{|\bar{r} - \bar{r}'|^2} dv' \qquad \nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon}$$

In other words, for homogenous materials, **replace** ε_0 (the permittivity of free-space) with the more general permittivity value ε.

Example – 1

- At the center of a hollow dielectric sphere ($\varepsilon = \varepsilon_0 \varepsilon_r$) is placed a point charge Q. If the sphere has inner radius a and outer radius b, calculate $\vec{D}$, $\vec{E}$ and $\vec{P}$.



For $0 < r < a$

Gauss's law gives: $$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \Rightarrow \vec{P} = \vec{D} - \varepsilon_0\vec{E} = 0$$

For $a < r < b$

Gauss's law gives: $$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{a}_r$$

$$\vec{P} = \chi_e\varepsilon_0\vec{E} = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{Q}{4\pi r^2}\hat{a}_r$$

Example – 1 (contd.)

For $r > b$

Gauss's law gives: $\vec{D} = \dfrac{Q}{4\pi r^2}\hat{a}_r \Rightarrow \vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \Rightarrow \vec{P} = \vec{D} - \varepsilon_0\vec{E} = 0$

Therefore:

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r \qquad r > 0$$

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{a}_r & a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r & \text{otherwise} \end{cases}$$

$$\vec{P} = \begin{cases} \dfrac{\varepsilon_r - 1}{\varepsilon_r}\dfrac{Q}{4\pi r^2}\hat{a}_r & a < r < b \\ 0 & \text{otherwise} \end{cases}$$

Example – 2

Show that: $\vec{P} = \left(\varepsilon - \varepsilon_0\right)\vec{E}$ and $\vec{D} = \dfrac{\varepsilon_r}{\varepsilon_r - 1}\vec{P}$

Continuity Equation

- The charge conservation principle says: the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume.
- Therefore, current I_{out} coming out of the closed surface is:

$$I_{out} = \oint \vec{J}.\overline{dS} = \frac{-dQ_{in}}{dt}$$

Q_{in} is the charge enclosed by the closed surface

From Divergence Theorem

$$\oint \vec{J}.\overline{dS} = \int_v \left(\nabla.\vec{J}\right) dv$$

- We know that:

$$\frac{-dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v dv$$

If we agree to keep the volume constant

$$\frac{-dQ_{in}}{dt} = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

Continuity Equation (contd.)

- Therefore: $$\nabla.\vec{J} = \frac{\partial \rho_v}{\partial t}$$

Continuity Equation **or** Continuity of Current Equation

Continuity equation is derived from the principle of conservation of charge → It states that there can be no accumulation of charge at any point

For steady currents, $\frac{\partial \rho_v}{\partial t} = 0$, and therefore, $\nabla.\vec{J} = 0$

Total charge leaving the volume is the same as the charge entering the volume ← precursor to Kirchoff's Current Law

Electrostatic Boundary Conditions

- A vector field is said to be spatially continuous if it doesn't exhibit abrupt changes in either magnitude or direction as a function of position.
- Even though the electric field may be continuous in adjoining dissimilar media, it may well be discontinuous at the boundary between them.
- Boundary conditions specify how the components of fields tangential and normal to an interface between two media relate across the interface.
- To determine boundary conditions, we need to use Maxwell's equations:

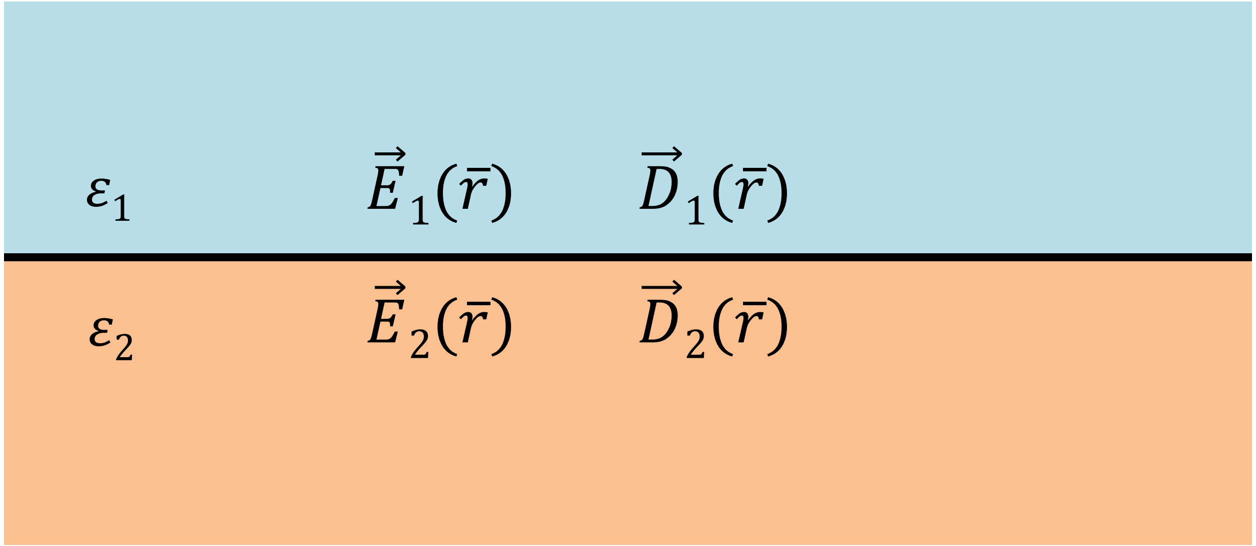
$$\oint \vec{E}.\overline{dl} = 0 \quad \Longleftrightarrow \quad \nabla \times \vec{E} = 0$$

$$\oint_S \vec{D}.\overline{dS} = Q_{enc} \quad \Longleftrightarrow \quad \nabla .\vec{D} = \rho_v$$

Needless to say, these boundary conditions are equally valid for Electrodynamics

Dielectric – Dielectric Boundary Conditions

- Consider the **interface** between two dissimilar **dielectric** regions:



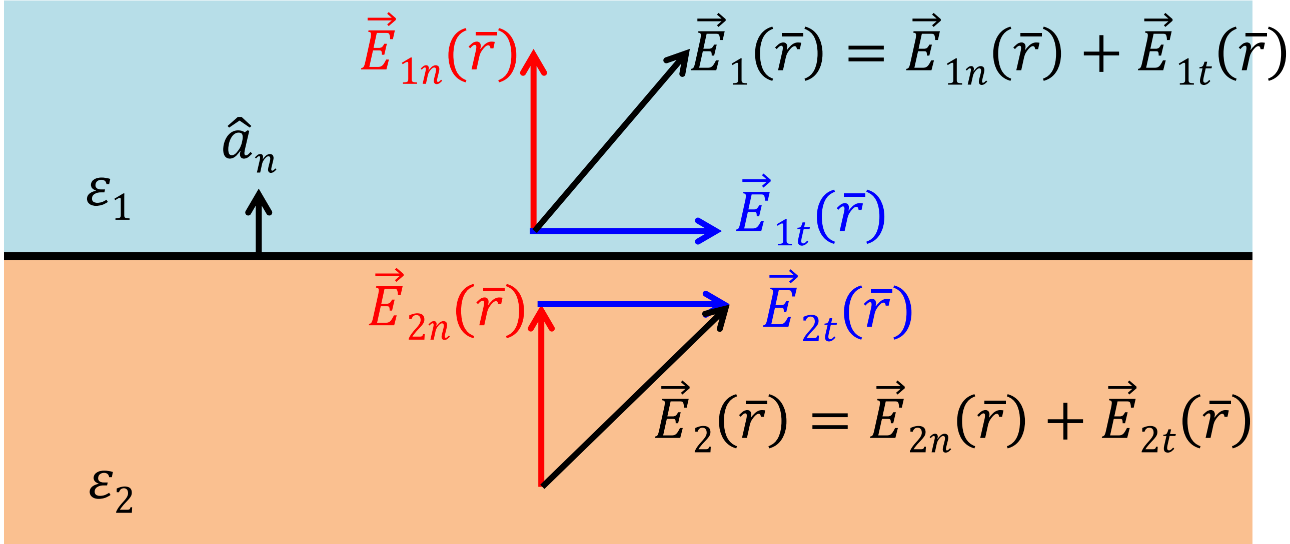
- Say that an **electric field** is present in both regions, thus producing also an electric flux density $\vec{D}(\bar{r}) = \varepsilon \vec{E}(\bar{r})$.

Q: How are the fields in dielectric **region 1** related to the fields in **region 2 ?**

A: They must satisfy the **dielectric boundary conditions** !

Dielectric – Dielectric Boundary Conditions (contd.)

- First, let's write the fields **at the dielectric interface** in terms of their **normal** $\vec{E}_n(\bar{r})$ and **tangential** $\vec{E}_t(\bar{r})$ vector components:



- Our first boundary condition states that the **tangential** component of the electric field is **continuous** across a boundary.

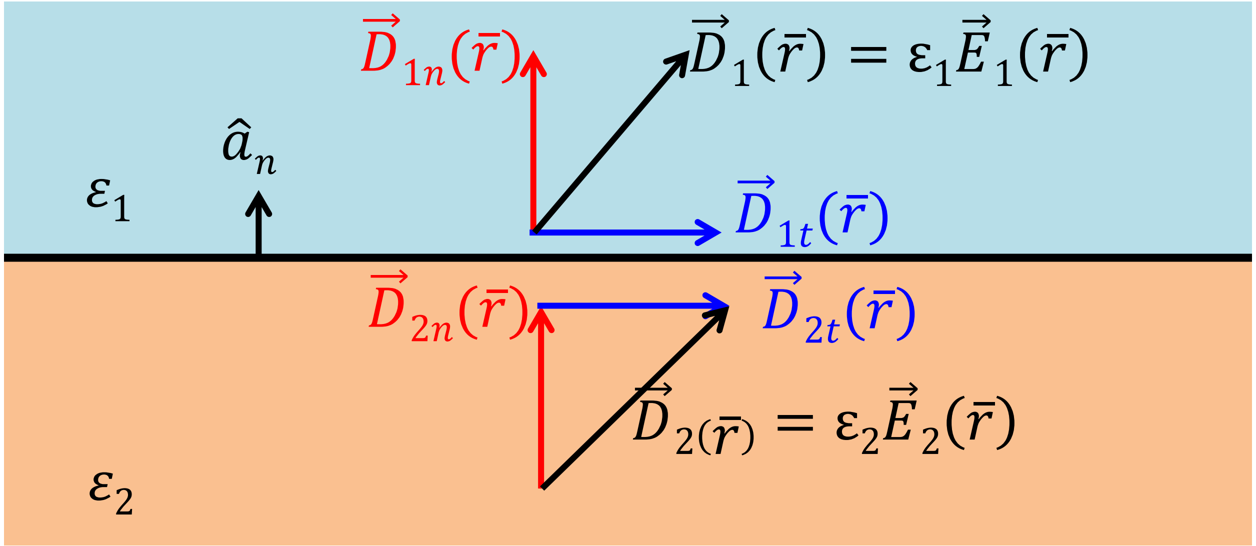
- In other words: $\vec{E}_{1t}(\bar{r}_b) = \vec{E}_{2t}(\bar{r}_b)$

where $\bar{r}_b$ denotes any point **on the boundary** (e.g., dielectric interface).

➡ The **tangential** component of the electric field at **one** side of the dielectric boundary is **equal** to the tangential component at the **other** side !

Dielectric – Dielectric Boundary Conditions (contd.)

- We can likewise consider the **electric flux densities** on the dielectric interface in terms of their **normal** and **tangential** components:



- The second dielectric boundary condition states that the **normal** vector component of the **electric flux density** is **continuous** across the dielectric boundary.

- In other words: $\vec{D}_{1n}(\bar{r}_b) = \vec{D}_{2n}(\bar{r}_b)$ ← where $\bar{r}_b$ denotes any point **on the boundary** (e.g., dielectric interface).

Dielectric – Dielectric Boundary Conditions (contd.)

- Since $\vec{D}(\bar{r}) = \varepsilon\vec{E}(\bar{r})$, these boundary conditions can **likewise** be expressed as:

$$\frac{\vec{D}_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{\vec{D}_{2t}(\bar{r}_b)}{\varepsilon_2}$$

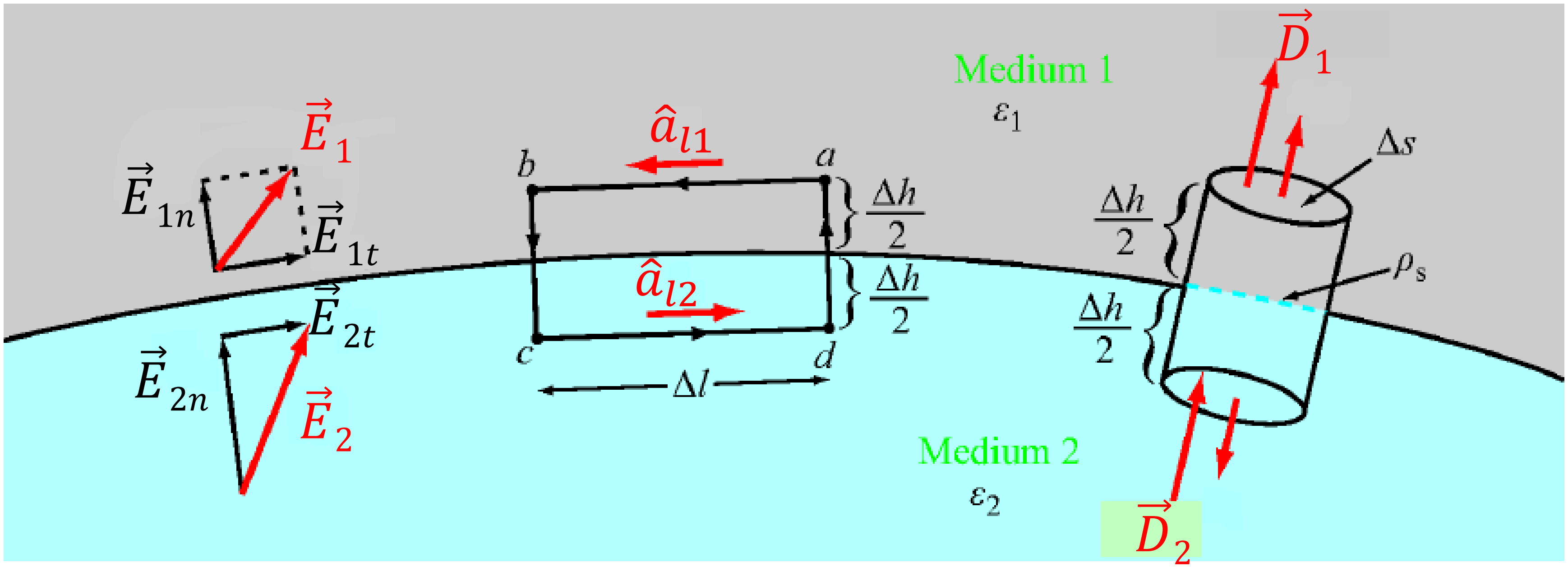
$$\varepsilon_1 \vec{E}_{1n}(\bar{r}_b) = \varepsilon_2 \vec{E}_{2n}(\bar{r}_b)$$

MAKE SURE YOU UNDERSTAND THIS:

These boundary conditions describe the relationships of the vector fields **at the dielectric interface** only (i.e., at points $r = \bar{r}_b$)!!!! They say **nothing** about the value of the fields at points above or below the interface.

Dielectric – Dielectric Boundary Conditions (contd.)

Proof



- To derive the boundary conditions for tangential components of $\vec{E}$ and $\vec{D}$, let us consider the closed rectangular loop $abcda$.
- The line integral along this closed loop is ZERO.
- If $\Delta h \rightarrow 0$, the contributions to the line integral by the segments bc and da vanish.

Dielectric – Dielectric Boundary Conditions (contd.)

- Therefore: $$\oint_C \vec{E}.\overline{dl} = \int_a^b \vec{E}_1.\hat{a}_{l1} dl + \int_c^d \vec{E}_2.\hat{a}_{l2} dl = 0$$

Where, $\hat{a}_{l1}$ and $\hat{a}_{l2}$ are the unit vectors along segments ab and cd.

- Next, we decompose $\vec{E}_1$ and $\vec{E}_2$ into components normal and tangential to the boundary:

$$\vec{E}_1 = \vec{E}_{1n} + \vec{E}_{1t} \qquad \vec{E}_2 = \vec{E}_{2n} + \vec{E}_{2t}$$

- We also know that: $\hat{a}_{l1} = -\hat{a}_{l2}$
- Thus the contour integral can be simplified to:

$$\left(\vec{E}_1 - \vec{E}_2\right).\hat{a}_{l1} = 0 \Rightarrow \vec{E}_{1t} = \vec{E}_{2t}$$

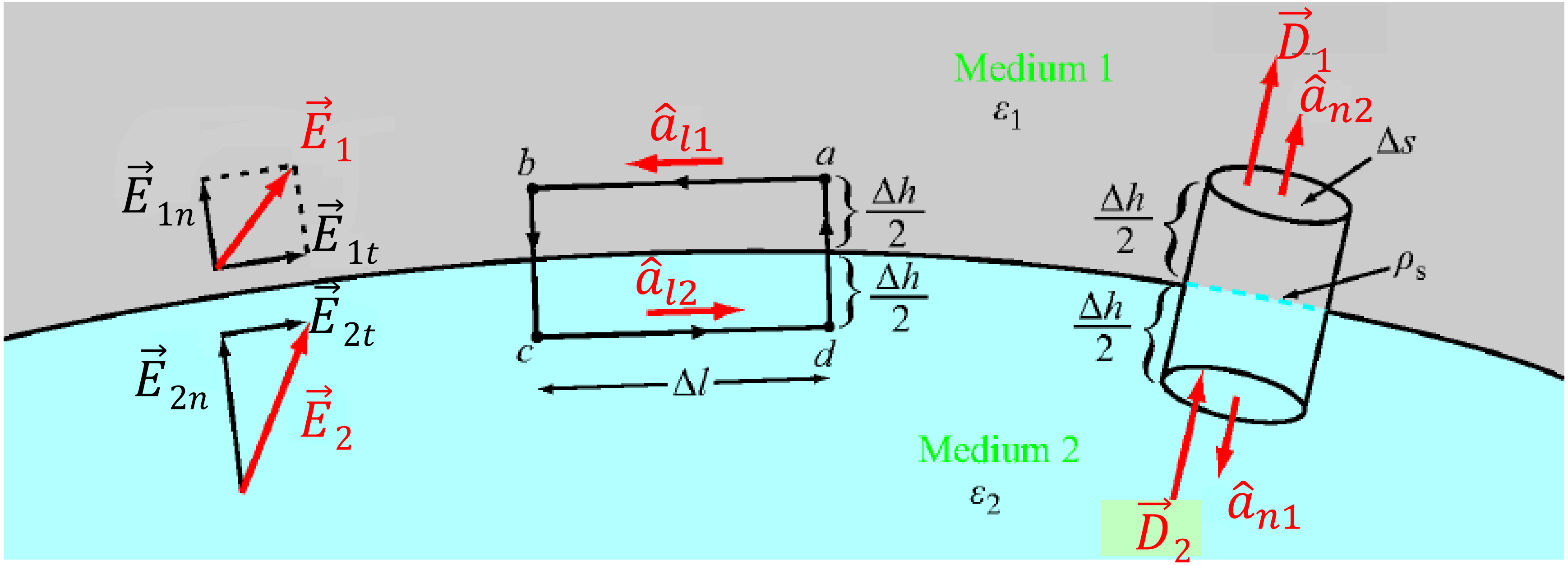
Thus the tangential component of the electric field is continuous across the boundary between any two media

- Upon decomposing $\vec{D}_1$ and $\vec{D}_2$ into components normal and tangential to the boundary and noting that $\vec{D}_{1t} = \varepsilon_1 \vec{E}_{1t}$ and $\vec{D}_{2t} = \varepsilon_2 \vec{E}_{2t}$, the boundary condition on the tangential component of the electric flux density is:

$$\frac{\vec{D}_{1t}}{\varepsilon_1} = \frac{\vec{D}_{2t}}{\varepsilon_2}$$

Dielectric – Dielectric Boundary Conditions (contd.)

- Now, apply Gauss's law to determine boundary conditions on the normal components of $\vec{E}$ and $\vec{D}$.



- The total outward flux through the three surfaces of the small cylinder must equal the total charge enclosed in the cylinder.
- By letting the cylinder's height $\Delta h \to 0$, the contribution to the total flux through the side surface goes to ZERO.

Dielectric – Dielectric Boundary Conditions (contd.)

- Even if each of the two media happens to contain free charge densities, the only free charge remaining in the collapsed cylinder is that distributed on the boundary ($Q_{enc} = \Delta s \times \rho_s$).

$$\oint_S \vec{D}.\overline{dS} = Q_{enc} = \Delta s \times \rho_s \quad \Rightarrow \quad \int_{top} \vec{D}_1.\hat{a}_{n1} dS + \int_{bottom} \vec{D}_2.\hat{a}_{n2} dS = \Delta s \times \rho_s$$

- It is important to remember that the normal unit vector at the surface of any medium is always defined to be in the outward direction away from that medium.

- Since, $\hat{a}_{n1} = -\hat{a}_{n2}$

$$\left(\vec{D}_1 - \vec{D}_2\right).\hat{a}_{n2} = \rho_s$$

- If $\vec{D}_{1n}$ and $\vec{D}_{2n}$ denotes the normal components of $\vec{D}_1$ and $\vec{D}_2$ along $\hat{a}_{n2}$

$$\vec{D}_{1n} - \vec{D}_{2n} = \rho_s$$

Dielectric – Dielectric Boundary Conditions (contd.)

- If no free charge exist at the boundary (i.e., charges are not deliberately placed at the boundary) then:

$$\vec{D}_{1n} - \vec{D}_{2n} = 0 \quad \Rightarrow \quad \vec{D}_{1n} = \vec{D}_{2n}$$

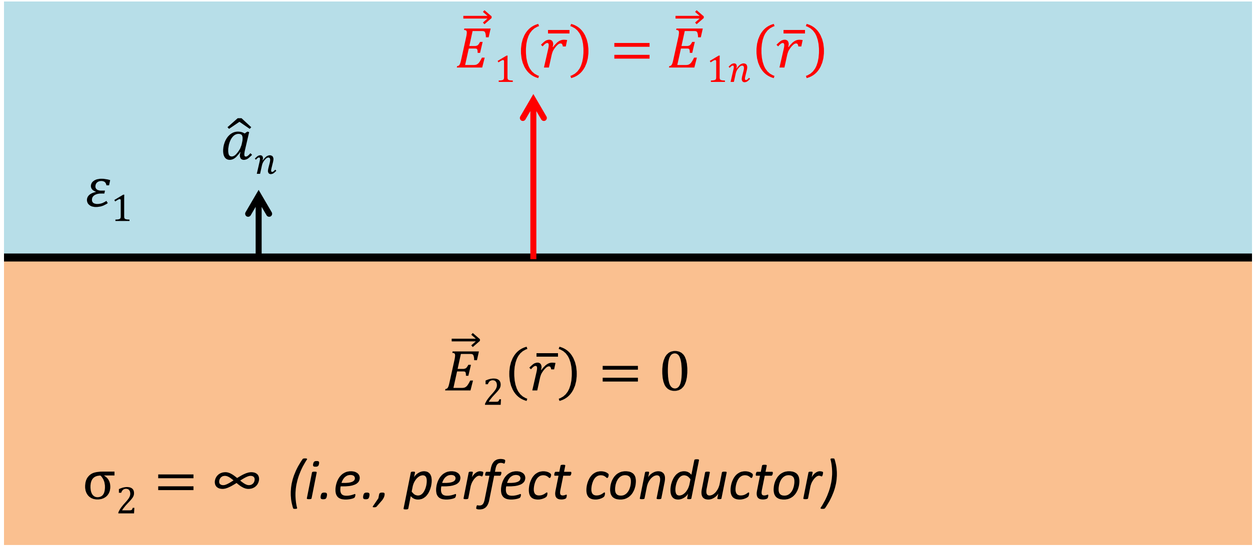
- Thus the normal component of $\vec{D}$ is continuous across the interface, that is D_n undergoes no change at the boundary.

- Furthermore: $\varepsilon_1 \vec{E}_{1n} = \varepsilon_2 \vec{E}_{2n}$

- The boundary conditions are usually applied in finding the electric field on one side of the boundary given the field on the other side.
- Beside this, we can use the boundary conditions to determine the "refraction" of the electric field across the interface.

Conductor – Dielectric Boundary Conditions

- Consider the case where region 2 is a **perfect conductor**:



- Recall $\vec{E}(\bar{r}) = 0$ in a perfect conductor. This of course means that **both** the tangential and normal component of $\vec{E}_2(\bar{r})$ are also equal to **zero**:

$$\vec{E}_{2t}(\bar{r}) = 0 = \vec{E}_{2n}(\bar{r})$$

- And, since the **tangential** component of the electric field is **continuous** across the boundary, we find that **at the interface**:

$$\vec{E}_{1t}(\bar{r}_b) = \vec{E}_{2t}(\bar{r}_b) = 0$$

Conductor – Dielectric Boundary Conditions (contd.)

- **Think about what this means!** The **tangential** vector component in the dielectric (at the dielectric/conductor boundary) is **zero**. Therefore, the electric field **at the boundary** only has a **normal** component:

$$\vec{E}_1(\bar{r}_b) = \vec{E}_{1n}(\bar{r}_b)$$

- Therefore, we can say:

The **electric field** on the surface of a **conductor** is **orthogonal** (i.e., normal) to the conductor.

Q1: What about the **electric flux density** $\vec{D}_1(\bar{r})$?

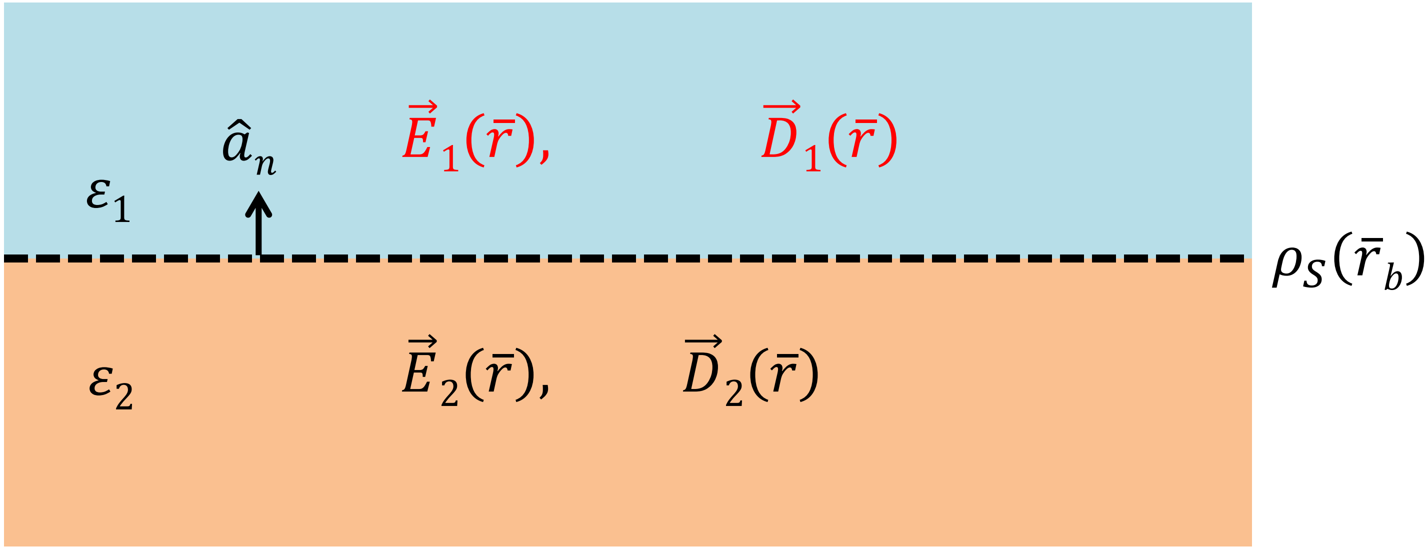
A1: The relation $\vec{D}_1(\bar{r}) = \varepsilon_1 \vec{E}_1(\bar{r})$ is still of course valid, so that the **electric flux density** at the surface of the conductor must **also be orthogonal** to the conductor. For boundary with surface charge density (ρ_s), $\vec{D}_{1n}(\bar{r}) = \varepsilon_1 \vec{E}_{1n}(\bar{r}) = \rho_s$.

Q2: But, we learnt that the **normal** component of the **electric flux density** is **continuous** across an interface. If $\vec{D}_{2n}(\bar{r}) = 0$, why isn't $\vec{D}_{1n}(\bar{r}) = 0$?

A2: Great question! The answer comes from a more **general** form of the **boundary condition**.

Conductor – Dielectric Boundary Conditions (contd.)

- Consider again the interface of two **dissimilar dielectrics**. This time, however, there is some **surface charge distribution** $\rho_S(\bar{r}_b)$ (i.e., **free** charge!) **at the dielectric interface**:



- The **boundary condition** for this situation turns out to be:

$$\hat{a}_n \cdot \left[\vec{D}_{1n}(\bar{r}_b) - \vec{D}_{2n}(\bar{r}_b)\right] = \rho_S(\bar{r}_b) \quad \Rightarrow \quad D_{1n}(\bar{r}_b) - D_{2n}(\bar{r}_b) = \rho_S(\bar{r}_b)$$

Conductor – Dielectric Boundary Conditions (contd.)

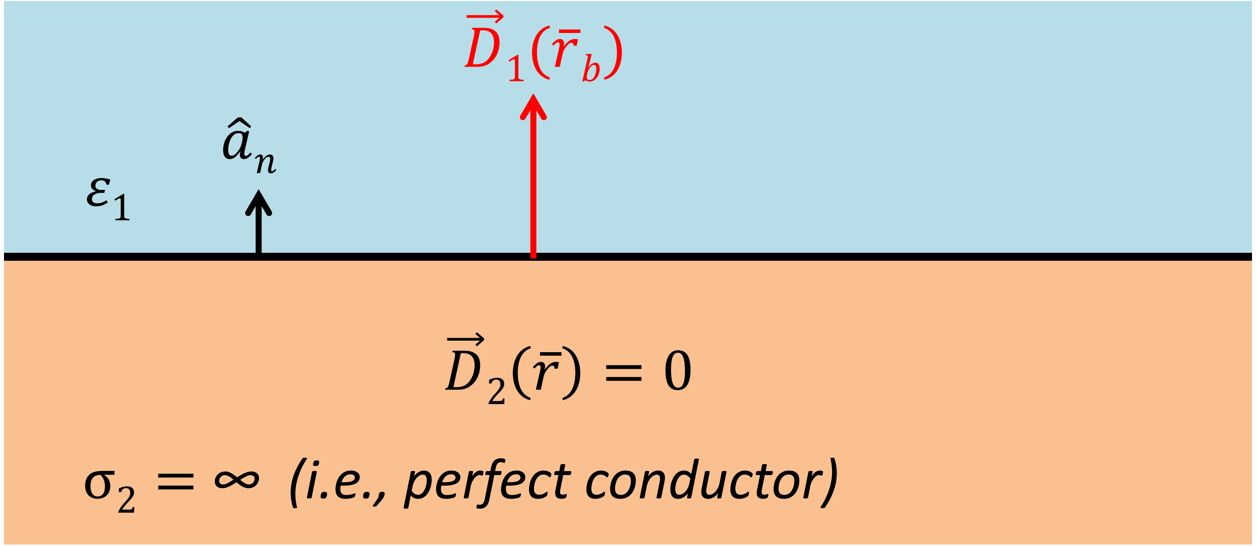
- Note that **if** $\rho_S(\bar{r}_b)$ =0, this boundary condition returns (both physically and mathematically) to the case studied earlier—the **normal** component of the electric flux density is **continuous** across the interface.
- This more **general** boundary condition is useful for the dielectric/**conductor** interface. Since $\overrightarrow{D}_2(\bar{r}) = 0$ in the conductor, we find that:

$$\hat{a}_n . \overrightarrow{D}_{1n}(\bar{r}_b) = \rho_S(\bar{r}_b) \quad \Rightarrow \quad D_{1n}(\bar{r}_b) = \rho_S(\bar{r}_b)$$

In other words, the **normal** component of the **electric flux density** at the **conductor surface** is equal to the **charge density** on the conductor surface.

Conductor – Dielectric Boundary Conditions (contd.)

- Note in a perfect conductor, there is **plenty** of **free** charge available to form this charge density! Therefore, we find in **general** that $\vec{D}_{1n}(\bar{r}) \neq 0$ at the surface of a conductor.



Conductor – Dielectric Boundary Conditions (contd.)

Summary:

$$\vec{E}_{1t}(\bar{r}_b) = 0$$

$$\vec{D}_{1t}(\bar{r}_b) = 0$$

$$\vec{D}_{1n}(\bar{r}_b) = \rho_S(\bar{r}_b)$$

$$\vec{E}_{1n}(\bar{r}_b) = \frac{\rho_S(\bar{r}_b)}{\varepsilon_1}$$

Again, these boundary conditions describe the fields **at the conductor/dielectric interface**. They say **nothing** about the value of the fields at locations above this interface.

Conductor – Dielectric Boundary Conditions (contd.)

- Thus under static conditions, the following conclusions can be made about a perfect conductor:

1. No electric field may exist within a conductor, i.e.,

$$\rho_v = 0, \qquad \vec{E} = 0$$

2. Since, $\vec{E} = -\nabla V = 0$, there can be no potential difference between any two points in the conductor; that is, a conductor is an equipotential body.
3. An electric field must be external to the conductor and must be normal to its surface. i.e.,

$$\vec{D}_t = \varepsilon_0 \varepsilon_r \vec{E}_t = 0, \qquad \vec{D}_n = \varepsilon_0 \varepsilon_r \vec{E}_n = \rho_s$$

An important use of this concept is in the design of Electrostatic Shielding

Conductor – Free Space Boundary Conditions

- It is a special case of conductor-dielectric boundary conditions.
- Replace by $\varepsilon_r = 1$ in the expressions to get:

$$\vec{D}_t = \varepsilon_0 \vec{E}_t = 0, \qquad \vec{D}_n = \varepsilon_0 \vec{E}_n = \rho_s$$

It should be noted once again that the electric field must approach a conducting surface normally.