Lecture – 7 Date: 29.01.2015

- Energy Density in Electrostatic Field (contd.)
- Conduction and Convection Current
- Dielectrics and Conductors
- Ohm’s Law
- Resistor
- Perfect Conductor

Example – 1

- If $V = \rho^2 z sin\phi$, calculate the energy within the region defined by $1 < \rho < 4, -2 < z < 2, 0 < \phi < \frac{\pi}{3}$.

Start:

$$\vec{E} = -\nabla V \quad \Rightarrow \vec{E} = -\left(\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$\therefore \vec{E} = -\left(2\rho z \sin\phi \hat{a}_\rho + \rho z \cos\phi \hat{a}_\phi + \rho^2 \sin\phi \hat{a}_z\right)$$

Therefore:

$$W_E = \frac{1}{2}\varepsilon_0 \int_v \left|\vec{E}\right|^2 dv$$

$$\frac{2W_E}{\varepsilon_0} = \iiint_v \left(4\rho^2 z^2 \sin^2\phi \hat{a}_\rho + \rho^2 z^2 \cos^2\phi \hat{a}_\phi + \rho^4 \sin^2\phi \hat{a}_z\right)\rho d\phi dz d\rho$$

$$\therefore W_E = \frac{1507.67}{2}\left(\frac{10^{-9}}{36\pi}\right)$$

Energy Density in Electrostatic Field (contd.)

$$W_E = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right)dv$$

$$W_E = \frac{1}{2}\int_v \rho_v V dv$$

What these expressions mean is that it takes energy to assemble a charge distribution $\rho_v(\bar{r})$, or equivalently, an electric field $\vec{E}(\bar{r})$. This energy is stored until it is released— the charge density returns to zero.

Q: Is this energy stored in the fields $\vec{E}(\bar{r})$ and $\vec{D}(\bar{r})$, or by the charge $\rho_v(\bar{r})$??

A: One equation for W_E would suggest that the energy is stored by the **fields**, while the other by the **charge**.

It turns out, **either** interpretation is correct! The fields $\vec{E}(\bar{r})$ and $\vec{D}(\bar{r})$ **cannot** exist without a charge density $\rho_v(\bar{r})$, and knowledge of the fields allow us to determine **completely** the charge density.

In other words, charges and the fields they create are “inseparable pairs”, since both must be present, we can attribute the stored energy to **either** quantity.

Convection and Conduction Current

- The current through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt}$$

- Now, if the current ΔI flows through a planar surface ΔS then:

$$\frac{\Delta I}{\Delta S} = J$$

Current Density

$$\Rightarrow \Delta I = J \Delta S$$

When current density is perpendicular to the surface

- For the case when current density is not normal to the surface:

$$\Delta I = \vec{J} . \overline{\Delta S}$$

Total current flowing through the surface

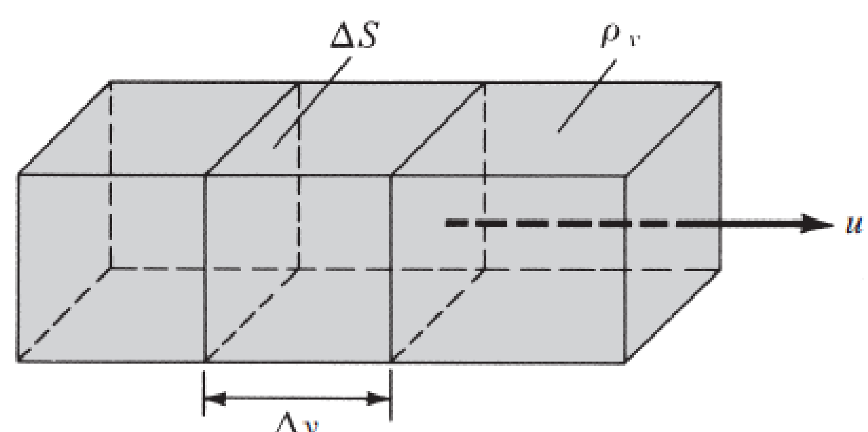
$$I = \int_S \vec{J} . \overline{dS}$$

current "I" through S is the flux of current density $\vec{J}$

Convection and Conduction Current (contd.)

- "I" can be produced in three ways and therefore three kinds of current density exist: Convection Current Density, Conduction Current Density, and Displacement Current Density.
- The derived expression for current density is valid for any type of current.
- Convection current doesn't involve conductors and as a consequence doesn't satisfy Ohm's Law.
- It occurs when current flows through an insulating medium such as liquid, rarefied gas, or a vacuum.
- A beam of electrons in a vacuum tube, for example, is a convection current.
- For example, if there is a charge flow, of density ρ_v, at velocity $\vec{u} = u_y \hat{a}_y$ then:

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t}$$



Convection and Conduction Current (contd.)

$$\Rightarrow \Delta I = \rho_v \Delta S \frac{\Delta y}{\Delta t} = \Delta S \rho_v u_y$$

Y-directed current density

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y$$

Generic Expression

$$\vec{J} = \rho_v \vec{u}$$

Convection Current Density

- **Conduction current** requires conductor.
- A conductor is characterized by a large number of free electrons that provide conduction current due to an applied electric field.
- The force due to an electric field $\vec{E}$ on an electron with charge $-e$ is:

$$\vec{F} = -e\vec{E}$$

Convection and Conduction Current (contd.)

- Since the electron isn't free in space, it will not experience an average acceleration under the influence of electric field.
- Instead, it suffers constant collisions with the atomic lattice and drifts from one atom to another.
- If electron of mass m is moving in an electric field $\vec{E}$ with an average drift velocity $\vec{u}$ then:

$$\frac{m\vec{u}}{\tau} = -e\vec{E} \quad \Rightarrow \quad \vec{u} = -\frac{e\tau}{m}\vec{E}$$

τ is average time between collisions

- If there are n electrons per unit volume:

$$\rho_v = -ne \quad \Rightarrow \quad \vec{J} = \rho_v \vec{u} = \frac{ne^2\tau}{m}\vec{E} \quad \Rightarrow \quad \vec{J} = \sigma\vec{E}$$

$$\frac{ne^2\tau}{m} = \sigma$$

Conductivity of Conductor

Point form of Ohm's Law

Example – 2

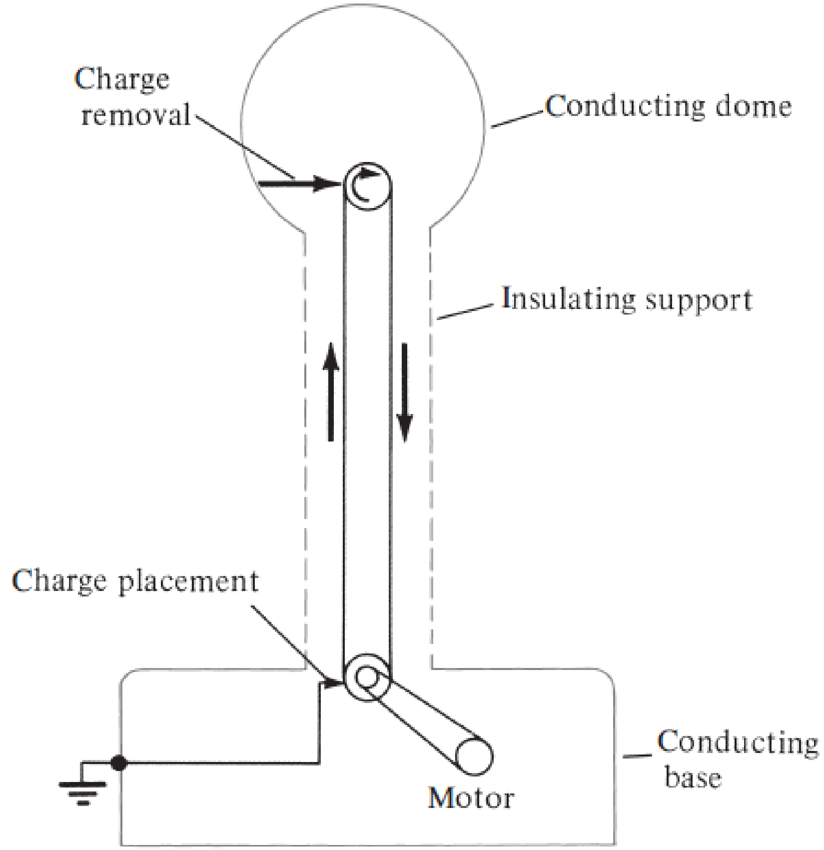
- For the current density $\vec{J} = 10z\sin^2\phi\hat{a}_\rho\ A/m^2$, find the current through the cylindrical surface $\rho = 2, 1 \le z \le 5\ m$.

$$\overrightarrow{dS} = \rho d\phi dz\hat{a}_\rho \quad \Rightarrow \quad I = \int_S \vec{J}.\overrightarrow{dS} \quad \Rightarrow \quad I = \int_{\phi=0}^{2\pi}\int_{z=1}^{5} 10z\sin^2\phi\rho dzd\phi|_{\rho=2}$$

$$\Rightarrow I = \int_{\phi=0}^{2\pi}\int_{z=1}^{5} 10z\sin^2\phi\rho dzd\phi|_{\rho=2} = 10(2)\left[\frac{z^2}{2}\right]_1^5 \int_0^{2\pi}\frac{1}{2}(1-\cos 2\phi)d\phi \quad \Rightarrow \quad \therefore I = 240\pi = 754\text{ A}$$

Example – 3

- A typical example of convective charge transport is found in the Van de Graaf generator where charge is transported on a moving belt from the base to the dome as shown in Figure.
- If a surface charge density $10^{-7}\ C/m^2$ is transported by the belt at a velocity of $2\ m/s$, calculate the charge collected in $5s$. Take the width of the belt as $10\ cm$.

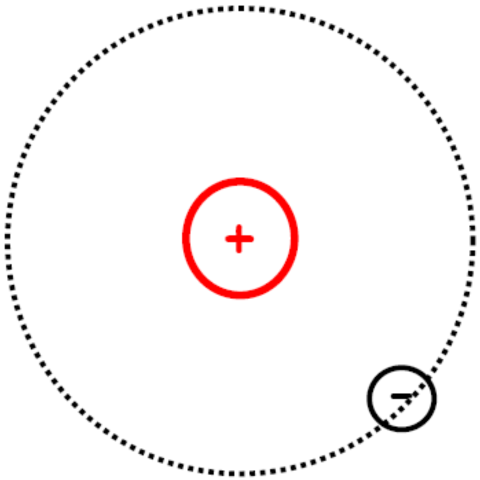


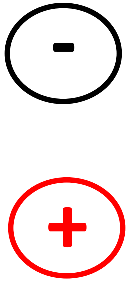
$$I = (\rho_S w)u \quad \Rightarrow \quad Q = It = (\rho_S w)ut \quad \Rightarrow \quad \Rightarrow Q = (10^{-7} \times 0.1) \times 2 \times 5$$

$$\therefore Q = 100nC$$

Dielectrics and Conductors

- Consider a very **simple** model of an **atom**:





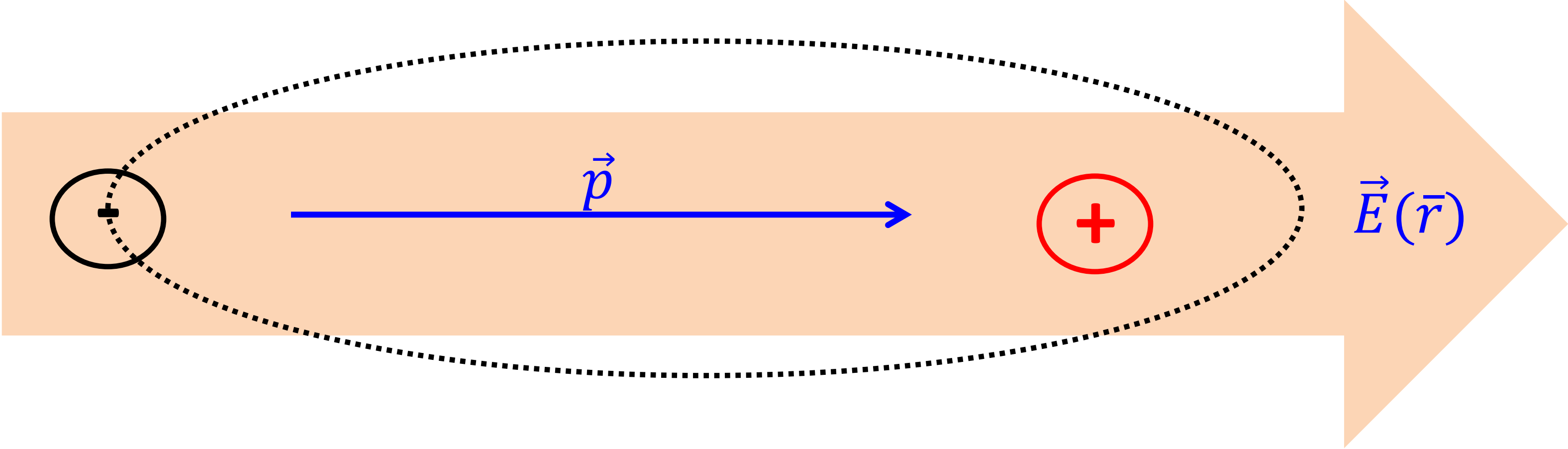
→ electron (negative charge)

→ nucleus (positive charge)

- Say an **electric field** is applied to this atom.
  - Note the field will apply a **force** on both the positively charged nucleus and the negatively charged electron.
  - However, these forces will move these particles in **opposite** directions!
  - This will lead to two situations.

Dielectrics and Conductors (contd.)

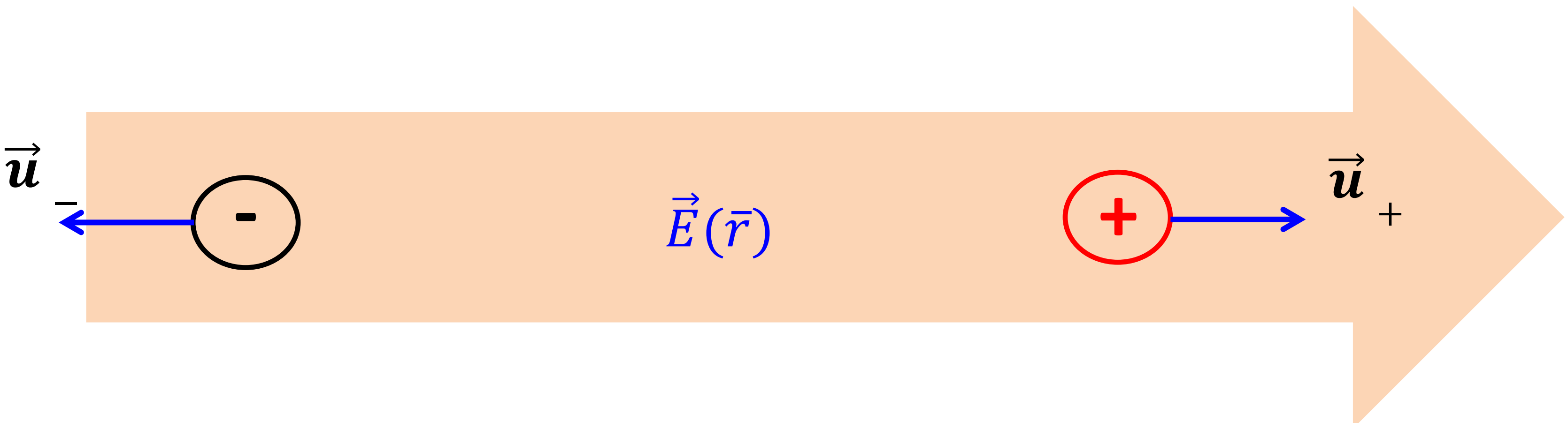
- In the **first** case, the atom may **stretch**, but the electron will remain **bound** to the atom:



<u>Note, an **electric dipole** has been created!</u>

Dielectrics and Conductors (contd.)

- For the **second** case, the electron may **break free** from the atom, creating a positive ion and a **free electron**. We call these free charges, and the electric field will cause them to **move** in opposite directions.



- **Moving charge**! We know what moving charge is.

Moving charge is **electric current** $\vec{J}(\bar{r})$.

These two examples provide a simple demonstration of what occurs when an electric field is applied to some **material** (e.g., plastic, copper, water, oxygen).

Dielectrics and Conductors (contd.)

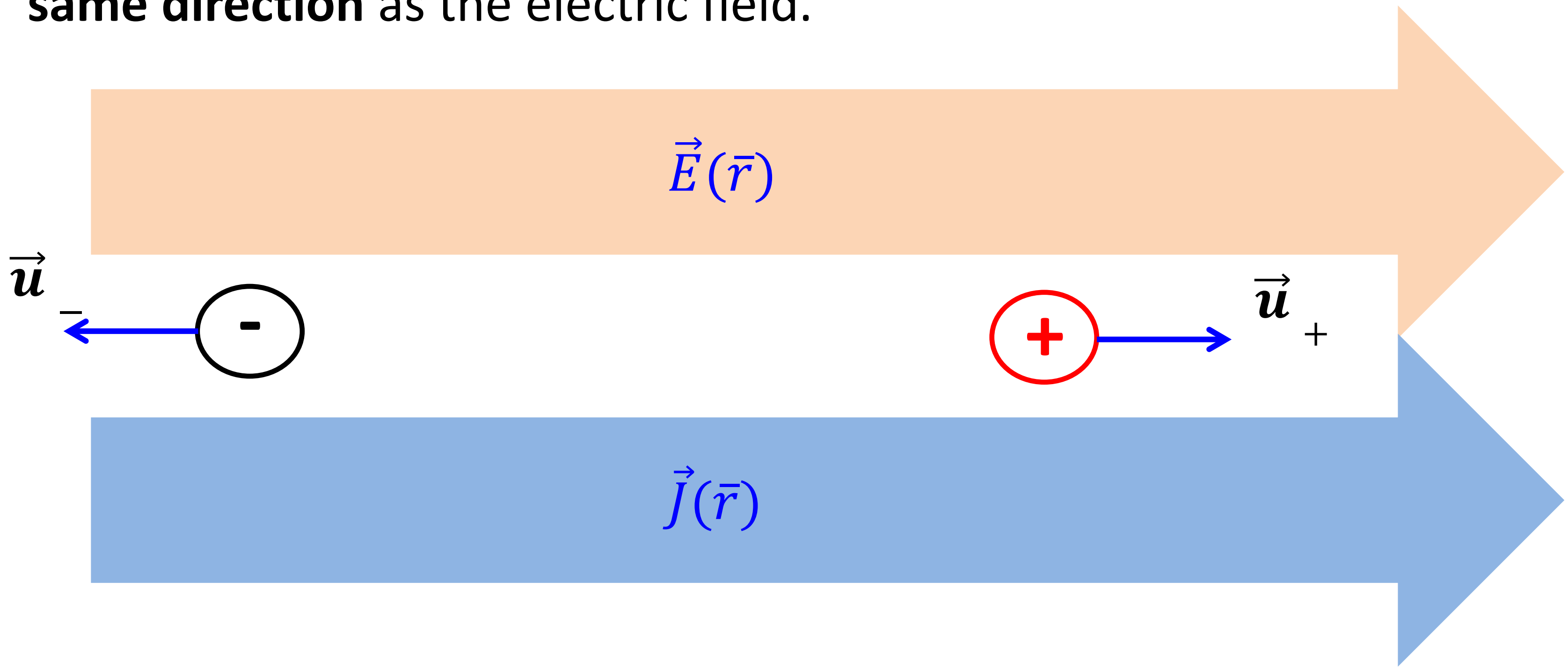
1. Materials where the charges remain bound (and thus dipoles are created) are called **insulator** (or **dielectric**) materials.
2. Materials where the electrons are free to move are called **conductors**.

Of course, materials consists of molecules with **many electrons**, and in general some electrons are **bound** and some are **free**. As a result, there are no **perfect** conductors or **perfect** insulators, although some materials are **very** close!

Additionally, some materials lie between being a good conductor or a good insulator. We call these materials **semiconductors** (e.g., Silicon).

Ohm’s Law

- Recall that a positively charged particle will move in the direction of an electric field, whereas a negative charge will move in the opposite direction. Both types of charge, however, result in **current** moving in the **same direction** as the electric field.



Ohm's Law (contd.)

Q: So, the direction of current density $\vec{J}(\bar{r})$ and electric field $\vec{E}(\bar{r})$ are the same. The question then is, how are their **magnitudes** related?

A: They are related by **Ohm's Law**:

$$\vec{J}(\bar{r}) = \sigma(\bar{r})\vec{E}(\bar{r})$$

The scalar value $\sigma(\bar{r})$ is called the material's **conductivity**.

Note: the units of conductivity are:

$$\sigma(\bar{r}) = \frac{\vec{J}(\bar{r})}{\vec{E}(\bar{r})} \quad \left(\frac{Ampere}{m^2}\right)\left(\frac{Volts}{m}\right)^{-1}$$

$$= \frac{\vec{J}(\bar{r})}{\vec{E}(\bar{r})} \quad \left(\frac{Ampere}{Volts * m}\right)$$

$$= \frac{\vec{J}(\bar{r})}{\vec{E}(\bar{r})} \quad \left(\frac{1}{Ohm * m}\right)$$

In other words, the unit of conductivity is **conductance/unit length.**

Ohm's Law (contd.)

- We emphasize that conductivity $\sigma(\bar{r})$ is **a material parameter**. For example, the conductivity of **copper** is:

$$\sigma_{copper} = 5.8\times 10^{7}\left[\frac{1}{\Omega m}\right]$$

- and the conductivity of **polyethylene** (a plastic) is:

$$\sigma_{polyethylene} = 1.5\times 10^{-12}\left[\frac{1}{\Omega m}\right]$$

Note the **vast** difference in conductivity between these two materials. Copper is a **conductor** and polyethylene is an **insulator**.

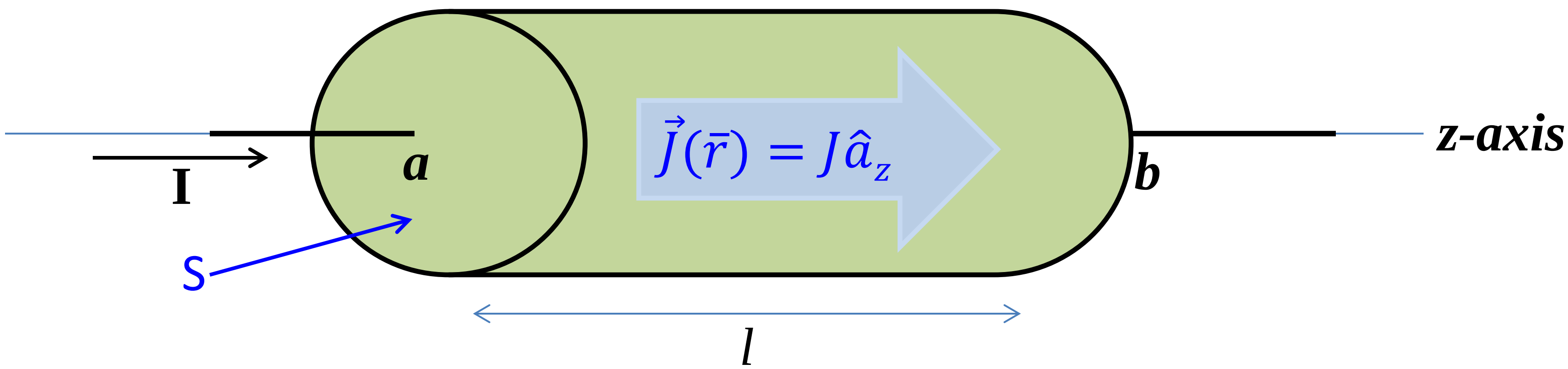
A **perfect insulator (i.e., dielectric)** is a material with $\sigma = 0$. In contrast, a **perfect conductor** is a material with $\sigma = \infty$.

Alternatively, **we can say**: **For a perfect dielectric** $\vec{J} = 0$, **whereas for a perfect conductor** $\vec{E} = 0$.

Georg Simon Ohm (1789-1854) was the German physicist who in 1827 discovered the law that the current flowing through a conductor is proportional to the voltage and inversely proportional to the resistance. Ohm was then a professor of mathematics in Cologne. His work was **coldly** received! The Prussian minister of education announced that "a professor who preached such heresies was unworthy to teach science." Ohm resigned his post, went into academic exile for several years, and then left Prussia and became a professor in Bavaria.

Resistors

- Consider a **uniform** cylinder of material with mediocre to poor to pathetic **conductivity** $\sigma(\bar{r}) = \sigma$.
- This cylinder is centered on the z-axis, and has **length** l. The **surface area** of the ends of the cylinder is S.



- Say the cylinder has **current** "I" flowing into it (and thus out of it), producing a current **density** $\vec{J}(\bar{r})$.
- By the way, we can refer such a cylinder as a **resistor**!

Resistors (contd.)

Q: What is the **resistance** R of this resistor, given length l, cross-section area S, and conductivity σ?

A: Let's first begin with the circuit form of Ohm's Law:

$$R = \frac{V}{I}$$

where V is the potential difference between the two ends of the resistor (i.e., the voltage across the resistor), and I is the current through the resistor.

- From **electrostatics**, we know that the potential difference V is:

$$V = V_{ab} = \int_a^b \vec{E}(\overline{r}).\overline{dl}$$

- and the current "I" is:

$$I = \iint_S \vec{J}(\overline{r}).\overline{dS}$$

Resistors (contd.)

- Thus, we can **combine** these expressions and find resistance R, expressed in terms of electric field $\vec{E}(\bar{r})$ within the resistor, and the current density $\vec{J}(\bar{r})$ within the resistor:

$$R = \frac{V}{I} = \frac{\int_a^b \vec{E}(\bar{r}).\overline{dl}}{\iint_S \vec{J}(\bar{r}).\overline{dS}}$$

- Lets evaluate **each integral** in this expression to determine the resistance R of the device described earlier!

1. **The voltage** V is the potential difference V_{ab} between point a and point b:

$$V = V_{ab} = \int_a^b \vec{E}(\bar{r}).\overline{dl}$$

Q: But, what is the electric field $\vec{E}(\bar{r})$?

A: The electric field within the resistor can be determined from **Ohm's Law**:

$$\vec{E}(\bar{r}) = \vec{J}(\bar{r}) \big/ \sigma(\bar{r})$$

Resistors (contd.)

- We can assume that the **current density** is approximately **constant** across the cross section of the cylinder:

$$\vec{J}(\bar{r}) = J\hat{a}_z$$

- Likewise, we know that the conductivity of the resistive material is a **constant**:

$$\sigma(\bar{r}) = \sigma$$

- As a result, the electric field **within** the resistor is:

$$\vec{E}(\bar{r}) = \frac{\vec{J}(\bar{r})}{\sigma(\bar{r})} = \frac{J}{\sigma}\hat{a}_z$$

- Therefore, **integrating** in a straight line along the z-axis from point a to point b, we find the potential difference V to be:

$$V = \int_a^b \vec{E}(\bar{r}).\overline{dl} = \frac{J}{\sigma}\int_{Z_a}^{z_b} \hat{a}_z.\hat{a}_z dz = \frac{J}{\sigma}\int_{Z_a}^{z_b} dz = \frac{Jl}{\sigma}$$

Resistors (contd.)

2. We likewise know that the current I through the resistor is found by evaluating the **surface integral**:

$$I = \iint_S J\hat{a}_z . \hat{a}_z ds_z = J \iint_S ds_z = JS$$

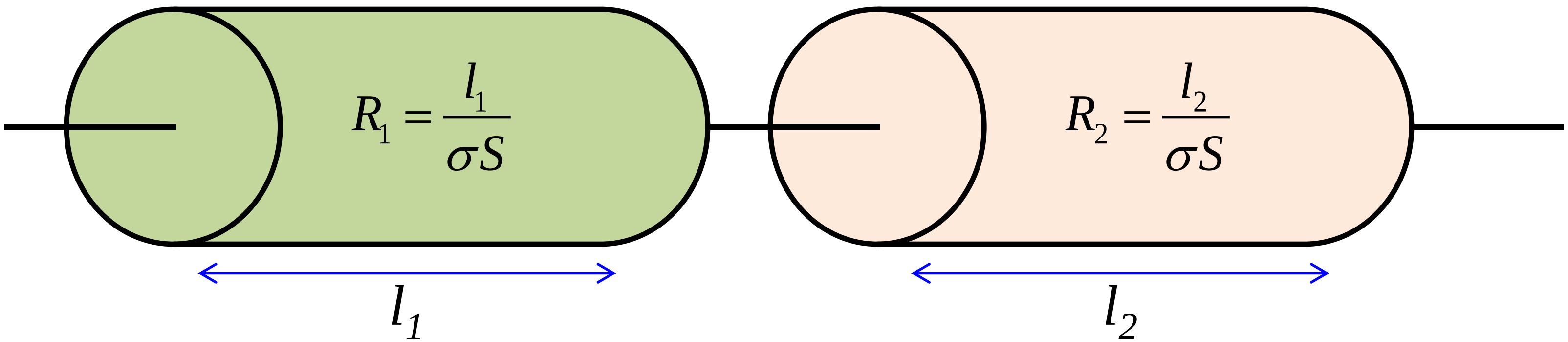
- Therefore, the resistance R of **this particular** resistor is:

$$R = \frac{V}{I} = \left(\frac{Jl}{\sigma}\right)\left(\frac{1}{JS}\right) = \frac{l}{\sigma S}$$

- An interesting result! Consider a resistor as sort of a "clogged pipe". **Increasing** the cross-sectional area S makes the pipe bigger, allowing for **more current** flow. In other words, the resistance of the pipe decreases, as predicted by the above equation.
- Likewise, increasing the **length** l simply increases the length of the "clog". The current encounters resistance for a longer distance, **thus the value of R increases with increasing length l**. Again, this behavior is predicted by the equation shown above.

Resistors (contd.)

- For **example**, consider the case where we add two resistors together:



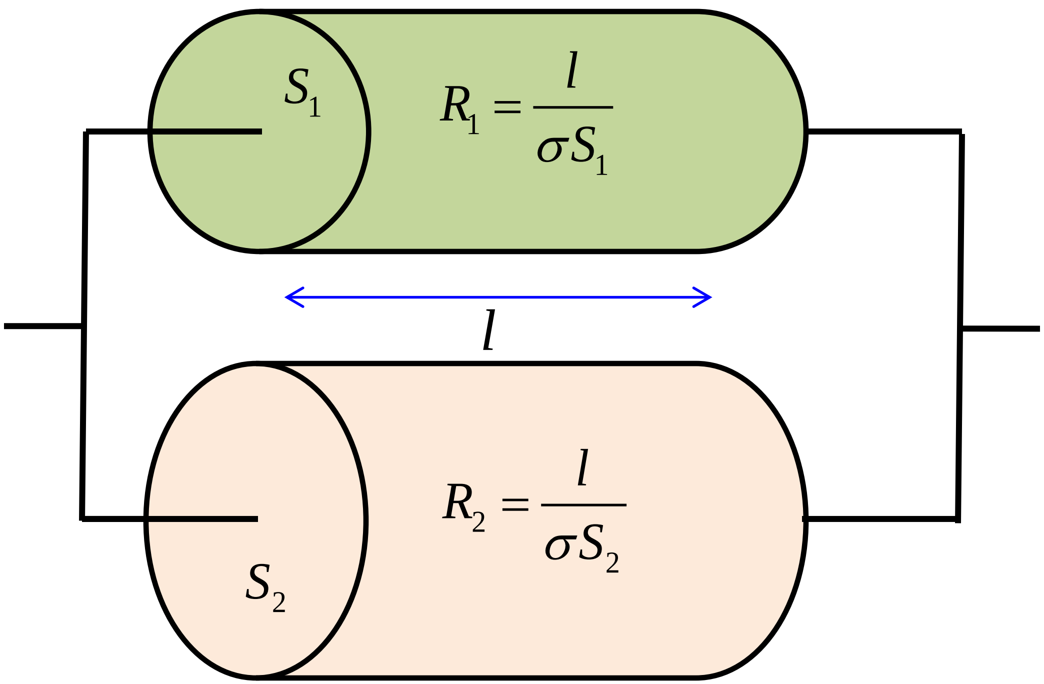
- We can view this case as a single resistor with a length l_1+l_2, resulting in a total resistance of:

$$R_{total} = \frac{l_1 + l_2}{\sigma S} \Rightarrow R_{total} = \frac{l_1}{\sigma S} + \frac{l_2}{\sigma S} \Rightarrow \therefore R_{total} = R_1 + R_2$$

But, this result is not the **least bit** surprising, as the two resistors are connected in **series**!

Resistors (contd.)

- Now let's consider the case where two resistors are connected in a different manner:



- We can view this as a single resistor with a total cross sectional area of S_1+S_2. Thus, its total resistance is:

$$R_{total} = \frac{l}{\sigma(S_1 + S_2)}$$

$$= \left[\frac{\sigma(S_1 + S_2)}{l}\right]^{-1} \quad = \left[\frac{\sigma S_1}{l} + \frac{\sigma S_2}{l}\right]^{-1}$$

$$\therefore R_{total} = \left[\frac{1}{R_1} + \frac{1}{R_2}\right]^{-1}$$

Again, this should be no surprise, as these two resistors are connected in **parallel**.

Resistors (contd.)

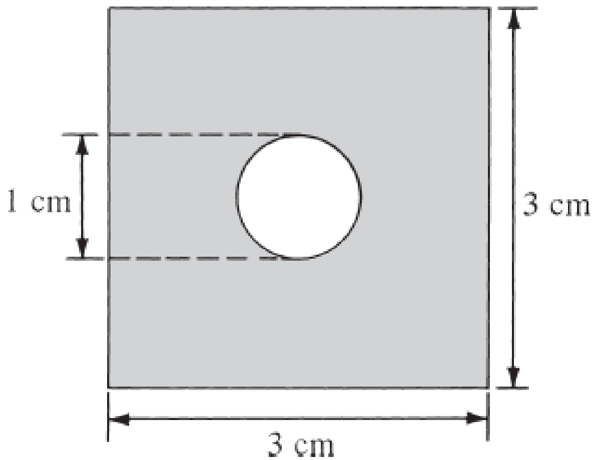
IMPORTANT NOTE: The result R= $l/\sigma S$ is valid **only** for the resistor whose conductivity is a **constant** $(\sigma(\bar{r}) = \sigma)$.

- If the conductivity is **not** a constant, then we **must** evaluate the potential difference across the resistor with the more **general** expression:

$$V_{ab} = \int_a^b \vec{E}(\bar{r}).\overline{dl} \quad \Rightarrow \quad = \int_a^b \frac{\vec{J}(\bar{r})}{\sigma(\bar{r})}.\overline{dl}$$

Example – 4

- A lead ($\sigma = 5 \times 10^6\ S/m$) bar of square cross section has a hole bored along its length of 4m so that the cross section becomes as shown below. Find the resistance between the square ends.



- Since the cross section of the bar is uniform: $R = \dfrac{l}{\sigma S}$

- Where: $S = d^2 - \pi r^2 \;\Rightarrow\; S = (3)^2 - \pi\left(\dfrac{1}{2}\right)^2 \;\Rightarrow\; S = \left(9 - \dfrac{\pi}{4}\right) cm^2$

- Therefore: $$R = \frac{4}{5\times10^6 \times \left(9 - \dfrac{\pi}{4}\right)\times 10^{-4}} = 974\,\mu\Omega$$

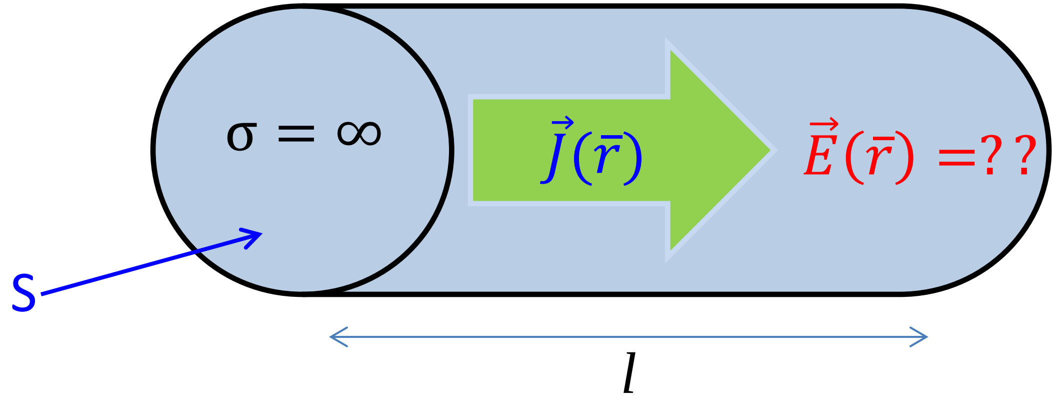
Example – 5

Example – 6

Example – 7

Perfect Conductors

- Consider some current with density $\vec{J}(\bar{r})$, flowing within some material with **perfect conductivity** (i.e., $\sigma = \infty$)!



Q: What is the **electric field** $\vec{E}(\bar{r})$ within this perfectly conducting material?

A: Well, we know from **Ohm's Law** that the electric field is related to the material conductivity and current density as:

$$\vec{E}(\bar{r}) = \frac{\vec{J}(\bar{r})}{\sigma}$$

Thus, as the material conductivity approaches **infinity**, we find:

$$\lim_{\sigma \to \infty} \vec{E}(\bar{r}) = \frac{\vec{J}(\bar{r})}{\sigma} = 0$$

Perfect Conductors (contd.)

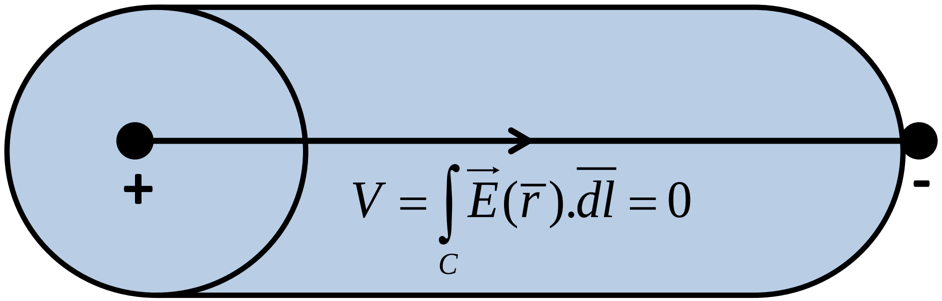
- The **electric field** within a perfectly conducting material is always equal to **zero**!
- **This makes sense** when you think about it! Since the material offers **no resistance**, we can move charges through it **without** having to apply any **force** (i.e., and electric field).

This is just like a skater moving across **frictionless** ice! It can continue to move with great velocity, even though **no force** is being applied!

Consider what this means with regards to a **wire** made of a **perfectly conducting** material (an often applied assumption).

Perfect Conductors (contd.)

- The electric potential difference between either end of a perfectly conducting wire is **zero**!



Since the electric field within a perfect wire is **zero**, the voltage across any perfect wire is also **zero**, regardless of the current flowing through it.