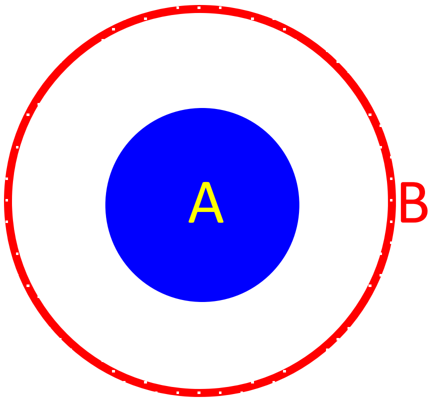
Lecture – 6

Date: 22.01.2015

- Applications of Gauss Law
- Work Done by Electric Field
- Potential Difference
- Potential Gradient
- Electric Dipole
- Equipotential Surfaces
- Energy Density in Electrostatic Fields
- Conduction and Convection Current

Example – 1

A blue sphere A is contained within a red spherical shell B. There is a charge Q_A on the blue sphere and charge Q_B on the red spherical shell.



- The electric field in the region between the spheres is completely independent of Q_B the charge on the red spherical shell.

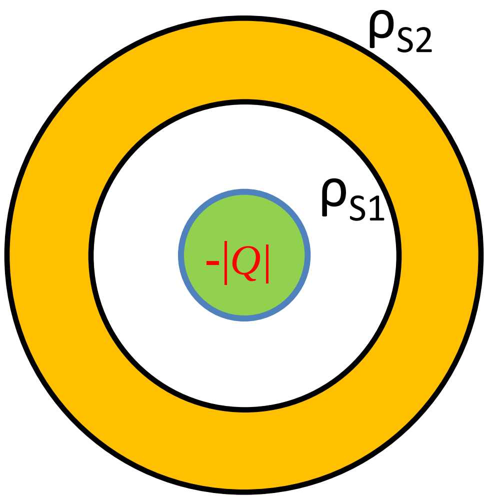
True

False

Example – 2

Consider the following topology:

A) A solid non-conducting sphere carries a total charge Q = -3 mC distributed evenly throughout. It is surrounded by an *uncharged* conducting spherical shell.

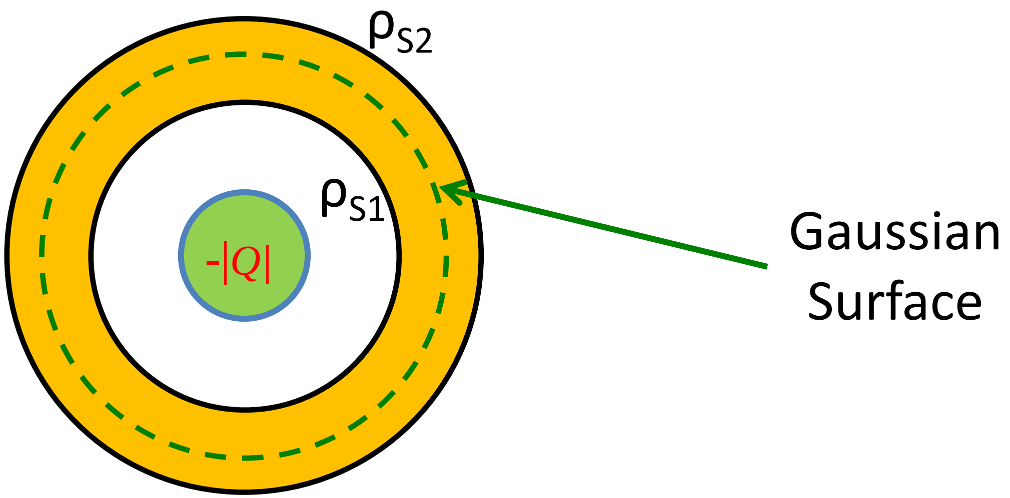


- What is the surface charge density ρ_{S1} on the inner surface of the conducting shell?

(c) $\rho_{S1} < 0$ (c) $\rho_{S1} = 0$ (c) $\rho_{S1} > 0$

Example – 2 (contd.)

- Inside the conductor, we know the field $\vec{E} = 0$
- Select a Gaussian surface inside the conductor
  - Since $\vec{E} = 0$ on this surface, the total enclosed charge must be 0.
  - Therefore, the surface charge density on the inner surface of the conducting shell must be positive, to cancel the charge -|Q|.



(a) $\rho_{s1} < 0$ (b) $\rho_{s1} = 0$ (c) $\rho_{S1} > 0$

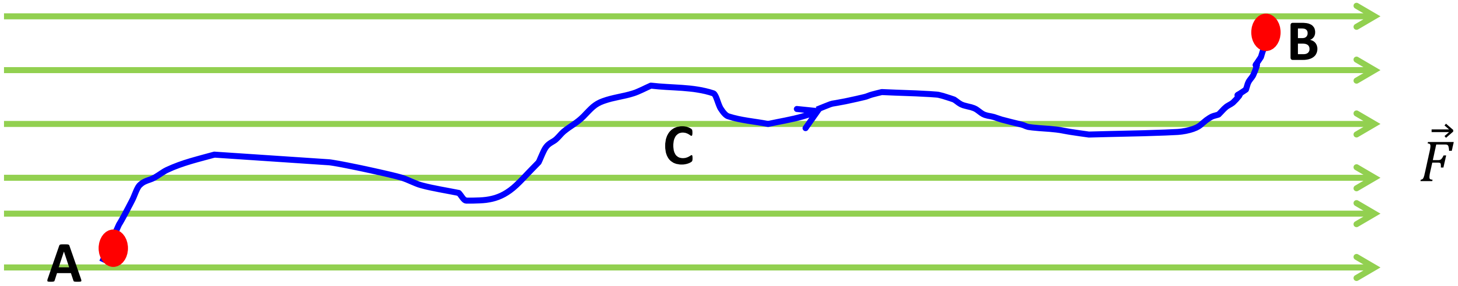
Alternative Formulation for $\vec{E}$ Determination

- Through Coulomb's Law
- Through Gauss's Law – if charge distribution is symmetric
- Through a Scalar Quantity (?) – easier to handle

Work done by Electric Field

- An important application of the line integral is the calculation of **work**. Say there is some vector field $\vec{F}(\bar{r})$ that exerts a **force** on some object.

  Q: How much **work** (W) is done by this vector field if the object moves from point A to B, along contour C??



A: We can find out by evaluating the line integral: $$W_{AB} = \int_C \vec{F}(\bar{r}).\overline{dl}$$

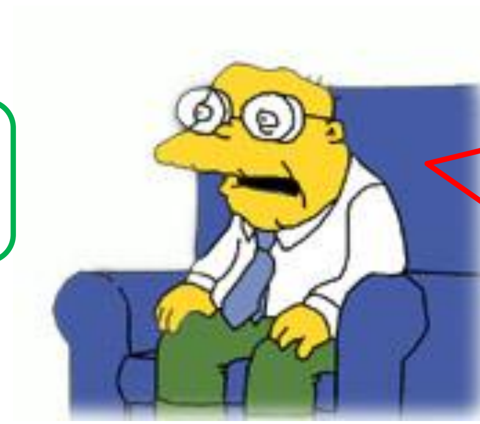
- Say this object is a **charged particle** with charge Q, and the force is applied by a static **electric field** $\vec{E}(\bar{r})$. We **know** the force on the charged particle is:

$$\vec{F}(\bar{r}) = Q\vec{E}(\bar{r})$$

Work done by Electric Field (contd.)

- and thus the work done **by the electric field** in moving a charged particle along some contour C is:

$$W_{AB} = \int_C \vec{F}(\overline{\mathrm{r}}).\overline{dl} = Q\int_C \vec{E}(\overline{\mathrm{r}}).\overline{dl}$$



Q: Oooh, I don't like evaluating contour integrals; isn't there some **easier** way?

A: **Yes** there is! Recall that a **static** electric field is a **conservative** vector field. Therefore, we can write any static electric field as the **gradient** of a specific **scalar** field V($\bar{r}$):

$$\vec{E}(\overline{\mathrm{r}}) = -\nabla V(\overline{\mathrm{r}})$$

Work done by Electric Field (contd.)

- We can then evaluate the work integral as: $W_{AB} = Q\int_C \vec{E}(\overline{r}).\overline{dl} = -Q\int_C \nabla V(\overline{r}).\overline{dl}$

$$\Rightarrow W_{AB} = -Q\left[V(\overline{r}_B) - V(\overline{r}_A)\right]$$

$$\therefore W_{AB} = Q\left[V(\overline{r}_A) - V(\overline{r}_B)\right]$$

- We **define**: $V(\overline{r}_A) - V(\overline{r}_B) \doteq V_{AB}$

- Therefore: $W_{AB} = QV_{AB}$

So what the heck **is** V_{AB}? Does it mean any thing? Do we use it in engineering?

First, consider what W_{AB} is!

The value W_{AB} represents the work done **by** the electric field **on** charge Q when moving it from point A to point B. This is **precisely** the same concept as when a **gravitational force** moves an object from one point to another.

Work done by Electric Field (contd.)

- The work done by the gravitational field in this case is equal to the **difference** in **potential energy** between the object at these two points.
- The value W_{AB} represents the **same** thing! It is the **difference** in **potential energy** between the charge at point A and at B.
  - Great, now we know what W_{AB} is. But the question was, **WHAT IS** V_{AB} !?!
  - That's easy! Just rearrange the equation:

$$V_{AB} = \frac{W_{AB}}{Q}$$

See? The value V_{AB} is equal to the difference in potential energy, **per coulomb of charge**!

- In other words V_{AB} represents the difference in potential energy for **each** coulomb of charge in Q.
- Another way to look at it: V_{AB} is the difference in potential energy if the particle has a charge of **1 Coulomb** (i.e., Q =1).

Work done by Electric Field (contd.)

- Note that V_{AB} can be expressed as:

$$V_{AB} = -\int_C \vec{E}(\bar{r}).\overline{dl} = V(\bar{r}_A) - V(\bar{r}_B)$$

where point A lies at the **beginning** of contour C, and B lies at the **end**.

- We refer to the **scalar field** V($\bar{r}$) as the **electric potential function**, or the electric potential **field**.
- We likewise refer to the scalar value V_{AB} as the **electric potential difference**, or simply the **potential difference** between point A and point B.
- Note that V_{AB} (and therefore V($\bar{r}$)), has units of:

$$V_{AB} = \frac{W_{AB}}{Q}\left[\frac{Joules}{Coulomb}\right]$$

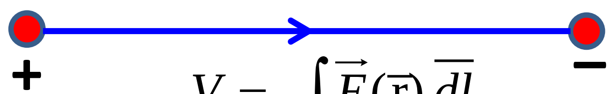
- Joules/Coulomb is a rather **awkward** unit, so we will use the other name for it—**VOLTS!**

$$1\left[\frac{Joules}{Coulomb}\right] \doteq 1 Volt$$

Work done by Electric Field (contd.)

Q: Hey! We used volts in **circuits** class. Is this the **same** thing ?
A: It is **precisely** the same thing !

- Perhaps this will help. Say A and B are two points somewhere on a circuit. But let's call these points something different, say point + and point – .



$$V = -\int_C \vec{E}(\bar{r}).\overline{dl}$$

- Therefore, V represents the **potential difference** (in volts) **between** point (i.e., **node**) + and point (**node**) – . Note this value can be either **positive** or **negative**.

Q: But, does this mean that circuits produce **electric fields**?

Work done by Electric Field (contd.)

A: Absolutely! Anytime you can measure a **voltage** (i.e., a potential difference) between two points, an electric field **must** be present!

Potential Difference (contd.)

- Note that V_{AB} can be expressed as:

$$V_{AB} = -\int_A^B \vec{E}(\bar{r}).\overline{dl} = V(\bar{r}_A) - V(\bar{r}_B)$$

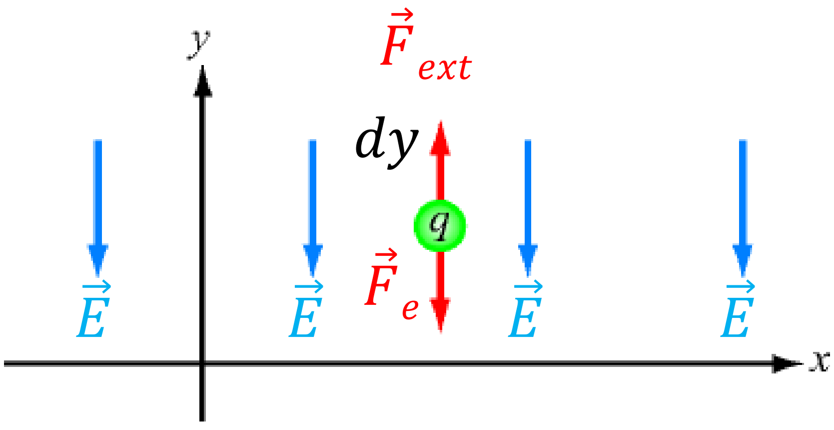
where point A lies at the **beginning** of contour C, and B lies at the **end**.

- The potential at any point is the potential difference between that point and a chosen point (or reference point) at which the potential is zero (usually ground !).

$$V = -\int_\infty^r \vec{E}(\bar{r}).\overline{dl}$$

Polarity of Electric Potential and Fields are Opposite?

- Lets consider the simple case of a positive charge q in a uniform electric field $\vec{E} = -E\hat{a}_y$
- The presence of $\vec{E}$ exerts a force $\vec{F}_e = q\vec{E}$ on the charge in the **−y direction**



- To move the charge along the **+y direction** against $\vec{F}_e$, there is a need of external force $\vec{F}_{ext}$ to counteract $\vec{F}_e$
- To move q without acceleration:

$$\vec{F}_{ext} = -\vec{F}_e = -q\vec{E}$$

- The work done in moving the charge a vector differential distance $\overline{dy} = dy\hat{a}_y$:

$$dW = \vec{F}_{ext}.\left(dy\hat{a}_y\right) = -q\vec{E}.\overline{dy}$$

- Now, the electric potential is the energy per unit charge:

$$dV = \frac{dW}{q}$$

$$dV = -\vec{E}.\overline{dy}$$

Conclusion of our Premise !

Electric Potential for Point Charge

- Recall that a point charge Q, **located at the origin** ($(\bar{r}' = 0)$), produces a static electric field:

$$\vec{E}(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

- Now, we know that this field is the **gradient** of some scalar field:

$$\vec{E}(\bar{r}) = -\nabla V(\bar{r})$$

Q: What is the **electric potential** function V($\bar{r}$) generated by a **point charge** Q, located at the origin?

A: We find that it is:

$$V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r}$$

Q: Where did **this** come from? How do we know that this is the correct solution?

A: We can show it is the correct solution by **direct substitution**!

Electric Potential for Point Charge (contd.)

Verification:

$$\vec{E}(\bar{r}) = -\nabla V(\bar{r}) = -\nabla\left(\frac{Q}{4\pi\varepsilon_0 r}\right) = -\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\varepsilon_0 r}\right)\hat{a}_r + 0 + 0$$

$$\therefore \vec{E}(\bar{r}) = -\nabla V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

The **correct** result!

Q: What if the charge is **not** located at the **origin** ?

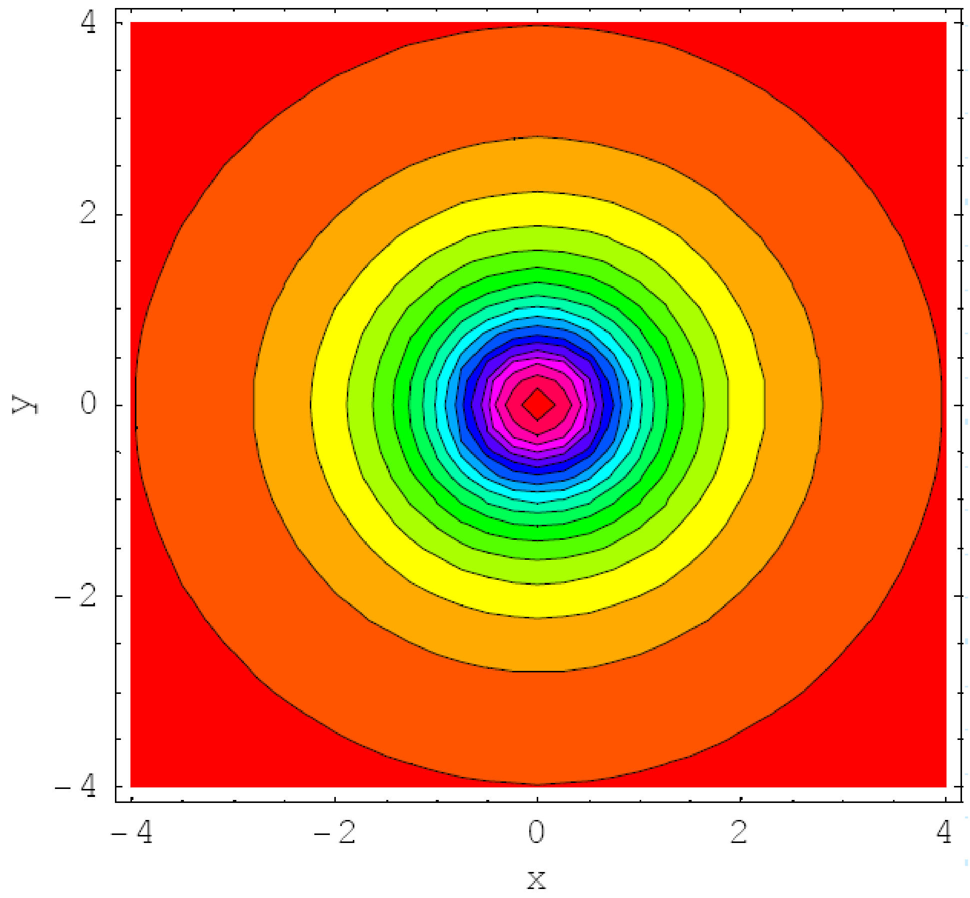
A: Substitute r with $|\bar{r} - \bar{r}'|$, and we get:

$$V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 |\bar{r} - \bar{r}'|}$$

where, as before, the position vector $\bar{r}'$ denotes the location of the **charge** Q, and the position vector $\bar{r}$ denotes the location in space where the electric potential function is **evaluated**.

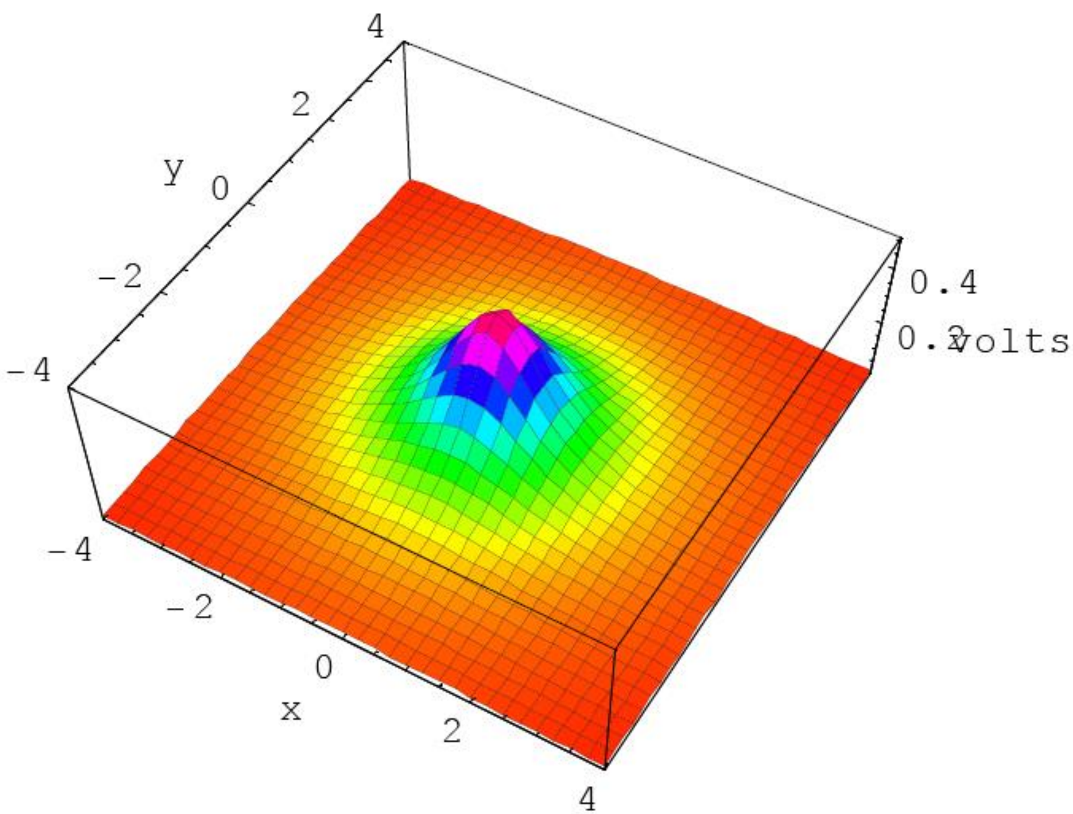
Electric Potential for Point Charge (contd.)

- The **scalar** function $V(\bar{r})$ for a point charge can be shown graphically as a **contour plot**:



Electric Potential for Point Charge (contd.)

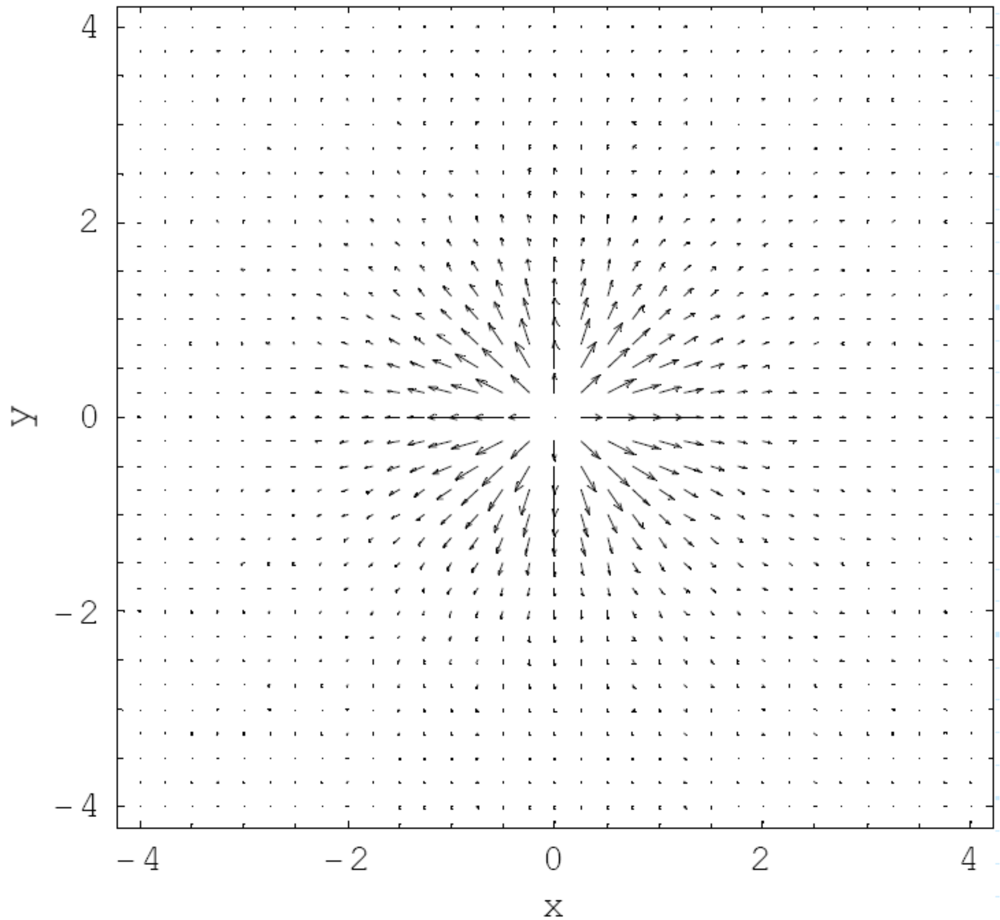
- Or, in **three** dimensions as:



Note the electric potential **increases** as we get **closer** to the point charge (located at the origin). It appears that we have "**mountain**" of electric potential; an appropriate analogy, considering that the potential energy of a mass in the Earth's gravitational field increases with altitude (i.e., height)!

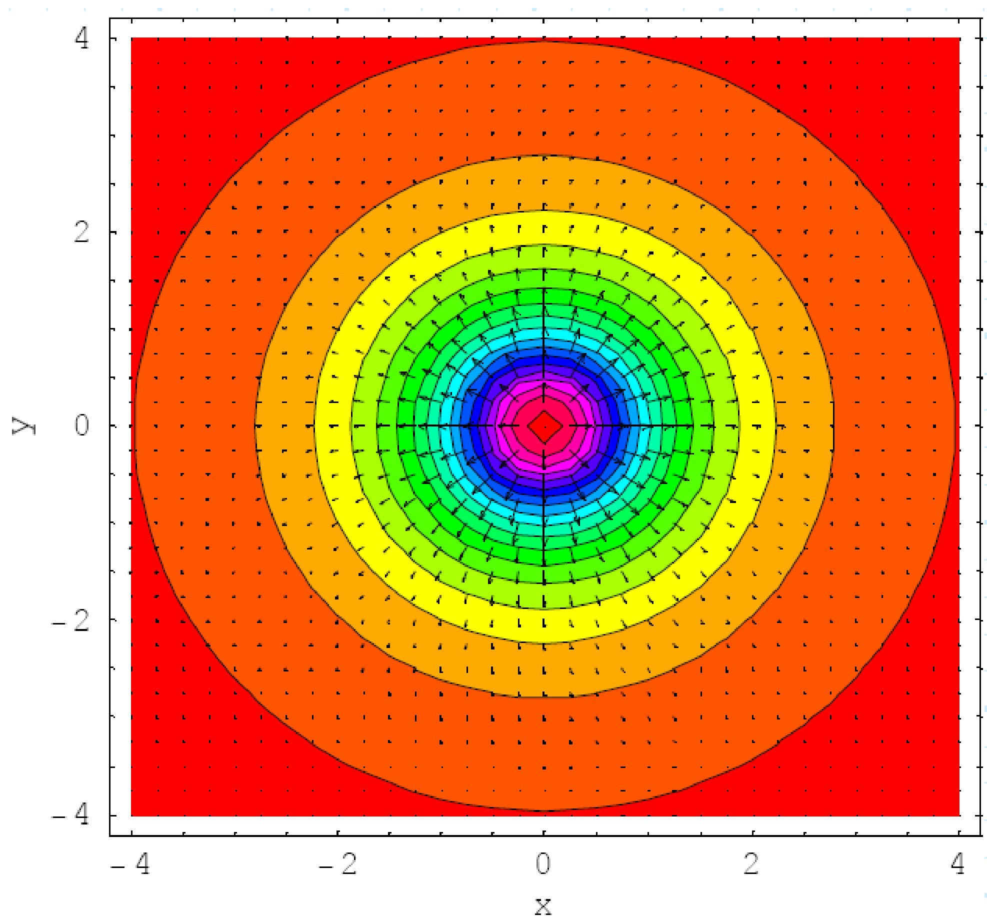
Electric Potential for Point Charge (contd.)

- Recall the **electric field** produced by a point charge is a **vector** field that looks like:



Electric Potential for Point Charge (contd.)

- **Combining** the electric **field** plot with the electric **potential** plot, we get:



Given our understanding of the **gradient**, the above plot makes perfect sense! Do **you** see why ?

Electric Potential Function for Charge Densities

- Recall the total static electric field produced by 2 **different** charges (or charge densities) is just the **vector sum** of the fields produced by each:

$$\vec{E}(\bar{r}) = \vec{E}_1(\bar{r}) + \vec{E}_2(\bar{r})$$

- Since the fields are conservative, we can write this as:

$$\vec{E}(\bar{r}) = \vec{E}_1(\bar{r}) + \vec{E}_2(\bar{r})$$

$$\Rightarrow -\nabla V(\bar{r}) = -\nabla V_1(\bar{r}) - \nabla V_2(\bar{r})$$

$$\therefore -\nabla V(\bar{r}) = -\nabla\left(V_1(\bar{r}) + V_2(\bar{r})\right)$$

- Therefore, we find:

$$V(\bar{r}) = V_1(\bar{r}) + V_2(\bar{r})$$

In other words, **superposition** also holds for the electric potential function! The total electric potential field produced by a collection of charges is simply the **sum** of the electric potential produced by **each**.

Electric Potential Function for Charge Densities (contd.)

- Consider now some **distribution** of charge, $\rho_v(\bar{r})$. The amount of charge dQ, contained within **small volume** dv, located at position $\bar{r}'$, is:

$$dQ = \rho_v(\bar{r}')dv'$$

- The **electric potential function** produced by this charge is therefore:

$$dV(\bar{r}) = \frac{dQ}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|} = \frac{\rho_v(\bar{r}')dv'}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|}$$

- Therefore, **integrating** across all the charge in some **volume** v, we get:

$$V(\bar{r}) = \iiint_v \frac{\rho_v(\bar{r}')}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|}dv'$$

- Likewise, for **surface** or **line** charge density:

$$V(\bar{r}) = \iint_S \frac{\rho_S(\bar{r}')}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|}dS'$$

$$V(\bar{r}) = \int_C \frac{\rho_l(\bar{r}')}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|}dl'$$

Note that these integrations are **scalar** integrations—typically they are **easier** to evaluate than the integrations resulting from **Coulomb's Law**.

- Once we find the electric potential function $V(\bar{r})$, we can **then** determine the total **electric field** by taking the gradient:

$$\vec{E}(\bar{r}) = -\nabla V(\bar{r})$$

Electric Potential Function for Charge Densities (contd.)

- Thus, we now have **three** (!) potential methods for determining the **electric field** produced by some **charge distribution** $\rho_v(\bar{r})$.
  1. Determine $\vec{E}(\bar{r})$ from **Coulomb's Law**.
  2. If $\rho_v(\bar{r})$ is symmetric, determine $\vec{E}(\bar{r})$ from **Gauss's Law**.
  3. Determine the **electric potential function** $V(\bar{r})$, and then determine the electric field as $\vec{E}(\bar{r}) = -\nabla V(\bar{r})$.

Q: Yikes! Which of the three should we use??

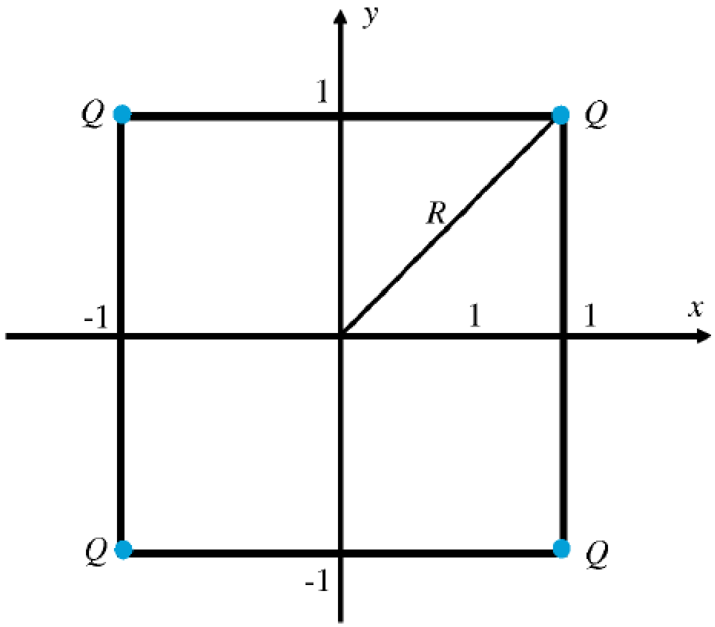
A: To a certain extent, it does **not matter**! All three will provide the **same** result (although $\rho_v(\bar{r})$ **must** be symmetric to use method 2!).

However, **if** the charge density is symmetric, we will find that using Gauss's Law (method 2) will **typically** result in much less work!

Otherwise (i.e., for **non**-symmetric $\rho_v(\bar{r})$), we find that **sometimes** method 1 is easiest, but in **other** cases method 3 is a bit less stressful (i.e., **you** decide!).

Example – 3

- Determine the electric potential at the origin due to four 20-mC charges residing in free space at the corners of a $2m \times 2m$ square centered about the origin in the x–y plane.



- For four identical charges all equidistant from the origin:

$$V(\bar{r}) = \frac{4Q}{4\pi\varepsilon_0 R}$$

$$R = |\bar{r} - \bar{r}'| = \sqrt{2}m$$

$$\therefore V(\bar{r}) = \frac{4 \times 20 \times 10^{-6}}{4\pi\varepsilon_0 \sqrt{2}} = \frac{\sqrt{2} \times 10^{-5}}{\pi\varepsilon_0} \quad \text{(V)}$$

Example – 4

- A spherical shell of radius R has a uniform surface charge density ρ_S. Determine the electric potential at the center of the shell.

$$V(\bar{r}) = \iint_S \frac{\rho_S(\bar{r}')}{4\pi\varepsilon_0 \left|\bar{r} - \bar{r}'\right|} dS'$$

$$V(\bar{r}) = \frac{\rho_S}{4\pi\varepsilon_0} \iint_S \frac{dS'}{\left|\bar{r} - \bar{r}'\right|}$$

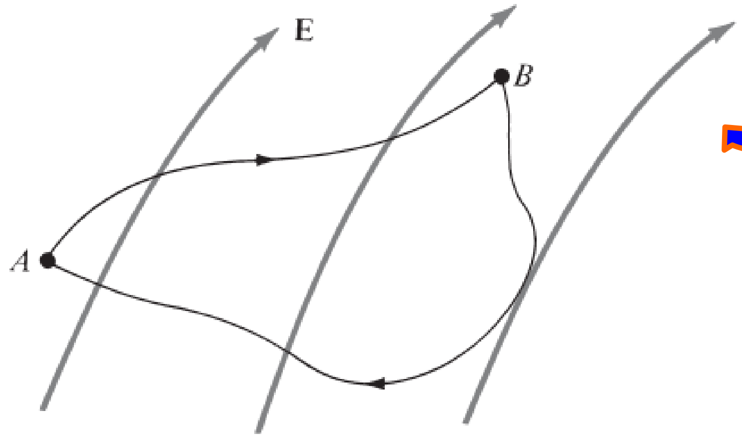
$$V(\bar{r}) = \frac{\rho_S}{4\pi\varepsilon_0 R} \iint_S dS'$$

$$\therefore V(\bar{r}) = \frac{\rho_S}{4\pi\varepsilon_0 R}\left(4\pi R^2\right) = \frac{\rho_S R}{\varepsilon_0}$$

Relationship between $\vec{E}$ and V

- We have learnt that the electrostatic field is conservative and therefore, following is true for the given situation:

$$V_{BA} = -V_{AB} \Rightarrow V_{BA} + V_{AB} = \oint_L \vec{E}.\overline{dl} = 0$$



Line integral of $\vec{E}$ along a closed path is zero

- Lets apply Stoke's theorem: $\oint_L \vec{E}.\overline{dl} = 0 = \int_S \left(\nabla \times \vec{E}\right).\overline{dS} \Rightarrow \boxed{\nabla \times \vec{E} = 0}$

Conservative or Irrotational

Relationship between $\vec{E}$ and V (contd.)

Intergal Form

$$V_{BA} + V_{AB} = \oint_L \vec{E}.\overline{dl} = 0$$

Maxwell’s equations for Electrostatics

$$\nabla \times \vec{E} = 0$$

Differential Form

- We defined potential as: $V = -\int \vec{E}.\overline{dl} \qquad \Rightarrow dV = -\vec{E}.\overline{dl}$

$$\Rightarrow dV = -E_x dx - E_y dy - E_z dz$$

- Alternatively we can also write: $dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$

- Comparison gives: $E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$

- Therefore: $\vec{E} = -\nabla V$

Relationship between $\vec{E}$ and V (contd.)

$$\vec{E} = -\nabla V$$

The electric field intensity is the gradient of V. The negative sign shows that the direction of $\vec{E}$ is opposite to the direction in which V increases $\leftrightarrow$ $\vec{E}$ is directed from higher to lower levels of V

It provides another tool to determine electric field apart from Coulomb's and Gauss's laws $\rightarrow$ $\vec{E}$ can be obtained if the scalar function V is known

Example – 5

- Determine Electric Field due to potential: $V = \rho^2(z+1)sin\phi$

$$-\vec{E} = \nabla V \quad \Longrightarrow \quad -\vec{E} = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\Rightarrow -\vec{E} = 2\rho(z+1)\sin\phi\hat{a}_\rho + \rho(z+1)\cos\phi\hat{a}_\phi + \rho^2\sin\phi\hat{a}_z$$

$$\therefore \vec{E} = -2\rho(z+1)\sin\phi\hat{a}_\rho - \rho(z+1)\cos\phi\hat{a}_\phi - \rho^2\sin\phi\hat{a}_z$$

Example – 6

- Determine Electric Field due to potential: $V = e^{-r} sin\theta cos2\phi$

$$-\vec{E} = \nabla V \quad \Rightarrow \quad -\vec{E} = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

$$\Rightarrow -\vec{E} = -e^{-r}\sin\theta\cos 2\phi\hat{a}_r + \frac{1}{r}e^{-r}\cos\theta\cos 2\phi\hat{a}_\theta + \frac{e^{-r}}{r}(-2\sin 2\phi)\hat{a}_\phi$$

$$\therefore \vec{E} = e^{-r}\sin\theta\cos 2\phi\hat{a}_r - \frac{1}{r}e^{-r}\cos\theta\cos 2\phi\hat{a}_\theta + \frac{2e^{-r}}{r}\sin 2\phi\hat{a}_\phi$$

Example – 7

- Given that $\vec{E} = (3x^2 + y)\hat{a}_x + x\hat{a}_y \; kV/m$, find the work done in moving a $-2\mu C$ charge from $(0, 5, 0)$ to $(2, -1, 0)$ by taking the straight line path:

(a) $(0, 5, 0) \;\rightarrow\; (2, 5, 0) \;\rightarrow\; (2, -1, 0)$
(b) $y = 5 - 3x$

Poisson's and Laplace's Equation

- From Gauss's Law: $\nabla.\vec{E} = \frac{\rho_v}{\varepsilon_0}$
- We have: $\vec{E} = -\nabla V$

$$\nabla.(\nabla V) = -\frac{\rho_v}{\varepsilon_0}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon_0}$$

(Poisson's Equation)

- If the medium under consideration contains no charge then: $\nabla^2 V = 0$

Laplace's Equations

These formulations are extremely useful for determining the electrostatic potential V in regions with boundaries on which V is known, such as the regions between the plates of a capacitor with specified voltage difference across it.

Electric Dipole

- An electric dipole is formed when two point charges of equal magnitude but opposite signs are separated by a small distance → a pretty useful configuration!!!
- In practical situation, the distance from the point of interest is much greater than the separation.
- The potential at point $P(r,\theta,\phi)$ is:

$$V=\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1}-\frac{1}{r_2}\right]=\frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2-r_1}{r_1r_2}\right]$$

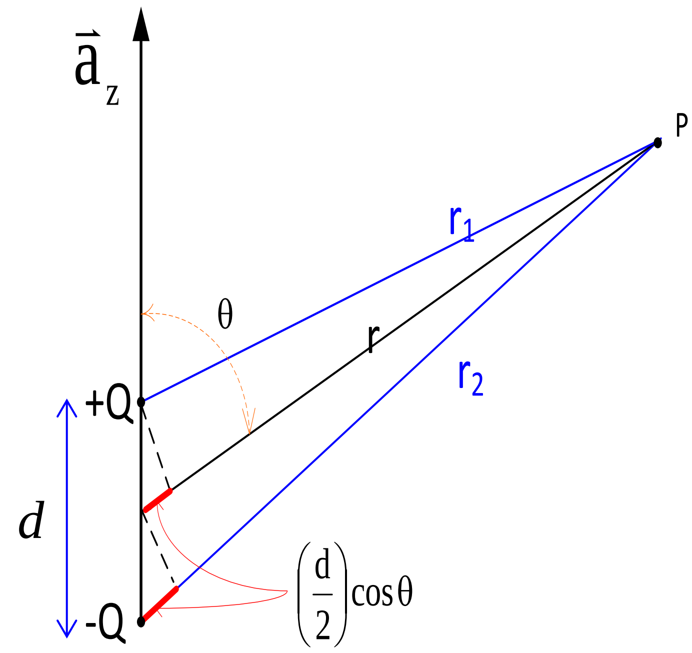
- Since $r \gg d$; r, r_1, and r_2 are almost parallel.

$$r_1=r-\frac{d}{2}\cos\theta$$

$$r_2=r+\frac{d}{2}\cos\theta$$

$$\therefore r_2-r_1=d\cos\theta$$

- Furthermore: $r_1r_2\approx r^2$



Electric Dipole (contd.)

$$\Rightarrow V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$$

- Since, $d\cos\theta = \bar{d}.\hat{a}_r$

$$\Rightarrow V = \frac{Q\bar{d}.\hat{a}_r}{4\pi\varepsilon_0 r^2}$$

$Q\bar{d} = \bar{p}$ is the dipole moment

$$\therefore V = \frac{\bar{p}.\hat{a}_r}{4\pi\varepsilon_0 r^2}$$

Dipole moment is directed from –Q to +Q

- If the dipole center is not at the origin but at r' then:

$$V = \frac{\bar{p}.\left(\bar{r}-\bar{r}'\right)}{4\pi\varepsilon_0\left|\bar{r}-\bar{r}'\right|^3}$$

- The electric field due to the dipole with center at the origin:

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right]$$

$$\vec{E} = \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

Electric Dipole (contd.)

$$\Rightarrow \vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\therefore \vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$p = \left|\overline{p}\right| = Qd$$

- It is important to notice that a point charge is a monopole and its $\vec{E}$ varies inversely as r^2 while its V varies inversely as r. **For a dipole**, the respective variations are $\vec{E} \propto \frac{1}{r^3}$ **and** $V \propto \frac{1}{r^2}$ while $\vec{E}$ due to successive higher-order multipoles vary inversely as r^4, r^5, r^6, …., and their corresponding V vary inversely as r^3, r^4, r^5, ……

Example – 8

- Point charges Q and $-Q$ are located at $(0,\frac{d}{2},0)$ and $\left(0,-\frac{d}{2},0\right)$. Show that at point (r,θ,ϕ), where $r \gg d$,

$$V=\frac{Qd\sin\theta\sin\phi}{4\pi\varepsilon_0 r^2}$$

- Find the corresponding $\vec{E}$ as well.

- The dipole is oriented along y-axis. Therefore:

$$V=\frac{Qd\sin\theta\sin\phi}{4\pi\varepsilon_0 r^2} \Leftarrow \bar{p}.\hat{a}_r = Qd\hat{a}_y.\hat{a}_r = Qd\sin\theta\sin\phi \Rightarrow \therefore V=\frac{Qd\sin\theta\sin\phi}{4\pi\varepsilon_0 r^2}$$

Now: $\vec{E}=-\nabla V \Rightarrow \vec{E}=-\frac{\partial V}{\partial r}\hat{a}_r-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{a}_\theta-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{a}_\phi$

$$\Rightarrow \vec{E}=\frac{Qd}{4\pi\varepsilon_0}\left\{\frac{2\sin\theta\sin\phi}{r^3}\hat{a}_r-\frac{\cos\theta\sin\phi}{r^3}\hat{a}_\theta-\frac{\cos\phi}{r^3}\hat{a}_\phi\right\}$$

$$\therefore \vec{E}=\frac{Qd}{4\pi\varepsilon_0 r^3}\left\{2\sin\theta\sin\phi\hat{a}_r-\cos\theta\sin\phi\hat{a}_\theta-\cos\phi\hat{a}_\phi\right\}$$

Equipotential Surfaces

- Equipotential surfaces are defined as surfaces over which the potential is constant.

$$V(\bar{r}) = constant$$

- At each point on the surface, the electric field is perpendicular to the surface since the electric field, being the gradient of potential, does not have component along a surface of constant potential.
- We have seen that any charge on a conductor must reside on its surface. These charges would move along the surface if there were a tangential component of the electric field. The electric field must therefore be along the normal to the surface of a conductor. The conductor surface is, therefore, an equipotential surface.
- Electric field lines are perpendicular to equipotential surfaces (or curves) and point in the direction from higher potential to lower potential.
- In the region where the electric field is strong, the equipotentials are closely packed as the gradient is large.

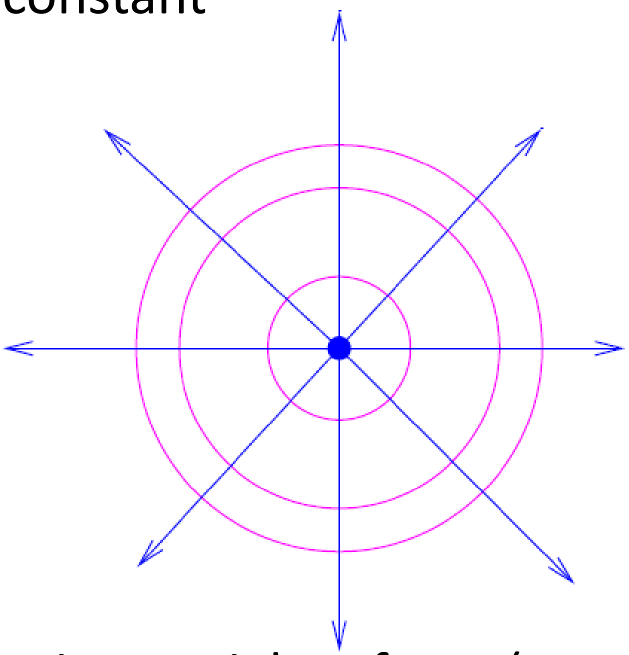
Equipotential Surfaces (contd.)

Example – 9: Determine the equipotential surface for a point charge.

- Let the point charge q be located at the origin. The equation to the equipotential surface is given by:

$$V(x, y.z) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{x^2 + y^2 + z^2}} = V_0 \quad = \text{constant}$$

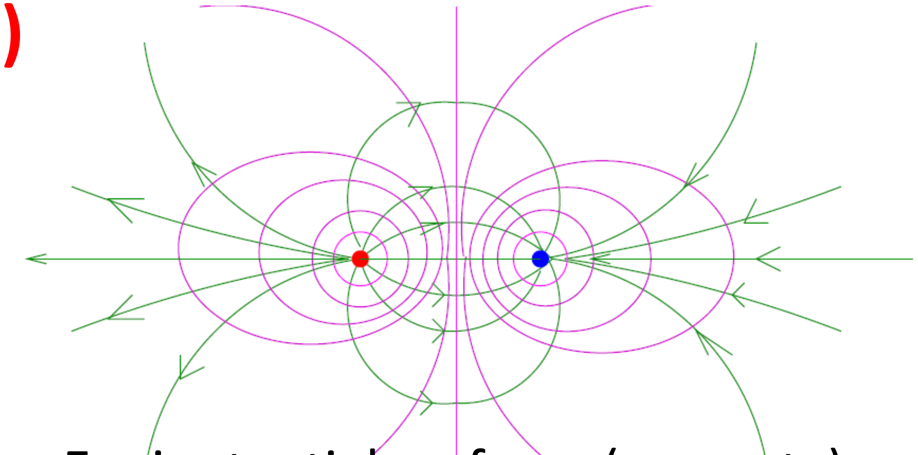
Thus the surfaces are concentric spheres with the origin (the location of the charge) as the centre.



Equipotential surfaces (magenta) and field lines (blue) for a positive charge

Equipotential Surfaces (contd.)

- The equipotential surfaces of an electric dipole is:



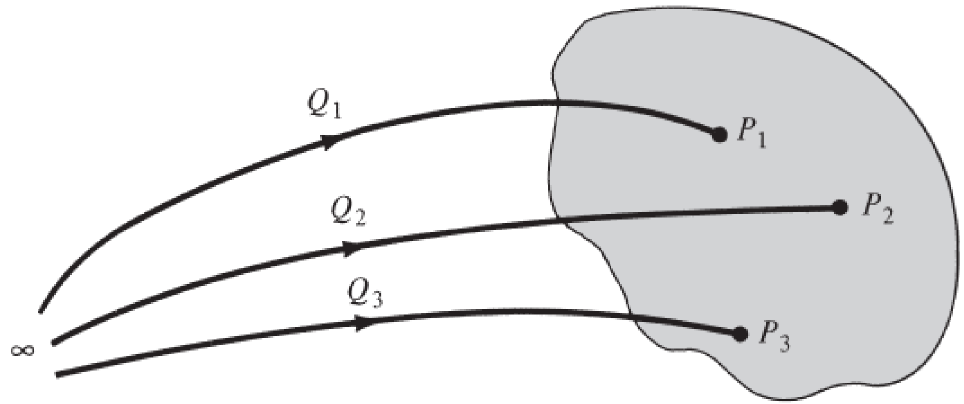
Equipotential surfaces (magenta) and field lines (green)

Importance of Equipotential Surfaces will be apparent when we discuss conducting bodies in $\vec{E}$

A typical use of field lines and equipotential surfaces is found in the diagnosis of human heart. The human heart beats in response to an electric potential difference across it. The heart can be characterized as a dipole with the field map similar to that of an electric dipole. Such a field map is useful in detecting abnormal heart position.

Energy Density in Electrostatic Field

- To determine the energy in an assembly of charges, let us first determine the amount of work needed to assemble them.
- Suppose, 3 point charges Q_1, Q_2 and Q_3 need to be assembled in empty space.



- No work is required to transfer Q_1 from infinity to P_1 as the space is free from any charge and thus without any electric field.
- The work done in transferring Q_2 from infinity to P_2 is Q_2V_{21}.
- The work done in bringing Q_3 from infinity to P_3 is $Q_3(V_{32}+V_{31})$.
- Therefore: $W_E = W_1 + W_2 + W_3$ → $W_E = 0 + Q_2V_{21} + Q_3\left(V_{31} + V_{32}\right)$

Energy Density in Electrostatic Field (contd.)

- If the charges were positioned in reverse order, then:

$$W_E = W_3 + W_2 + W_1 \Rightarrow W_E = 0 + Q_2 V_{23} + Q_1 \left(V_{12} + V_{13} \right)$$

- Lets combine the two expressions to get:

$$2W_E = Q_1 \left(V_{12} + V_{13} \right) + Q_2 \left(V_{21} + V_{23} \right) + Q_3 \left(V_{31} + V_{32} \right) \Rightarrow 2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$\therefore W_E = \frac{1}{2} \left(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \right)$$

where V_1, V_2 and V_3 are the total potentials at P_1, P_2 and P_3 respectively. In general, if there are n point charges then:

$$\therefore W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k$$

Energy Density in Electrostatic Field (contd.)

- For continuous charge distributions:

$$W_E = \frac{1}{2}\int_L \rho_l V dl$$

$$W_E = \frac{1}{2}\int_S \rho_S V dS$$

$$W_E = \frac{1}{2}\int_v \rho_v V dv$$

- We know from Maxwell's equation for electrostatics: $\rho_v = \nabla.\vec{D}$

- Therefore: $$W_E = \frac{1}{2}\int_v \rho_v V dv = \frac{1}{2}\int_v \left(\nabla.\vec{D}\right) V dv$$

- We also know the relationship: $$\nabla.V\vec{D} = \vec{D}.\nabla V + V\left(\nabla.\vec{D}\right)$$

$$\Rightarrow V\left(\nabla.\vec{D}\right) = \nabla.V\vec{D} - \vec{D}.\nabla V$$

- Thus: $$W_E = \frac{1}{2}\int_v \left(\nabla.V\vec{D}\right) dv - \frac{1}{2}\int_v \left(\vec{D}.\nabla V\right) dv$$

Energy Density in Electrostatic Field (contd.)

- Application of Divergence Theorem leads to:

$$W_E = \frac{1}{2}\oint_S \left(V\vec{D}\right).\overline{dS} - \frac{1}{2}\int_v \left(\vec{D}.\nabla V\right)dv$$

For large surface

$$\frac{1}{2}\oint_S \left(V\vec{D}\right).\overline{dS} \to 0$$

- Thus:

$$W_E = -\frac{1}{2}\int_v \left(\vec{D}.\nabla V\right)dv$$

$\vec{E} = -\nabla V$

$$W_E = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right)dv$$

- We know that $\vec{D} = \varepsilon_0\vec{E}$:

$$\therefore W_E = \frac{1}{2}\int_v \varepsilon_0 E^2 dv$$

- Therefore energy density w_E [in J/m^3) is:

$$W_E = \frac{dW_E}{dv} = \frac{1}{2}\vec{D}.\vec{E} = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}$$

Example – 10

- If $V = \rho^2 z sin\phi$, calculate the energy within the region defined by $1 < \rho < 4, -2 < z < 2, 0 < \phi < \frac{\pi}{3}$

Example – 11

- Point charges $Q_1 = 1\,nC, Q_2 = -2\,nC, Q_3 = 3\,nC, and\ Q_4 = -4\,nC$ are positioned one at a time and in that order at $(0,0,0)$, $(1,0,0)$, $(0,0,-1)$, and $(0,0,1)$, respectively. Calculate the energy in the system after each charge is positioned.

Electrostatic Discharge (ESD)

- It refers to the sudden transfer of static charge between objects at different electrostatic potential.
- For example, the **"zap"** you feel while walking on a synthetic carpet and then touching a metal doorknob.
- **Design of mechanism to protect** electronic devices, systems, and equipments against the static electricity is extremely important.

Please go through the additional materials posted on course URL to know about ESD, its impact, and the associated issues and solutions

New Beginning

- We have been studying the electrostatics of **free space** (i.e., a vacuum).

But, the universe is full of **stuff!**

Q: Does stuff (material) affect our electrostatics knowledge?
A: ???

Convection and Conduction Current

- The current through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt}$$

- Now, if the current ΔI flows through a planar surface ΔS then:

$$\frac{\Delta I}{\Delta S} = J$$

Current Density

$$\Rightarrow \Delta I = J \Delta S$$

When current density is perpendicular to the surface

- For the case when current density is not normal to the surface:

$$\Delta I = \vec{J}.\overline{\Delta S}$$

Total current flowing through the surface

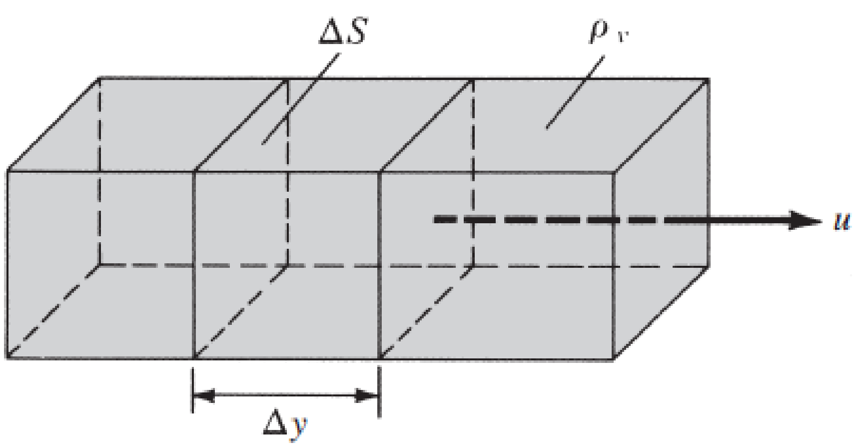
$$I = \int_S \vec{J}.\overline{dS}$$

current "I" through S is the flux of current density $\vec{J}$

Convection and Conduction Current (contd.)

- "I" can be produced in three ways and therefore three kinds of current density exist: Convection Current Density, Conduction Current Density, and Displacement Current Density.
- The derived expression for current density is valid for any type of current.
- Convection current doesn't involve conductors and as a consequence doesn't satisfy Ohm's Law.
- It occurs when current flows through an insulating medium such as liquid, rarefied gas, or a vacuum.
- A beam of electrons in a vacuum tube, for example, is a convection current.
- For example, if there is a charge flow, of density ρ_v, at velocity $\vec{u} = uy\hat{a}_y$ then:

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t}$$



Convection and Conduction Current (contd.)

$$\Rightarrow \Delta I = \rho_v \Delta S \frac{\Delta y}{\Delta t} = \Delta S \rho_v u_y$$

Y-directed current density

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y$$

Generic Expression

$$\vec{J} = \rho_v \vec{u}$$

Convection Current Density

- **Conduction current** requires conductor.
- A conductor is characterized by a large number of free electrons that provide conduction current due to an applied electric field.
- The force due to an electric field $\vec{E}$ on an electron with charge $-e$ is:

$$\vec{F} = -e\vec{E}$$

Convection and Conduction Current (contd.)

- Since the electron isn't free in space, it will not experience an average acceleration under the influence of electric field.
- Instead, it suffers constant collisions with the atomic lattice and drifts from one atom to another.
- If electron of mass m is moving in an electric field $\vec{E}$ with an average drift velocity $\vec{u}$ then:

$$\frac{m\vec{u}}{\tau} = -e\vec{E} \quad \Rightarrow \quad \vec{u} = -\frac{e\tau}{m}\vec{E}$$

τ is average time between collisions

- If there are n electrons per unit volume:

$$\rho_v = -ne \quad \Rightarrow \quad \vec{J} = \rho_v \vec{u} = \frac{ne^2\tau}{m}\vec{E} \quad \Rightarrow \quad \vec{J} = \sigma\vec{E}$$

$$\frac{ne^2\tau}{m} = \sigma$$

Conductivity of Conductor

Point form of Ohm's Law

Example – 12

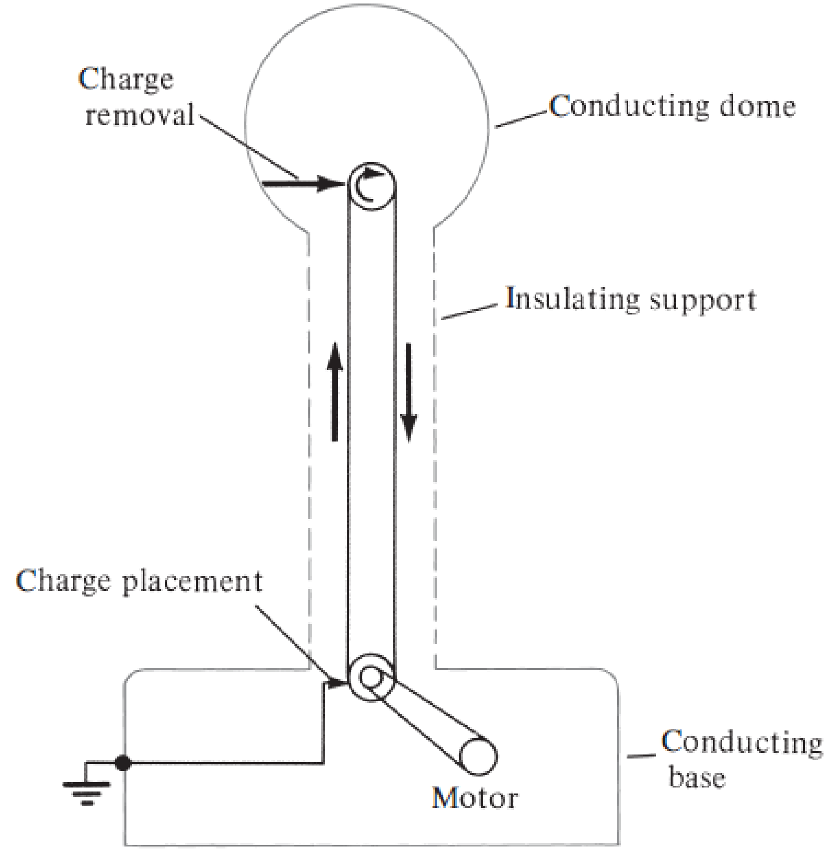
- For the current density $\vec{J} = 10z\sin^2\phi\,\hat{a}_\rho\ A/m^2$, find the current through the cylindrical surface $\rho = 2, 1 \le z \le 5\ m$.

$$\overrightarrow{dS} = \rho\, d\phi\, dz\, \hat{a}_\rho \;\Rightarrow\; I = \int_S \vec{J}.\overrightarrow{dS} \;\Rightarrow\; I = \int_{\phi=0}^{2\pi}\int_{z=1}^{5} 10z\sin^2\phi\,\rho\, dz\, d\phi\,\Big|_{\rho=2}$$

$$\Rightarrow I = \int_{\phi=0}^{2\pi}\int_{z=1}^{5} 10z\sin^2\phi\,\rho\, dz\, d\phi\,\Big|_{\rho=2} = 10(2)\left[\frac{z^2}{2}\right]_1^5 \int_0^{2\pi} \frac{1}{2}\left(1-\cos 2\phi\right) d\phi \;\Rightarrow\; \therefore I = 240\pi = 754\ \text{A}$$

Example – 13

- A typical example of convective charge transport is found in the Van de Graaf generator where charge is transported on a moving belt from the base to the dome as shown in Figure.
- If a surface charge density $10^{-7}\ C/m^2$ is transported by the belt at a velocity of $2\ m/s$, calculate the charge collected in $5s$. Take the width of the belt as $10\ cm$.



$$I = (\rho_S w)u \quad \rightarrow \quad Q = It = (\rho_S w)ut \quad \rightarrow \quad \Rightarrow Q = (10^{-7} \times 0.1) \times 2 \times 5$$

$$\therefore Q = 100nC$$